Subdiffusive fractional limit of a jump-renewal equation

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Abstract

In this paper, we consider an age-structured jump model that arises as a description of continuous time random walks with infinite mean waiting time between jumps. We prove that under a suitable rescaling, this equation converges in the long time large scale limit to a time fractional subdiffusion equation.

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1 Introduction

A distinctive feature of Brownian motion is that the variance of the travelled distance, or mean squared displacement (MSD), scales linearly with time. However, accumulated experimental recordings of the motion of biomolecules in living cells evidence random motions where the MSD grows sublinearly, as a powerlaw in time, say as t^α with $0 < \alpha < 1$. Such a nonlinear growth of the MSD is usually referred to as anomalous diffusion, and more precisely as subdiffusion in the case of a sublinear growth. Subdiffusion has been experimentally reported in various biological systems, from actin networks [1] to active intracellular transport [7], or in single-molecule tracking experiments of transcription factors in the nucleus [14].

A very abundant literature has proposed mathematical models able to account for subdiffusion in biology. We refer firstly to several textbooks (and the references therein), namely, two introductory books [9, 10] or a more cell-oriented modelling point of view in [4, chap. 6-7]. A variety of points of view has been expressed regarding what models may be at the origin of the experimental records showing anomalous diffusion [5, part III]; the interested reader will find a collection of applicative examples in [13, 22, 26]. Historically, the first tool proposed to model mathematically such behaviors has been the so called continuous time random walk (CTRW) [18], a generalization of random walk where the duration between two random successive jumps, during which the walker maintains his position, is modelled as a random variable with associated distribution. Assuming separability, such a process is therefore characterized by two probability distributions, one for the jumps in the position space,

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typically a kernel w on $\mathbb{R}^d$, and a second one for the waiting or residence time, that we will call ϕ, on the time half line $[0, \infty)$.

We denote by $r(t, a, x)$ the density of particles, molecules, or individuals, which, at time t, have been resident at position x for the last a time units. This density distribution r satisfies the following nonlocal partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t} r(t,a,x) + \dfrac{\partial}{\partial a} r(t,a,x) + \beta(a) r(t,a,x) = 0, & t, a > 0, \quad x \in \mathbb{R}^d, \\ \mathbf{r}(t,x) := r(t,0,x) = \displaystyle\int_0^\infty \int_{\mathbb{R}^d} w(x-y)\beta(a) r(t,a,y)\, dy da, & \\ r(0,a,x) = r^0(a,x), & \end{cases} \tag{1}$$

where β is a jump rate, or escape rate. Upon arriving at a new position at time 0, the probability of not jumping before time t, sometimes called the *survival function*, is given by

$$\psi(t) = e^{-\int_0^t \beta(s) ds}.$$

The density probability function of the times of jumps is then given by

$$\phi(t) = \beta(t)\psi(t) = \beta(t)\, e^{-\int_0^t \beta(s) ds} = -\psi'(t).$$

This model falls into the class of age-structured models, since the residence time a behaves like an "age" for the particles, which is by definition reset to zero when a jump occurs. A similar equation appears in the context of kinetic models in domains with holes in [2]. For more on age-structured modelling, we refer to the classical textbook [19].

Equation (1) was first proposed by [23] as an alternative to the generalized master equation (GME) which is often used to describe CTRW, see for instance, among many others, [12, 17, 21]. As noticed in [24], unlike the GME which does not incorporate age variable, Equation (1) corresponds to a Markovian description of the random walk. At the mesoscopic level it means that Equation (1) enjoys a semigroup property. This makes it more relevant than GME in particular for incorporating reaction terms, see [25, 27, 16].

We suppose that w is a probability measure on $\mathbb{R}^d$ with finite second moment and zero first moment, namely

$$w \geq 0, \qquad \int_{\mathbb{R}^d} w(dx) = 1, \qquad \int_{\mathbb{R}^d} x w(dx) = 0, \qquad \sigma^2 := \int_{\mathbb{R}^d} |x|^2 w(dx) < \infty. \tag{2}$$

The mean waiting time of a trapped particle before escaping is given by

$$T := \int_0^\infty t\phi(t) dt = \int_0^\infty \psi(t) dt.$$

When this quantity is infinite, the underlying random process is expected to exhibit a subdiffusive behaviour, and diffusive when it is finite (recall that we are in the case $\sigma^2 < \infty$), see for instance [17]. Here we are interested in the subdiffusive case and we will then assume that $T = +\infty$. More precisely we suppose that β is a bounded continuous function on $[0, \infty)$ such that

$$\psi(t) = \Psi t^{-\alpha} + O(t^{-\alpha-\delta}) \qquad \text{as} \quad t \to +\infty \tag{3}$$

for some constants $\Psi > 0$, $\alpha \in (0,1)$ and $\delta \in (0, 1-\alpha)$. A prototypical example is provided by the jump rate functions $\beta(a) = \alpha/(K+a)$, with $K > 0$, for which $\psi(t) = 1/(1+t/K)^\alpha$.

For such coefficients, we have the following results about the behavior of the solutions to Equation (1), and more precisely about the behavior of the marginal

$$\rho(t,x) := \int_0^\infty r(t,a,x)\, da.$$

Proposition 1. *Under Assumptions* (2) *and* (3)*, the solutions to Equation* (1) *satisfy*

$$\int_{\mathbb{R}^d} \rho(t,x)\, dx = \int_{\mathbb{R}^d} \rho^0(x)\, dx, \qquad \forall t \geq 0, \tag{4}$$

$$\int_{\mathbb{R}^d} x\rho(t,x)\, dx = \int_{\mathbb{R}^d} x\rho^0(x)\, dx, \qquad \forall t \geq 0, \tag{5}$$

and, if $\int_{\mathbb{R}^d} |x|^2 \rho^0(x)\, dx < \infty$,

$$\int_{\mathbb{R}^d} |x|^2 \rho(t,x)\, dx \sim \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{\sigma^2}{\Psi} \Big(\int_{\mathbb{R}^d} \rho^0(x) dx \Big) t^\alpha, \qquad \text{as } t \to \infty. \tag{6}$$

The result in eq. (6) emphasizes the fact that the MSD of the underlying random walk grows sublinearly (more precisely as t^α), thus showing that the long time limit of the age-structured model eq. (1) exhibits subdiffusion.

This result also motivates performing a time-space rescaling $(t,a,x) \to (t/\epsilon^{2/\alpha}, a, x/\epsilon)$. More precisely we consider the equation

$$\begin{cases} \epsilon^{2/\alpha} \dfrac{\partial}{\partial t} r_\epsilon(t,a,x) + \dfrac{\partial}{\partial a} r_\epsilon(t,a,x) + \beta(a) r_\epsilon(t,a,x) = 0, & t,a > 0, \quad x \in \mathbb{R}^d, \\ \mathbf{r}_\epsilon(t,x) := r_\epsilon(t,0,x) = \displaystyle\int_0^\infty \int_{\mathbb{R}^d} w_\epsilon(x-y) \beta(a) r_\epsilon(t,a,y)\, dy da, & \\ r_\epsilon(0,a,x) = r_\epsilon^0(a,x), & \end{cases} \tag{7}$$

where we have set

$$w_\epsilon(z) = \epsilon^{-d}\, w(z/\epsilon).$$

For a survival function ψ satisfying (3) and a jump kernel satisfying (2), it was obtained by [27], see also [16, Chap. 2.3], through a formal derivation based on asymptotic expansion in the Laplace (in time) – Fourier (in space) domain, that the dynamics of the marginal ρ of the solutions to Equation (1) is prescribed in the large scale and large time regime by the time fractional heat equation

$$\partial_t^\alpha \rho(t,x) = D_\alpha\, \Delta \rho(t,x). \tag{8}$$

In this equation, $D_\alpha > 0$ is a (sub)diffusion coefficient and ∂_t^α stands for the Caputo fractional derivative defined for $f \in C^1([0,\infty))$ by

$$\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds$$

with $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ the Gamma function. In the last years, many mathematical properties of such time fractional parabolic equations have been obtained, see for instance [8, 15, 28].

The main objective of the present paper is to derive rigorously Equation (8) as the limit of Equation (7) when $\epsilon \to 0$. Prior to that, we need to give a definition of what we call a solution to Equation (8). It can be shown, see Corollary 6 in Section 2, that for $f \in C^1([0,\infty))$ and $g \in C([0,\infty))$, the equation

$$\partial_t^\alpha f = g$$

is equivalent to the integrated form

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Noticing that the second formulation requires less regularity on f, this motivates our definition in a mild sense of solution to Equation (8). Concerning the regularity in space, we only require $\rho(t,\cdot)$ to be a finite positive measure. We then consider $\mathcal{M}(\mathbb{R}^d) = (C_0(\mathbb{R}^d))'$ the space of finite signed measures that we endow with its weak-$*$ topology, namely

$$\rho_n \rightharpoonup \rho \quad \text{in } \mathcal{M}(\mathbb{R}^d) \qquad \Longleftrightarrow \qquad \forall \varphi \in C_0(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \varphi(x)\rho_n(dx) \to \int_{\mathbb{R}^d} \varphi(x)\rho(dx),$$

where $C_0(\mathbb{R}^d)$ is the set of continuous functions on $\mathbb{R}^d$ that tend to zero at infinity. The positive cone of $\mathcal{M}(\mathbb{R}^d)$, which is made of the finite positive measures on $\mathbb{R}^d$, is denoted as $\mathcal{M}_+(\mathbb{R}^d)$.

Definition 2. *We say that $\rho \in C([0,\infty), \mathcal{M}(\mathbb{R}^d))$ is a solution to Equation* (8) *with initial condition $\rho(0) = \rho^0 \in \mathcal{M}(\mathbb{R}^d)$ if for any $\varphi \in C_c^2(\mathbb{R}^d)$ and all $t \geq 0$*

$$\int_{\mathbb{R}^d} \varphi(x)\rho(t,dx) = \int_{\mathbb{R}^d} \varphi(x)\rho^0(dx) + \frac{D_\alpha}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \int_{\mathbb{R}^d} \Delta\varphi(x)\rho(s,dx)\, ds. \tag{9}$$

We now state the main result of the paper.

Theorem 3. *Suppose that Assumptions* (2) *and* (3) *hold, that*

$$r_\epsilon^0 \geq 0 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \int_0^\infty r_\epsilon^0(a,x)(1+a)^\alpha da dx \leq M \tag{10}$$

for some $M > 0$ and all $\epsilon \in (0,1)$, and that

$$\int_{\mathbb{R}^d} \int_0^\infty r_\epsilon^0(a,x)\varphi(x)\, da dx \to \int_{\mathbb{R}^d} \varphi(x)\rho^0(dx) \qquad \text{as } \epsilon \to 0 \tag{11}$$

for some $\rho^0 \in \mathcal{M}_+(\mathbb{R}^d)$ and all $\varphi \in C_0(\mathbb{R}^d)$.

Then there exists a sequence $(\epsilon_n)_{n\geq 1}$ decreasing towards zero and a solution $\rho \in C([0,\infty), \mathcal{M}(\mathbb{R}^d))$ to Equation (8) *with initial condition $\rho(0) = \rho^0$, in the sense of Definition 2, with*

$$D_\alpha = \frac{\sigma^2}{2d\Psi\Gamma(1-\alpha)}$$

such that for all $T > 0$, the solution ρ_{ϵ_n} to Equation (7) *with $\epsilon = \epsilon_n$ satisfies*

$$\rho_{\epsilon_n} \to \rho \qquad \text{in } C([0,T], \mathcal{M}(\mathbb{R}^d)) \text{ as } n \to \infty.$$

Let us make a few comments on this theorem.

Typical examples of initial distributions that verify (11) are:
- the case of well prepared initial conditions where $r^0_\epsilon(a,x)=r^0(a,x)$ is independent of ϵ,
- the case of rescaling Equation (1) with $r_\epsilon(t,a,x)=\epsilon^{-d}r(t/\epsilon^{2/\alpha},a,x/\epsilon)$ which satisfies (7) with initial condition $r^0_\epsilon(a,x)=\epsilon^{-d}r^0(a,x/\epsilon)$ and verifies the convergence (11) with $\rho^0=\delta_0$.

A similar result as Theorem 3 was obtained recently in [20] by means of Laplace transform in time. Contrary to their approach, our method consists in deriving estimates in the physical domain. In particular our result does not rely on the injectivity of the Laplace transform.

A different scaling of Equation (1) was performed in [6] together with a Hopf-Cole transformation, yielding a Hamilton-Jacobi limiting equation. The Hamiltonian encodes the subdiffusive character of the equation through its behavior at the origin, which is a powerlaw with power $2/\alpha$.

The last comment is about the solutions of Equation (8) in the sense of Definition 2, which satisfy the same moments estimates as for the solutions of Equation (1).

Proposition 4. *The solutions of Equation* (8) *satisfy* (4), (5), *and*

$$\int_{\mathbb{R}^d}|x|^2\rho(t,dx)=\int_{\mathbb{R}^d}|x|^2\rho^0(dx)+\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{\sigma^2}{\Psi}\Big(\int_{\mathbb{R}^d}\rho^0(dx)\Big)t^\alpha,\qquad\forall t\geq 0. \tag{12}$$

In the next section, we provide preliminary results that will be useful for proving Theorem 3 and we give the proofs of Propositions 1 and 4. The last section is devoted to the proof of Theorem 3.

2 Preliminary results and proofs of the propositions

Integrating Equation (1) in the space variable, we readily obtain that the quantity

$$n(t,a):=\int_{\mathbb{R}^d}r(t,a,x)\,dx$$

verifies the equation

$$\begin{cases}\partial_t n+\partial_a n+\beta(a)n=0,\\ \mathbf{n}(t):=n(t,0)=\displaystyle\int_0^\infty\beta(a)n(t,a)da,\\ n(0,a)=n^0(a).\end{cases} \tag{13}$$

This equation is sometimes known as the conservative renewal equation, in the sense that the integral $\int_0^\infty n(t,a)da$ is preserved along time. It has been widely studied in the literature. Most of the studies are in the case where $\liminf_{a\to\infty}\beta(a)>0$, see for instance [11] and the references therein. For the case where β tends to zero as α/a when $a\to\infty$, which is the case we are considering here, see [3] and the references therein.

For $f,g\in L^1_{loc}(0,\infty)$, we define the convolution $f*g\in L^1_{loc}(0,\infty)$ by

$$f*g(t)=\int_0^t f(s)g(t-s)\,ds=\int_0^t f(t-s)g(s)\,ds.$$

We recall the useful commutativity and associativity properties $f*g=g*f$, $(f*g)*h=f*(g*h)$.

We also define, for any $\nu > 0$, the function $Y_\nu \in L^1_{loc}(0,\infty)$ by

$$Y_\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)},$$

so that for $\alpha \in (0,1)$ we have

$$\partial_t^\alpha f(t) = Y_{1-\alpha} * f'(t).$$

We recall a classical result about the convolution of such functions Y_ν.

Lemma 5. *For any $\mu, \nu > 0$ one has*

$$Y_\mu * Y_\nu = Y_{\mu+\nu}.$$

In particular, $Y_\alpha * Y_{1-\alpha} = Y_1 = 1$.

Proof. It is based on the standard formula

$$\int_0^1 (1-s)^{\mu-1} s^{\nu-1} ds = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. \tag{14}$$

For the sake of completeness we recall the classical proof of this formula through a change of variable. Starting from

$$\Gamma(\mu)\Gamma(\nu) = \int_0^\infty u^{\mu-1} e^{-u} du \int_0^\infty v^{\nu-1} e^{-v} dv$$

and using the change of variable $(u,v) = (sw, (1-s)w)$, the function $\varphi : (0,1) \times (0,\infty) \to (0,\infty)^2$ defined by $\varphi(s,w) = ((1-s)w, sw)$ being a bijection with Jacobian determinant equal to $-w$, we get that

$$\begin{aligned}
\Gamma(\mu)\Gamma(\nu) &= \int_0^\infty \int_0^1 (1-s)^{\mu-1} w^{\mu-1} e^{-(1-s)w} (sw)^{\nu-1} e^{-sw} w \, ds dw \\
&= \int_0^\infty w^{\mu+\nu-1} e^{-w} dw \int_0^1 (1-s)^{\mu-1} s^{\nu-1} ds,
\end{aligned}$$

and the result. Now for $t > 0$ we have by using the change of variable $s \to ts$ that

$$\int_0^t (t-s)^{\mu-1} s^{\nu-1} ds = t^{\mu+\nu-1} \int_0^1 (1-s)^{\mu-1} s^{\nu-1} ds$$

which readily gives by using (14)

$$Y_\mu * Y_\nu(t) = Y_{\mu+\nu}(t).$$

$\square$

Corollary 6. *For $f \in C^1([0,\infty),\mathbb{R})$ and $g \in C([0,\infty),\mathbb{R})$ we have*

$$\Big[\ \forall t \geq 0, \quad \partial_t^\alpha f(t) = g(t)\ \Big] \quad \Longleftrightarrow \quad \Big[\ \forall t \geq 0, \quad f(t) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds\ \Big].$$

Proof. The equivalence reads

$$Y_{1-\alpha} * f' = g \quad \Longleftrightarrow \quad f - f(0) = Y_\alpha * g.$$

From the left to the right, it suffices to convolve by Y_α, use Lemma 5 and note that $Y_1 * f'(t) = f(t) - f(0)$. For going from the right to the left, a convolution with $Y_{1-\alpha}$ first yields

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(t-s) - f(0)}{s^\alpha} ds = \int_0^t g(s)\, ds,$$

and a differentiation in t then gives the result. □

We now give useful estimates on the convolution of the boundary condition $\mathbf{n}$ of Equation (13) with Y_ν. We start with a result on the convolution of $\mathbf{n}$ with ψ, which extends the result in [20, Prop. 3].

Lemma 7. *Assume that $\psi(t) \searrow 0$ as $t \to \infty$, which is equivalent to considering $\beta \notin L^1(0, \infty)$. Then the boundary condition $\mathbf{n}(t)$ of Equation* (13) *satisfies*

$$\mathbf{n} * \psi(t) \to \int_0^\infty n^0(a)\, da \qquad \text{as} \quad t \to +\infty.$$

Proof. Using the method of characteristics, we have

$$n(t,a) = \mathbf{n}(t-a)\psi(a)\mathbb{1}_{t \geq a} + n^0(a-t)\frac{\psi(a)}{\psi(a-t)}\mathbb{1}_{t<a}.$$

Injecting this into the boundary condition we get

$$n(t,0) = \int_0^\infty \beta(a) n(t,a)\, da = \int_0^t \phi(a)\mathbf{n}(t-a)\, da + \int_t^\infty \frac{\phi(a)}{\psi(a-t)} n^0(a-t)\, da,$$

which also reads

$$\mathbf{n}(t) = \phi * \mathbf{n}(t) + \int_0^\infty \frac{\phi(a+t)}{\psi(a)} n^0(a)\, da. \tag{15}$$

Noticing that

$$Y_1 * \phi = 1 - \psi = Y_1 - \psi$$

we deduce from (15) that

$$\begin{aligned}
\psi * \mathbf{n}(t) = Y_1 * (\mathbf{n} - \phi * \mathbf{n})(t) &= \int_0^t \int_0^\infty \frac{\phi(a+s)}{\psi(a)} n^0(a)\, da ds \\
&= \int_0^\infty \frac{n^0(a)}{\psi(a)} \int_0^t \phi(a+s)\, ds da \\
&= \int_0^\infty \frac{n^0(a)}{\psi(a)} \big[\psi(a) - \psi(a+t)\big]\, da \\
&= \int_0^\infty n^0(a)\, da - \int_0^\infty \frac{\psi(a+t)}{\psi(a)} n^0(a)\, da \to \int_0^\infty n^0(a)\, da,
\end{aligned}$$

where we have used the dominated convergence theorem for the last convergence. Indeed, since ψ is a non-increasing function that goes to zero at infinity, we have $\psi(a+t)/\psi(a) \leq 1$ and $\psi(a+t)/\psi(a) \to 0$ as $t \to \infty$ for any $a \geq 0$. □

Remark 8. *Note that the assumption on ψ in Lemma 7 is weaker than* (3).

However, under Assumption (3) *and supposing besides, in the spirit of* (10), *that*

$$\int_0^\infty |n^0(a)|(1+a)^\alpha da \le M, \tag{16}$$

the convergence can be quantified. Indeed, (3) *ensures the existence of $C>0$ such that*

$$\frac{\psi(a+t)}{\psi(a)} \le C\left(\frac{1+a}{1+a+t}\right)^\alpha,$$

so that

$$\left|\psi * \mathbf{n}(t) - \int n^0\right| \le \int_0^\infty \frac{\psi(a+t)}{\psi(a)}|n^0(a)|\,da \le CM2^\alpha(1+t)^{-\alpha}.$$

Corollary 9. *Assume* (3)*. Then for any $\mu > 1-\alpha$ we have*

$$\mathbf{n} * Y_\mu(t) \sim \frac{\int_0^\infty n^0(a)\,da}{\Psi\Gamma(1-\alpha)}\,Y_{\mu+\alpha}(t) \qquad \text{as } t \to +\infty.$$

Proof. Since Equation (13) is linear, we can suppose without loss of generality that $n^0 \ge 0$, so that $n(t,\cdot) \ge 0$ for all $t \ge 0$.

We start by computing, for all $\nu > 0$,

$$\begin{aligned}
Y_\nu * \psi(t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\psi(s)\,ds \\
&= \frac{t^\nu}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}\psi(ts)\,ds \\
&= \frac{t^{\nu-\alpha}}{\Gamma(\nu)}\int_0^1 (1-s)^{\nu-1}s^{-\alpha}(\Psi + (ts)^\alpha\psi(ts) - \Psi)\,ds.
\end{aligned}$$

Since Assumption (3) ensures the existence of $C>0$ such that $|(ts)^\alpha\psi(ts) - \Psi| \le C(1+ts)^{-\delta}$, we infer, using (14) and the fact that $(1+ts)^{-\delta} \le (1+t)^{-\delta}s^{-\delta}$ when $s \in (0,1)$, that

$$Y_\nu * \psi(t) = \Psi\Gamma(1-\alpha)Y_{1+\nu-\alpha}(t) + \tau_\nu(t)t^{\nu-\alpha} \tag{17}$$

with

$$|\tau_\nu(t)| \le C\frac{\Gamma(1-\alpha-\delta)}{\Gamma(1+\nu-\alpha-\delta)}(1+t)^{-\delta} \xrightarrow[t\to\infty]{} 0.$$

Then we have

$$Y_{1+\nu-\alpha} = \frac{1}{\Psi\Gamma(1-\alpha)}Y_\nu * \psi - \frac{\Gamma(1+\nu-\alpha)}{\Psi\Gamma(1-\alpha)}\,\tau_\nu Y_{1+\nu-\alpha}.$$

We make the convolution with $\mathbf{n}$ and get

$$Y_{1+\nu-\alpha} * \mathbf{n}(t) = \frac{1}{\Psi\Gamma(1-\alpha)}Y_\nu * \psi * \mathbf{n}(t) - \frac{\Gamma(1+\nu-\alpha)}{\Psi\Gamma(1-\alpha)}(\tau_\nu Y_{1+\nu-\alpha}) * \mathbf{n}(t).$$

In the right hand side, since $\nu > 0$ and $\mathbf{n} * \psi \to \int n^0$ from Lemma 7, we have firstly

$$\begin{aligned}\frac{1}{\Psi\Gamma(1-\alpha)} Y_\nu * \psi * \mathbf{n}(t) &= \frac{1}{\Psi\Gamma(1-\alpha)} \frac{t^\nu}{\Gamma(\nu)} \int_0^1 (1-s)^{\nu-1} (\psi * \mathbf{n})(ts)\, ds \\ &\sim \frac{1}{\Psi\Gamma(1-\alpha)} \frac{t^\nu}{\Gamma(\nu)} \left(\int n^0\right) \frac{1}{\nu} = \frac{1}{\Psi\Gamma(1-\alpha)} \left(\int n^0\right) Y_{1+\nu}(t).\end{aligned}$$

For the second term we write, for some $\eta \in (0,1)$ to be chosen later,

$$\begin{aligned}\Gamma(1+\nu-\alpha)\,(\tau_\nu Y_{1+\nu-\alpha}) * \mathbf{n}(t) &= \int_0^t \tau_\nu(s) s^{\nu-\alpha} \mathbf{n}(t-s)\, ds \\ &= \int_0^{t^\eta} \tau_\nu(s) s^{\nu-\alpha} \mathbf{n}(t-s)\, ds + \int_{t^\eta}^t \tau_\nu(s) s^{\nu-\alpha} \mathbf{n}(t-s)\, ds = I_1 + I_2\end{aligned}$$

Since Equation (13) preserves the integral of the solutions along time and since β is positive and bounded by assumption, we have

$$0 \le \mathbf{n}(t) \le \|\beta\|_\infty \int_0^\infty n^0(a)\, da$$

and we can thus bound the first integral by

$$|I_1| \le \int_0^{t^\eta} |\tau_\nu(s) s^{\nu-\alpha} \mathbf{n}(t-s)|\, ds \le \|\tau_\nu\|_\infty \|\mathbf{n}\|_\infty \frac{(t^\eta)^{1+\nu-\alpha}}{1+\nu-\alpha}.$$

Choosing $\eta > 0$ small enough so that $\eta(1+\nu-\alpha) < \nu$, we get that $I_1 = o(t^\nu) = o(Y_{1+\nu})$. The second integral can be controlled by

$$\begin{aligned}|I_2| &\le \int_{t^\eta}^t |\tau_\nu(s) s^{\nu-\alpha} \mathbf{n}(t-s)|\, ds \\ &\le C \frac{\Gamma(1-\alpha-\delta)}{\Gamma(1+\nu-\alpha-\delta)} (1+t^\eta)^{-\delta} \int_{t^\eta}^t s^{\nu-\alpha} \mathbf{n}(t-s)\, ds \\ &\le C \frac{\Gamma(1-\alpha-\delta)\Gamma(1+\nu-\alpha)}{\Gamma(1+\nu-\alpha-\delta)} (1+t^\eta)^{-\delta} Y_{1+\nu-\alpha} * \mathbf{n}(t),\end{aligned}$$

so that $I_2 = o(Y_{1+\nu-\alpha} * \mathbf{n})$. Combining everything together,

$$Y_{1+\nu-\alpha} * \mathbf{n}(t) = \frac{1}{\Psi\Gamma(1-\alpha)} \left(\int n^0\right) Y_{1+\nu}(t)\,(1+o(1)) + o(Y_{1+\nu}(t)) + o(Y_{1+\nu-\alpha} * \mathbf{n}(t)),$$

which implies the desired equivalence by taking $\nu = \mu + \alpha - 1$. $\square$

Remark 10. *Recalling Remark 8 and looking carefully at above proof, we can see that under Assumption* (16) *the result of Corollary 9 can be made more precise with the estimate*

$$\mathbf{n} * Y_\mu(t) = \frac{\int_0^\infty n^0(a)\, da}{\Psi\Gamma(1-\alpha)} Y_{\mu+\alpha}(t) + O(t^{-\alpha} + t^{\eta\mu} + t^{-\eta\delta}) \qquad \text{as } t \to \infty$$

for any choice of η *such that* $0 < \eta < 1 - \frac{1-\alpha}{\mu}$.

Corollary 9 will be crucial in our proof of Theorem 3, but it also allows us to prove Proposition 1.

Proof of Proposition 1. The first two identities (4) and (5) are directly obtained by integration of Equation (1), by using that $\int w = 1$ and $\int xw = 0$ respectively. For the equivalence of the second moment in (6), we first show that

$$\frac{d}{dt}\int_0^\infty\int_{\mathbb{R}^d}|x|^2 r(t,a,x)\,dxda = \sigma^2\,\mathbf{n}(t). \tag{18}$$

Using again that $\int w = 1$, we have by integration of Equation (1) multiplied by $|x|^2$ that

$$\begin{aligned}
\int_0^\infty\int_{\mathbb{R}^d}|x|^2 r(t,a,x)\,dxda &= \int_{\mathbb{R}^d}|x|^2\mathbf{r}(t,x)\,dx - \int_0^\infty\int_{\mathbb{R}^d}|x|^2\beta(a)r(t,a,x)\,dxda\\
&= \int_0^\infty\iint_{\mathbb{R}^{2d}} w(x-y)|x|^2\beta(a)r(t,a,y)\,dydxda\\
&\qquad - \int_0^\infty\iint_{\mathbb{R}^{2d}} w(x-y)|x|^2\beta(a)r(t,a,x)\,dydxda\\
&= \int_0^\infty \beta(a)\int_{\mathbb{R}^d} r(t,a,x)\Big(\int_{\mathbb{R}^d} w(x-y)(|y|^2-|x|^2)\,dy\Big)\,dxda.
\end{aligned}$$

Using again (2), we have

$$\begin{aligned}
\int_{\mathbb{R}^d} w(x-y)(|y|^2-|x|^2)\,dy &= \int_{\mathbb{R}^d} w(z)(|x-z|^2-|x|^2)\,dz\\
&= \int_{\mathbb{R}^d}|z|^2 w(z)\,dz - 2x\cdot\int_{\mathbb{R}^d} zw(z)\,dz = \sigma^2
\end{aligned}$$

and thus (18). Integrating (18) in time we obtain, using Corollary 9

$$\int_0^t\int_{\mathbb{R}^d}|x|^2 r(t,a,x)\,dxda = \sigma^2\int_0^t \mathbf{n}(s)\,ds = \sigma^2\,Y_1 * \mathbf{n}(t) \sim \frac{\sigma^2\int_0^\infty n^0(a)da}{\Psi\Gamma(1-\alpha)}Y_{1+\alpha}(t)$$

from which we get (6) due to the formula

$$\Gamma(1-\alpha)\Gamma(1+\alpha) = \frac{\pi\alpha}{\sin(\pi\alpha)} \tag{19}$$

and because $\int_0^\infty n^0(a)da = \int_0^\infty\int_{\mathbb{R}^d} r^0(a,x)dadx = \int_{\mathbb{R}^d}\rho^0(x)dx$. □

We end this section with the proof of Proposition 4.

Proof of Proposition 4. Let $\xi \in C^2([0,\infty),[0,\infty))$ such that $\xi(r) = 1$ if $0 \le r \le 1$ and $\xi(r) = 0$ if $r \ge 2$, and then define $\varphi \in C_c^2(\mathbb{R}^d)$ by $\varphi(x) = \xi(|x|)$

Using $\varphi_n(x) = \varphi(x/n)$ as a test function in (9) and dominated convergence, we get (4) by passing to the limit $n \to \infty$.

Similarly, considering the test functions $\varphi_n(x) = x_i\varphi(x/n)$, $1 \le i \le d$, which verify

$$\Delta\varphi_n(x) = 2n^{-1}\partial_i\varphi(x/n) + n^{-2}x_i\Delta\varphi(x/n),$$

we obtain (5) by passing to the limit $n \to \infty$ in (9), still by dominated convergence.

For (12) we proceed in the exact same way with the test function $\varphi_n(x) = |x|^2\varphi(x/n)$, which verifies

$$\Delta\varphi_n(x) = 2d\varphi(x/n) + 4n^{-1}x \cdot \nabla\varphi(x/n) + |x/n|^2\Delta\varphi(x/n).$$

Passing to the limit $n \to \infty$, again by dominated convergence, we get

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^2\rho(t, dx) &= \int_{\mathbb{R}^d} |x|^2\rho^0(dx) + 2dD_\alpha\Big(\int_{\mathbb{R}^d} \rho^0(dx)\Big)Y_\alpha * Y_1(t)\\ &= \int_{\mathbb{R}^d} |x|^2\rho^0(dx) + \frac{\sigma^2}{\Psi\Gamma(1-\alpha)}\Big(\int_{\mathbb{R}^d} \rho^0(dx)\Big)Y_{1+\alpha}(t)\\ &= \int_{\mathbb{R}^d} |x|^2\rho^0(dx) + \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{\sigma^2}{\Psi}\Big(\int_{\mathbb{R}^d} \rho^0(dx)\Big)t^\alpha,\end{aligned}$$

where we have used Lemma 5 in the second identity and (19) in the last one. □

3 Proof of the main theorem

In this section, we consider an initial distribution r^0_ϵ enjoying Assumption (10), and we will use the standard duality bracket notation

$$\langle h, \varphi\rangle := \int_{\mathbb{R}^d} h(x)\varphi(x)\, dx$$

for $h \in L^1(\mathbb{R}^d)$ and $\varphi \in L^\infty(\mathbb{R}^d)$, or $h \in \mathcal{M}(\mathbb{R}^d)$ and $\varphi \in C_b(\mathbb{R}^d)$.

In view of Definition 2, our first aim is to prove that for any $\varphi \in C^2_c(\mathbb{R}^d)$

$$\langle \rho_\epsilon(t,\cdot) - \rho^0_\epsilon, \varphi\rangle - D_\alpha\langle Y_\alpha * \rho_\epsilon(t,\cdot), \Delta\varphi\rangle \to 0 \qquad \text{as } \epsilon \to 0 \tag{20}$$

locally uniformly in $t \in [0,\infty)$. Then we will derive compactness estimates ensuring the existence of a sequence $(\rho_{\epsilon_n})_{n\in\mathbb{N}}$ which converges in $C([0, T], \mathcal{M})$ for all $T > 0$ to a limit $\rho \in C([0,\infty), \mathcal{M})$.

We first introduce an integral Laplace operator Δ^ϵ acting on the space variable as

$$\Delta^\epsilon h(x) = \epsilon^{-2}\int_{\mathbb{R}^d} w_\epsilon(z)\big(h(x - z) - h(x)\big)\, dz.$$

Since we do not assume that w is symmetric, we also define the dual operator $\check{\Delta}^\epsilon$ by

$$\check{\Delta}^\epsilon\varphi(x) = \epsilon^{-2}\int_{\mathbb{R}^d} \check{w}_\epsilon(z)\big(\varphi(x - z) - \varphi(x)\big)\, dz, \qquad \text{where } \check{w}(z) := w(-z).$$

We thus have

$$\langle \Delta^\epsilon h, \varphi\rangle = \langle h, \check{\Delta}^\epsilon\varphi\rangle. \tag{21}$$

Besides, under Assumption (2), we have for any $\varphi \in C^2_c(\mathbb{R}^d)$ the uniform convergence

$$\check{\Delta}^\epsilon\varphi \to \frac{\sigma^2}{2d}\Delta\varphi \qquad \text{as } \epsilon \to 0. \tag{22}$$

To prove (20) one thus has to show that

$$\langle \rho_\epsilon - \rho^0_\epsilon, \varphi\rangle - \frac{2d}{\sigma^2}D_\alpha\langle Y_\alpha * \rho_\epsilon, \check{\Delta}^\epsilon\varphi\rangle \to 0 \qquad \text{as } \epsilon \to 0 \tag{23}$$

in $C([0, T], \mathbb{R})$. We then use the following lemma.

Lemma 11. *We have*

$$\rho_\epsilon - \rho_\epsilon^0 = \Delta^\epsilon R_\epsilon$$

where

$$R_\epsilon(t,x) = \epsilon^{2-2/\alpha}\int_0^t\int_0^\infty \beta(a) r_\epsilon(s,a,x)\,dads.$$

Proof. Integrating (7) on $[0,t]$ in time and on $[0,\infty)$ in age and using the boundary condition we get

$$\begin{aligned}
\rho_\epsilon(t,x) - \rho_\epsilon^0(x) &= \epsilon^{-2/\alpha}\int_0^t \mathbf{r}_\epsilon(s,x)ds - \epsilon^{-2/\alpha}\int_0^t\int_0^\infty \beta(a)r_\epsilon(s,a,x)dads \\
&= \epsilon^{-2/\alpha}\int_0^t\int_0^\infty\int_{\mathbb{R}^d} \beta(a)w_\epsilon(z)r_\epsilon(s,a,x-z)\,dzdads - \epsilon^{-2/\alpha}\int_0^t\int_0^\infty \beta(a)r_\epsilon(s,a,x)dads \\
&= \epsilon^{2-2/\alpha}\int_0^t\int_0^\infty \beta(a)\Delta^\epsilon r_\epsilon(s,a,x)dads.
\end{aligned}$$

□

Using (21), one is thus led, to prove (23), to estimate

$$R_\epsilon - \frac{2dD_\alpha}{\sigma^2}\,Y_\alpha * \rho_\epsilon.$$

To do so, we separate the contributions of jumpers (that have jumped before time t) and non-jumpers (that have only aged). Integrating Equation (7) along the characteristics one gets

$$\begin{aligned}
r_\epsilon(t,a,x) &= r_\epsilon(t-\epsilon^{2/\alpha}a,0,x)e^{-\int_0^a \beta(a')\,da'}\,\mathbb{1}_{t\geq \epsilon^{2/\alpha}a} \\
&\quad + r_\epsilon(0,a-\epsilon^{-2/\alpha}t,x)e^{-\int_0^{\epsilon^{-2/\alpha}t}\beta(a'+a-\epsilon^{-2/\alpha}t)\,da'}\,\mathbb{1}_{t<\epsilon^{2/\alpha}a},
\end{aligned}$$

which also reads

$$r_\epsilon(t,a,x) = \mathbf{r}_\epsilon(t-\epsilon^{2/\alpha}a,x)\psi(a)\,\mathbb{1}_{t\geq\epsilon^{2/\alpha}a} + r_\epsilon^0(a-\epsilon^{-2/\alpha}t,x)\frac{\psi(a)}{\psi(a-\epsilon^{-2/\alpha}t)}\,\mathbb{1}_{t<\epsilon^{2/\alpha}a}.$$

This allows us to split ρ_ϵ as

$$\rho_\epsilon(t,x) = \rho_\epsilon^j(t,x) + \rho_\epsilon^{nj}(t,x), \tag{24}$$

where

$$\rho_\epsilon^j(t,x) = \int_0^{\epsilon^{-2/\alpha}t} r_\epsilon(t,a,x)\,da = \epsilon^{-2/\alpha}\int_0^t \psi(\epsilon^{-2/\alpha}(t-s))\mathbf{r}_\epsilon(s,x)ds$$

is the contribution of jumpers, and

$$\rho_\epsilon^{nj}(t,x) = \int_{\epsilon^{-2/\alpha}t}^\infty r_\epsilon(t,a,x)\,da = \int_0^\infty \frac{\psi(a+\epsilon^{-2/\alpha}t)}{\psi\,(a)} r_\epsilon^0(a,x)da$$

the one of non-jumpers. Similarly, one can write

$$R_\epsilon(t,x) = R_\epsilon^j(t,x) + R_\epsilon^{nj}(t,x) \tag{25}$$

with

$$
\begin{aligned}
R^j_\epsilon(t,x) &= \epsilon^{2-2/\alpha}\int_0^t\int_0^{\epsilon^{-2/\alpha}s}\beta(a)r_\epsilon(s,a,x)\,dads\\
&= \epsilon^{2-4/\alpha}\int_0^t\int_0^s\phi(\epsilon^{-2/\alpha}(s-s'))\mathbf{r}_\epsilon(s',x)\,ds'ds
\end{aligned}
$$

and

$$
\begin{aligned}
R^{nj}_\epsilon(t,x) &= \epsilon^{2-2/\alpha}\int_0^t\int_{\epsilon^{-2/\alpha}s}^{\infty}\beta(a)r_\epsilon(s,a,x)\,dads\\
&= \epsilon^{2-2/\alpha}\int_0^t\int_0^{\infty}\frac{\phi(a+\epsilon^{-2/\alpha}s)}{\psi(a)}r^0_\epsilon(a,x)\,dads.
\end{aligned}
$$

We will then estimate separately

$$
R^j_\epsilon-\frac{2dD_\alpha}{\sigma^2}\,Y_\alpha*\rho^j_\epsilon \qquad\text{and}\qquad R^{nj}_\epsilon-\frac{2dD_\alpha}{\sigma^2}\,Y_\alpha*\rho^{nj}_\epsilon.
$$

We start with the jumpers.

Lemma 12. *Under Assumption* (3)*, one has for all* $T>0$

$$
\sup_{0\le t\le T}\int_{\mathbb{R}^d}\Big|R^j_\epsilon-\frac{2dD_\alpha}{\sigma^2}Y_\alpha*\rho^j_\epsilon\Big|(t,x)\,dx = O(\epsilon^2)+O(\epsilon^{2\delta/\alpha}) \qquad \textit{as } \epsilon\to 0.
$$

Proof. We start by proving that

$$
\Big(R^j_\epsilon-\frac{2dD_\alpha}{\sigma^2}Y_\alpha*\rho^j_\epsilon\Big)(t,x) = \epsilon^2\,\mathbf{r}_\epsilon(\epsilon^{2/\alpha}\,\cdot\,,x)*\Big(1-\psi-\frac{2dD_\alpha}{\sigma^2}Y_\alpha*\psi\Big)(\epsilon^{-2/\alpha}t). \tag{26}
$$

On the one hand we have

$$
\begin{aligned}
R^j_\epsilon(t,x) &= \epsilon^{2-4/\alpha}\int_0^t\int_0^s\phi(\epsilon^{-2/\alpha}(s-s'))\mathbf{r}_\epsilon(s',x)ds'ds\\
&= \epsilon^{2-4/\alpha}\int_0^t\mathbf{r}_\epsilon(s',x)\int_{s'}^t\phi(\epsilon^{-2/\alpha}(s-s'))dsds'\\
&= \epsilon^{2-4/\alpha}\int_0^t\mathbf{r}_\epsilon(s',x)\int_0^{t-s'}\phi(\epsilon^{-2/\alpha}(s))dsds'\\
&= \epsilon^{2-2/\alpha}\int_0^t\mathbf{r}_\epsilon(s',x)(1-\psi(\epsilon^{-2/\alpha}(t-s'))ds'.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
Y_\alpha * \rho_\epsilon^j(t,x) &= \frac{\epsilon^{-2/\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \psi(\epsilon^{-2/\alpha}(s-s'))\mathbf{r}_\epsilon(s',x)\, ds' ds \\
&= \frac{\epsilon^{-2/\alpha}}{\Gamma(\alpha)} \int_0^t \mathbf{r}_\epsilon(s',x) \int_{s'}^t (t-s)^{\alpha-1}\psi(\epsilon^{-2/\alpha}(s-s'))\, ds ds' \\
&= \frac{\epsilon^{-2/\alpha}}{\Gamma(\alpha)} \int_0^t \mathbf{r}_\epsilon(s',x) \int_0^{t-s'} (t-s'-s)^{\alpha-1}\psi\big(\epsilon^{-2/\alpha}s\big)\, ds ds' \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \mathbf{r}_\epsilon(s',x) \int_0^{\epsilon^{-2/\alpha}(t-s')} (t-s'-\epsilon^{2/\alpha}s)^{\alpha-1}\psi(s)\, ds ds' \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t \mathbf{r}_\epsilon(s',x) \int_0^{\epsilon^{-2/\alpha}(t-s')} (\epsilon^{2/\alpha})^{\alpha-1}(\epsilon^{-2/\alpha}(t-s')-s)^{\alpha-1}\psi(s)\, ds ds' \\
&= \epsilon^{2-2/\alpha} \int_0^t \mathbf{r}_\epsilon(s',x)\, Y_\alpha * \psi(\epsilon^{-2/\alpha}(t-s')) ds'.
\end{aligned}
$$

This gives

$$
\begin{aligned}
\Big(R_\epsilon^j - \frac{2dD_\alpha}{\sigma^2} Y_\alpha * \rho_\epsilon^j\Big)(t,x) &= \epsilon^{2-2/\alpha} \int_0^t \mathbf{r}_\epsilon(s,x) \left(1 - \psi(\epsilon^{-2/\alpha}(t-s)) - \frac{2dD_\alpha}{\sigma^2}\, Y_\alpha * \psi(\epsilon^{-2/\alpha}(t-s))\right) ds \\
&= \epsilon^2 \int_0^{\epsilon^{-2/\alpha}t} \mathbf{r}_\epsilon(\epsilon^{2/\alpha}s,x) \left(1 - \psi(\epsilon^{-2/\alpha}t-s) - \frac{2dD_\alpha}{\sigma^2}\, Y_\alpha * \psi(\epsilon^{-2/\alpha}t-s)\right) ds,
\end{aligned}
$$

which is exactly (26). Now we use that (17) with $\nu = \alpha$ gives

$$
Y_\alpha * \psi(t) = \Psi\Gamma(1-\alpha) + O(t^{-\delta}).
$$

Recalling that $D_\alpha = \frac{\sigma^2}{2d\Psi\Gamma(1-\alpha)}$, we deduce the existence of $C > 0$ such that

$$
\left|1 - \frac{2dD_\alpha}{\sigma^2}\, Y_\alpha * \psi(t)\right| \le C Y_{1-\delta}(t). \tag{27}
$$

We now define the function n_ϵ by

$$
n_\epsilon(t,a) = \int_{\mathbb{R}^d} r_\epsilon(\epsilon^{2/\alpha}t, a, x)\, dx.
$$

Since

$$
\mathbf{n}_\varepsilon(t) = n_\epsilon(t,0) = \int_{\mathbb{R}^d} \mathbf{r}_\epsilon(\epsilon^{2/\alpha}t, x)\, dx,
$$

one obtains by integration of (26) in x and using (27) that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \left|R_\epsilon^j - \frac{2dD_\alpha}{\Gamma(\alpha)} Y_\alpha * \rho_\epsilon^j\right|(t,x)\, dx &\le \epsilon^2\, \mathbf{n}_\epsilon * \left|1 - \psi - \frac{2dD_\alpha}{\Gamma(\alpha)} Y_\alpha * \psi\right|(\epsilon^{-2/\alpha}t) \\
&\le \epsilon^2\, \mathbf{n}_\epsilon * \psi(\epsilon^{-2/\alpha}t) + \epsilon^2\, \mathbf{n}_\epsilon * \left|1 - \frac{2dD_\alpha}{\Gamma(\alpha)} Y_\alpha * \psi\right|(\epsilon^{-2/\alpha}t) \\
&\le \epsilon^2\, \mathbf{n}_\epsilon * \psi(\epsilon^{-2/\alpha}t) + C\epsilon^2\, \mathbf{n}_\epsilon * Y_{1-\delta}(\epsilon^{-2/\alpha}t).
\end{aligned} \tag{28}
$$

Because the function n_ϵ verifies Equation (13) with initial condition

$$n_\epsilon^0(a) = \int_{\mathbb{R}^d} r_\epsilon^0(a,x)\,dx,$$

which enjoys the convergence

$$\int_0^\infty n_\epsilon^0(a)\,da = \int_{\mathbb{R}^d} \rho_\epsilon^0(x)\,dx \underset{\epsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^d} \rho^0(x)\,dx,$$

one can use Lemma 7 and Corollary 9 to infer from (28) that

$$\int_{\mathbb{R}^d} \left| R_\epsilon^j - \frac{2dD_\alpha}{\Gamma(\alpha)} Y_\alpha * \rho_\epsilon^j \right| (t,x)\,dx = O(\epsilon^2) + O(\epsilon^{2\delta/\alpha})$$

locally uniformly in $t \in [0,\infty)$. □

We now turn to the non-jump part.

Lemma 13. *Assume that* (3) *holds. Then for all* $T > 0$

$$\sup_{0\le t\le T} \int_{\mathbb{R}^d} \left| R_\epsilon^{nj} - \frac{2dD_\alpha}{\sigma^2} Y_\alpha * \rho_\epsilon^{nj} \right| (t,x)\,dx = O(\epsilon^2) \qquad \text{as } \epsilon \to 0.$$

Proof. We start by computing

$$\begin{aligned}
\Big(R_\epsilon^{nj} - \frac{2D_\alpha}{\sigma^2} Y_\alpha * \rho_\epsilon^{nj}\Big)(t,x) &= \epsilon^{2-2/\alpha} \int_0^t \int_0^\infty \frac{\phi(a+\epsilon^{-2/\alpha}s)}{\psi(a)} r_\epsilon^0(a,x)dads \\
&\qquad - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty \frac{\psi(a+\epsilon^{-2/\alpha}s)}{\psi(a)} r_\epsilon^0(a,x)dads \\
&= \epsilon^2 \int_0^\infty \frac{\psi(a) - \psi(a+\epsilon^{-2/\alpha}t)}{\psi(a)} r_\epsilon^0(a,x)da \\
&\qquad - \epsilon^2 \int_0^{\epsilon^{-2/\alpha}t} \frac{(\epsilon^{-2/\alpha}t - s)^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty \frac{\psi(a+s)}{\psi(a)} r_\epsilon^0(a,x)dads \\
&= \epsilon^2 \int_0^\infty r_\epsilon^0(a,x) \frac{\psi(a) - \psi(a+\epsilon^{-2/\alpha}t) - (Y_\alpha * \psi(a+\cdot))(\epsilon^{-2/\alpha}t)}{\psi(a)} da.
\end{aligned}$$

To estimate this integral, we use that ψ is a non-increasing function to get

$$0 \le \frac{\psi(a) - \psi(a+\epsilon^{-2/\alpha}t)}{\psi(a)} \le 1$$

and

$$(Y_\alpha * \psi(a+\cdot))(\epsilon^{-2/\alpha}t) \le Y_\alpha * \psi(\epsilon^{-2/\alpha}t) \le C\, Y_\alpha * Y_{1-\alpha}(\epsilon^{-2/\alpha}t) = C,$$

where we have used Assumption (3) to ensure the existence of $C > 0$ such $\psi \le CY_{1-\alpha}$ in the second inequality, and Lemma 5 for the last equality. The result then follows from the fact r_ϵ^0 verifies the condition (10) and the fact that $(1+a)^\alpha \psi(a)$ is bounded on $[0,\infty)$, again as a consequence of (3). □

As a consequence of these lemmas, one obtains a quantified version of (23).

Corollary 14. *Under Assumptions* (2) *and* (3), *we have for all* $\varphi \in C_c^2(\mathbb{R}^d)$ *and all* $T > 0$

$$\sup_{0\le t\le T}\left|\langle \rho_\epsilon(t,\cdot) - \rho_\epsilon^0, \varphi\rangle - \frac{2dD_\alpha}{\sigma^2}\langle Y_\alpha * \rho_\epsilon(t,\cdot), \check{\Delta}^\epsilon\varphi\rangle\right| = O(\epsilon^2) + O(\epsilon^{2\delta/\alpha}) \qquad \text{as } \epsilon \to 0.$$

Proof. Using Lemma 11 one has for all $t \ge 0$

$$\begin{aligned}\left|\langle \rho_\epsilon(t,\cdot) - \rho_\epsilon^0, \varphi\rangle - \frac{2dD_\alpha}{\sigma^2}\langle Y_\alpha * \rho_\epsilon(t,\cdot), \check{\Delta}^\epsilon\varphi\rangle\right| &= \left|\langle \Delta^\epsilon R_\epsilon(t,\cdot), \varphi\rangle - \frac{2dD_\alpha}{\sigma^2}\langle Y_\alpha * \rho_\epsilon(t,\cdot), \check{\Delta}^\epsilon\varphi\rangle\right| \\ &\le \left\|\check{\Delta}^\epsilon\varphi\right\|_\infty \int_{\mathbb{R}^d}\left|R_\epsilon(t,x) - \frac{2dD_\alpha}{\sigma^2}Y_\alpha * \rho_\epsilon(t,x)\right| dx.\end{aligned}$$

The conclusion then follows from (22) and Lemmas 12 and 13, by using the splittings (24) and (25). ☐

As announced, the second step is to prove a time equicontinuity result on the family $\{\rho_\epsilon,\ 0 < \epsilon < 1\}$.

Lemma 15. *Assume* (2) *and* (3). *Then for all* $\varphi \in C_c^2(\mathbb{R}^d)$ *there exists a function* $\varpi \in C([0,\infty)^2, [0,\infty))$ *satisfying* $\varpi(t,t) = 0$ *for any* $t \ge 0$ *and: for all* $T > 0$ *there is a constant* $C > 0$ *such that*

$$|\langle \rho_\epsilon(t,\cdot), \varphi\rangle - \langle \rho_\epsilon(t',\cdot), \varphi\rangle| \le \varpi(t,t') + C\big(\epsilon^2 + \epsilon^{2\delta/\alpha}\big)$$

for all $t, t' \in [0,T]$ *and all* $\epsilon \in (0,1)$.

Proof. Due to Corollary 14, we only have to prove that

$$|\langle Y_\alpha * \rho_\epsilon(t,\cdot), \check{\Delta}^\epsilon\varphi\rangle - \langle Y_\alpha * \rho_\epsilon(t',\cdot), \check{\Delta}^\epsilon\varphi\rangle| \le \omega(t,t')$$

for all $t, t' \in [0,T]$ and $\epsilon \in (0,1)$. From (5) we have that for any $t \ge 0$

$$|\langle \rho_\epsilon(t,\cdot), \check{\Delta}^\epsilon\varphi\rangle| \le \|\rho_\epsilon(t,\cdot)\|_{L^1}\|\check{\Delta}^\epsilon\varphi\|_\infty = \|\rho_\epsilon^0\|_{L^1}\|\check{\Delta}^\epsilon\varphi\|_\infty$$

and we deduce that for $t' \ge t \ge 0$

$$\begin{aligned}|Y_\alpha * \langle \rho_\epsilon, \check{\Delta}^\epsilon\varphi\rangle(t) - Y_\alpha * \langle \rho_\epsilon, \check{\Delta}^\epsilon\varphi\rangle(t')| = \frac{1}{\Gamma(\alpha)}\bigg|&\int_t^{t'}(t'-s)^{\alpha-1}\langle \rho_\epsilon, \check{\Delta}^\epsilon\varphi\rangle(s)\,ds \\ &- \int_0^t \big((t-s)^{\alpha-1} - (t'-s)^{\alpha-1}\big)\langle \rho_\epsilon, \check{\Delta}^\epsilon\varphi\rangle(s)\,ds\bigg| \\ \le \frac{\|\rho_\epsilon^0\|_{L^1}\|\check{\Delta}^\epsilon\varphi\|_\infty}{\Gamma(\alpha)}&\left(\int_t^{t'}(t'-s)^{\alpha-1}ds + \int_0^t \big((t-s)^{\alpha-1} - (t'-s)^{\alpha-1}\big)ds\right) \\ \le \frac{\|\rho_\epsilon^0\|_{L^1}\|\check{\Delta}^\epsilon\varphi\|_\infty}{\alpha\Gamma(\alpha)}&\big(2(t'-t)^\alpha + t^\alpha - t'^\alpha\big).\end{aligned}$$

The result follows from the fact that

$$\|\rho_\epsilon^0\|_{L^1} \to \|\rho^0\|_{L^1} \quad \text{and} \quad \|\check{\Delta}^\epsilon\varphi\|_\infty \to \|\check{\Delta}\varphi\|_\infty \qquad \text{as } \epsilon \to 0.$$

☐

We are now in position to prove our main theorem.

Proof of Theorem 3. Because of Lemma (15), one can invoke Arzelà-Ascoli Theorem to infer, for any $T > 0$ and any $\varphi \in C_c^2(\mathbb{R}^d)$, the existence of a sequence (ϵ_n) decreasing to zero such that

$$\langle \rho_{\epsilon_n}(t, \cdot), \varphi \rangle \to \ell_\varphi(t) \qquad \text{in } C([0, T], \mathbb{R}).$$

Since we can find a sequence (φ_k) of $C_c^2(\mathbb{R}^d)$ functions which is dense in $C_0(\mathbb{R}^d)$, we can deduce by a diagonal argument the existence of a subsequence, still denoted by (ϵ_n), and the existence of $\rho \in C([0, T], \mathcal{M}(\mathbb{R}^d))$ such that

$$\rho_{\epsilon_n} \rightharpoonup \rho \qquad \text{in } C([0, T], \mathcal{M}(\mathbb{R}^d)). \tag{29}$$

Considering any $T \in \mathbb{N}$, we obtain, again by a diagonal argument, the existence of a subsequence, still denoted by (ϵ_n), and the existence of $\rho \in C([0, \infty), \mathcal{M}(\mathbb{R}^d))$ such that the convergence (29) holds for all $T > 0$. Since Corollary 14 guarantees that (20) holds for any $\varphi \in C_c^2(\mathbb{R}^d)$, we get by passing to the limit that ρ must verify Definition 2 of a solution to Equation (8). □

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