

# Limiting Hamilton-Jacobi equation for the large scale asymptotics of a subdiffusion jump-renewal equation.

Calvez, Vincent <sup>\*1</sup>, Gabriel, Pierre<sup>†2</sup>, and Mateos González, Álvaro <sup>‡1,3</sup>

<sup>1</sup>Université de Lyon, Unité de Mathématiques Pures et Appliquées, CNRS UMR 5669, École Normale Supérieure de Lyon, 15 parvis René Descartes, F-69007 Lyon, France

<sup>2</sup>Laboratoire de Mathématiques de Versailles, UVSQ, CNRS, Université Paris-Saclay, 45 avenue des États-Unis, F-78035 Versailles cedex, France

<sup>3</sup>Project Team Beagle, Inria Rhône-Alpes, 56 Blvd Niels Bohr, F-69603 Villeurbanne, France

September 22, 2016

## Abstract

Subdiffusive motion takes place at a much slower timescale than that of diffusion. As a preliminary step to studying reaction-subdiffusion pulled fronts, we consider here the hyperbolic limit  $(t, x) \rightarrow (t/\varepsilon, x/\varepsilon)$  of an age-structured equation describing the subdiffusive motion of, e.g., some protein inside a biological cell. Solutions of the rescaled equations are known to satisfy a Hamilton-Jacobi equation in the formal limit  $\varepsilon \rightarrow 0$ . In this work we derive uniform Lipschitz estimates, and establish the convergence towards the viscosity solution of the limiting Hamilton-Jacobi equation. Respectively, the two main obstacles overcome in the process are the non-existence of an integrable stationary measure, and the importance of memory terms in subdiffusion.

**Keywords:** age-structured PDE - renewal equation - anomalous diffusion - large deviations - WKB approximation - Hamilton-Jacobi equation

---

\*vincent.calvez@ens-lyon.fr

†pierre.gabriel@uvsq.fr

‡alvaro.mateos\_gonzalez@ens-lyon.fr

# 1 Introduction

## 1.1 Brief model description

Consistent experimental evidence stemming from recent methodological advances in cell biology such as in vivo single molecule tracking, report that the intra-cellular random motion of certain molecules often deviates from Brownian motion. Macroscopically, their mean squared displacement no longer scales linearly with time, but as a power law  $t^\mu$  for some exponent  $0 < \mu < 1$  [13, 5, 23, 6, 16]. This behaviour, due to crowding and trapping phenomena, is usually referred to as “anomalous” diffusion or “subdiffusion”. Refer to [15] for a review.

One of the main mechanisms recurrently evoked to explain the emergence of subdiffusion in cells are continuous time random walks (CTRW), a generalisation of random walks that couples a waiting time random process at each “jump” of the random walk [21]. CTRW can be used [17, 20] to derive macroscopic equations governing the spatio-temporal dynamics of the density of random walkers located at position  $x$  at time  $t$ :

$$\partial_t \rho(x, t) = D_\mu \mathcal{D}_t^{1-\mu} \Delta \rho(x, t).$$

Here,  $D_\mu$  is a generalised diffusion coefficient and  $\mathcal{D}_t^{1-\mu}(f)(t) = \frac{1}{\Gamma(\mu)} \frac{d}{dt} \int_0^t \frac{f(t')}{(t-t')^{1-\mu}} dt'$  is the Riemann-Liouville fractional derivative operator. Such a fractional dynamics formulation is very attractive for modelling in biology, in particular because of its apparent similarity with the classical diffusion equation. However, contrary to the diffusion equation, the Riemann-Liouville operator is non local in time. This is the “trace” of the non-Markovian property of the underlying CTRW process. Indeed, memory terms play a crucial role in subdiffusive processes. This non-Markovian property becomes a serious obstacle when one wants to couple subdiffusion with chemical reaction [14, 29, 9].

In this work, following [20], we take an alternative approach that rescues the Markovian property of the jump process at the expense of a supplementary structural age variable. We associate each random walker with a residence time (age, in short)  $a$ , which is reset when the random walker jumps to another location. In one dimension of space, we note  $n(t, x, a)$  the probability density function of walkers at time  $t$  that have been at location  $x$  exactly during the last span of time  $a$ . The dynamics of the CTRW are then described with an age-renewal equation with spatial jumps:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(a)n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, a = 0) = \int_0^\infty \beta(a') \int_{\mathbb{R}} \omega(x - x') n(t, x', a') dx' da' \\ n(t = 0, x, a) = n^0(x, a). \end{cases} \quad (1)$$

Throughout this work, we will only consider initial conditions compactly supported in age. Without loss of generality, we take:

$$\text{supp}(n^0(x, \cdot)) = [0, 1]. \quad (2)$$

The kernel  $\omega$  describes the spatial distribution of jumps (typically a Gaussian), and the function  $\beta(a)$  gives the jump rate. Since we are mostly interested here in the subdiffusive case (where the expectation of the residence time diverges), we will focus throughout this article on the case:

$$\begin{cases} \omega(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) & \text{Gaussian probability distribution} \\ \beta(a) = \frac{\mu}{1+a}, & 0 < \mu < 1 \end{cases} \quad (3)$$

where  $\mu \in (0, 1)$  is the subdiffusion exponent.

The distribution of residence times  $\Phi(a)$  is related to the jump rate as follow:

$$\Phi(a) = \beta(a) \exp\left(-\int_0^a \beta(s) ds\right) = \frac{\mu}{(1+a)^{1+\mu}},$$

which is a decreasing function.

*Remark (Diffusion limit).* Note that  $\int_0^\infty a\Phi(a) da$ , the mean residence time of particles between jumps, is infinite. However, there exist jump rates  $\beta$  different than that considered in this article for which the mean residence time is finite. The simplest example of this is the age-independent  $\beta \equiv K$  which leads to normal diffusion with diffusion coefficient  $D = \sigma^2/\mathbb{E}(\Phi) = K\sigma^2$  if  $\omega$  is a Gaussian of variance  $\sigma^2$ . This can be recovered by integrating equation (1) in age taking into account the boundary condition at age 0, and taking the diffusion limit  $(t, x) \rightarrow (t/\varepsilon^2, x/\varepsilon)$ . Similar manipulations are possible for  $\beta(a) = \mu/(1+a)$  for  $\mu > 1$ .

The age-structured approach that we follow has already been proposed in the CTRW literature [20, 8]. In [3], Berry, Lepoutre and the third author have studied the spatially-homogenous version of this problem by exhibiting an entropy structure of equation (1) in self-similar variables. They have proved the convergence of its solution to a self-similar profile, describing the aging of random walkers.

The motivation for our current work is the construction of tools that may allow us to better understand the behaviour of pulled reaction-subdiffusion fronts, covered by an extensive literature [11, 12, 28, 1, 26, 22, 18]. [28] gives a short and comprehensive review. We consider here the large scale asymptotics of equation (1) in the hyperbolic rescaling  $(t/\varepsilon, x/\varepsilon, a)$ , suitable for the study of constant speed fronts. Note that the mean field subdiffusive effects appear at a scale  $(\frac{t}{\varepsilon^{2/\mu}}, \frac{x}{\varepsilon}, a)$  and will not be captured by our analysis. This is consistent with the large deviations approach used to study rare events in probability theory. In the reaction-subdiffusion setting, pulled fronts (as opposed to pushed fronts driven by reaction kinetics) are indeed driven by the few particles which jump ahead of the front and not by the mean movement of particles. We refer to the seminal article by Evans and Souganidis [7] for the introduction of PDE tools inspired by large deviation methods in order to study geometric optics approximations for solutions of certain reaction-diffusion equations containing a small parameter [10].

## 1.2 Hyperbolic limit and derivation of the Hamilton-Jacobi equation.

Let us study the large scale asymptotics of the probability density function  $n$  in a hyperbolic scaling. We make the following ansatz (Hopf-Cole transform):

$$n_\varepsilon(t, x, a) = n(t/\varepsilon, x/\varepsilon, a) = \exp(-\phi_\varepsilon(t, x, a)/\varepsilon). \quad (4)$$

It enables us to measure accurately the behaviour of small tails of the p.d.f., reminiscent of large deviation principle theory. The function  $n_\varepsilon$  satisfies the following equation,

$$\begin{cases} \partial_t n_\varepsilon + \frac{1}{\varepsilon} \partial_a n_\varepsilon + \frac{1}{\varepsilon} \beta n_\varepsilon = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n_\varepsilon(t, x, 0) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) n_\varepsilon(t, x - \varepsilon z, a) dz da \\ n_\varepsilon(0, x, a) = n_\varepsilon^0(x, a) = n^0(x/\varepsilon, a). \end{cases} \quad (5)$$

We recall that  $\text{supp } n^0(x, \cdot) \subseteq [0, 1]$ , whence the upper limit  $1 + t/\varepsilon$  for the integral giving the boundary condition. For  $(t, x, a)$  such that  $\phi_\varepsilon(t, x, a) < \infty$ ,  $\phi_\varepsilon$  satisfies:

$$\begin{cases} \partial_t \phi_\varepsilon + \frac{1}{\varepsilon} \partial_a \phi_\varepsilon - \beta = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ \exp(-\phi_\varepsilon(t, x, 0)/\varepsilon) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(a) \omega(z) \exp(-\phi_\varepsilon(t, x - \varepsilon z, a)/\varepsilon) \, dz \, da \\ \phi_\varepsilon(0, x, a) = \phi_\varepsilon^0(x, a) = -\varepsilon \ln(n^0(x/\varepsilon, a)). \end{cases} \quad (6)$$

Let us denote by  $\psi_\varepsilon$  the boundary value at  $a = 0$ , which will be our main unknown:

$$\psi_\varepsilon(t, x) = \phi_\varepsilon(t, x, 0). \quad (7)$$

We compute the solution of equation (6) along characteristic lines:

$$\phi_\varepsilon(t, x, a) = \begin{cases} \psi_\varepsilon(t - \varepsilon a, x) + \varepsilon \int_0^a \beta(s) \, ds, & t > 0, \varepsilon a < t \\ \phi_\varepsilon^0(x, a - t/\varepsilon) + \varepsilon \int_{a-t/\varepsilon}^a \beta(s) \, ds, & t \geq 0, a \geq t/\varepsilon. \end{cases} \quad (8)$$

Injecting (8) into the  $a = 0$  boundary condition satisfied by  $\phi_\varepsilon$  in (6) now yields:

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) \, dz \, da \\ &\quad + \int_{t/\varepsilon}^{1+t/\varepsilon} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta\right) \, dz \, da. \end{aligned} \quad (9)$$

Let us formally derive the limiting Hamilton-Jacobi equation. Taking the formal limit of (9) when  $\varepsilon \rightarrow 0$  yields:

$$1 = \int_0^\infty \Phi(a) \exp(a \partial_t \psi_0(t, x)) \, da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) \, dz. \quad (10)$$

It is a Hamilton-Jacobi equation, since it is equivalent to:

$$\partial_t \psi_0(t, x) + H(\partial_x \psi_0)(t, x) = 0, \quad (11)$$

with  $H$  defined as follows, where  $\hat{\Phi}^{-1}$  is the inverse function of the Laplace transform of  $\Phi$ :

$$H(p) = -\hat{\Phi}^{-1}\left(\frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) \, dz}\right). \quad (12)$$

*Remark* (Alternative choice of  $\beta$ ). The limiting equation makes sense for a large class of functions  $\beta$ , including constant functions  $\beta = K$ . Indeed, the scaling considered here does not depend on the diffusive regime, whether it is anomalous or not. For such cases for which the large deviation scaling is problem-dependent, we refer to [4, 19].

*Remark* (Renormalisation by an instationary measure). The self-similar decay with an invariant profile in the space-homogenous case proved in [3] could argue for renormalising  $n$  by an instationary measure inspired by precisely the pseudo-equilibrium exhibited in that article. The limiting Hamilton-Jacobi equation obtained through this procedure is the same. Refer to section 4. However, the computations have to deal with parasitic terms in that case. That is why we have chosen not to renormalise  $n$ .

**Proposition 1.** *The Hamiltonian  $H$  defined in 12 is convex, but not strictly convex.*

*Proof.* Let us denote  $\hat{\omega}(p)$  the Laplace transform of  $\omega$  and  $p = \partial_x \psi_0$ . Differentiating equation (10) with respect to  $p$  yields:

$$0 = \int_0^\infty \int_{\mathbb{R}} \left( \nabla_p H(p) - \frac{z}{a} \right) a \Phi(a) \exp(-aH(p)) \omega(z) \exp(zp) dz da,$$

after which a second differentiation gives us:

$$0 = \int_0^\infty \int_{\mathbb{R}} a D_p^2 H(p) d\gamma(z, a) - \int_0^a \int_{\mathbb{R}} a^2 \left( \nabla_p H - \frac{z}{a} \right)^2 d\gamma(z, a),$$

where  $d\gamma(z, a) = \Phi(a) \omega(z) \exp(-aH(p)) \exp(zp) dz da$  is a non-negative measure. It follows that  $D_p^2 H \geq 0$ .

However, the Hamiltonian  $H$  is not strictly convex, since  $D_p^2 H(0) = 0$ . Indeed, since  $H(0) = \nabla_p H(0) = 0$  and  $a\Phi(a) \notin L^1$ , we get formally:

$$D_p^2 H(0) = \frac{\int_{\mathbb{R}} z^2 \omega(z) dz}{\int_0^\infty a \Phi(a) da} = 0.$$

□

**Proposition 2** (Behaviour at 0). *Suppose  $\omega$  is a Gaussian of variance  $\sigma^2$ . At  $p = 0$  the following equivalent holds:*

$$H(p) \sim_0 (\sigma p)^{2/\mu} (2\Gamma(1 - \mu))^{1/\mu}. \quad (13)$$

*Proof.* We have  $\Phi(a) = \mu(1+a)^{-1-\mu}$ , hence, thanks to equation (10):

$$\int_0^\infty \frac{\mu}{(1+a)^{1+\mu}} (\exp(-aH) - 1) da = \hat{\Phi}(H) - 1 = \exp(-(\sigma p)^2/2) - 1 \sim_{p=0} -(\sigma p)^2/2.$$

Denoting  $b = aH$ , since  $H(0) = 0$  the left hand side becomes:

$$\begin{aligned} & \frac{1}{H} \int_0^\infty \frac{\mu}{(1+b/H)^{1+\mu}} (e^{-b} - 1) db \\ &= H^\mu \int_0^\infty \frac{\mu}{(1+H/b)^{1+\mu}} b^{-1-\mu} (e^{-b} - 1) db \\ &\sim_{H=0} H^\mu \int_0^\infty \mu b^{-1-\mu} (e^{-b} - 1) db. \end{aligned}$$

Integrating that last expression by parts ends the proof. □

We have computed the Hamiltonian for  $p \in [0, 0.5]$ , different values of  $\mu$  and  $\sigma = 0.5$ , as depicted in Figure 1. Figure 2 shows how  $H$  and the asymptotic behaviour close to 0 proved in Proposition 2 grow apart for  $p$  large enough. For a visual representation of the evolution in time of the solution  $\psi_0$  of the Hamilton-Jacobi equation (9), refer to Figure 3, which is the result of a weighted essentially non-oscillatory (WENO) of order 5, Lax-Friedrichs numerical scheme. Refer to [27] for a review of such numerical methods. In Figure 3, the initial data taken for the first and third subfigures is the same, in order to illustrate how subdiffusion slows down significantly as time advances. The initial conditions in the second and third subfigures are such that  $H(\partial_x \psi) = \lambda(t)\psi$  for  $\partial_x \psi$  close to 0 for the respective Hamiltonians, hence the preserved

shape of the decaying profiles, most noticeable in the log scale. Indeed, injecting the Ansatz  $\psi_0(t, x) = c(t)x^\alpha$  into  $\partial_t \psi + \tilde{H}(\partial_x \psi)$ , with  $\tilde{H}$  given by the approached expression at 0 of  $H$  (13), leads to defining such initial conditions. We can see the effect of numerical diffusion close to  $\psi = 0$  in the first and second subfigures.

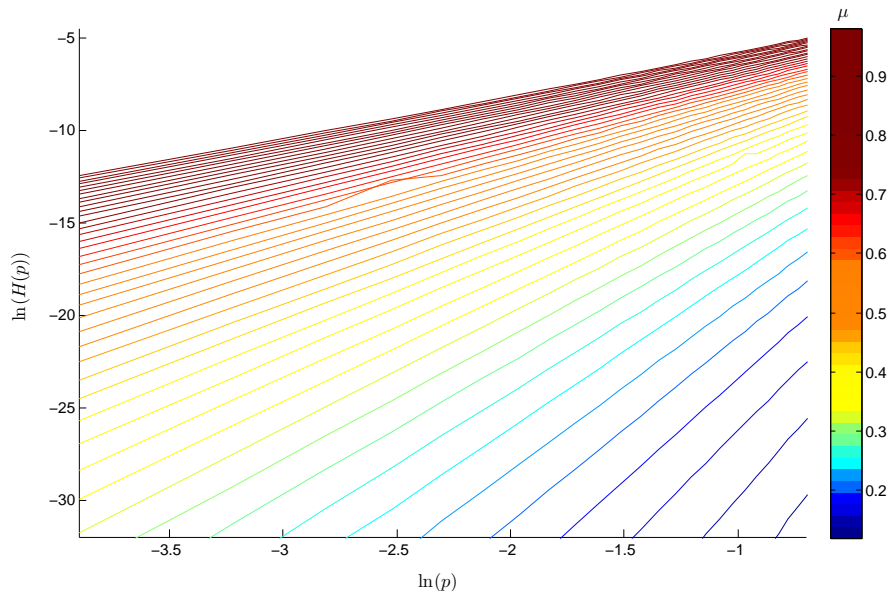


Figure 1:  $\ln(H(p))$  plotted against  $\ln(p)$  for  $p \in [0, 0.5]$  for values of  $\mu$  ranging from 0.12 (lower line) to 0.98 (upper line).  $H(p)$  behaves as a power of  $p$  for  $p \ll 1$ . This illustrates Proposition 2.

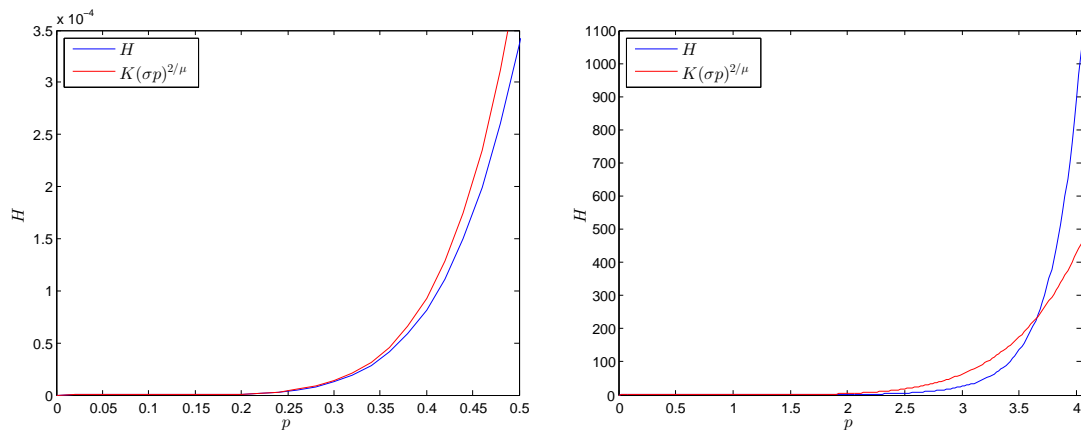
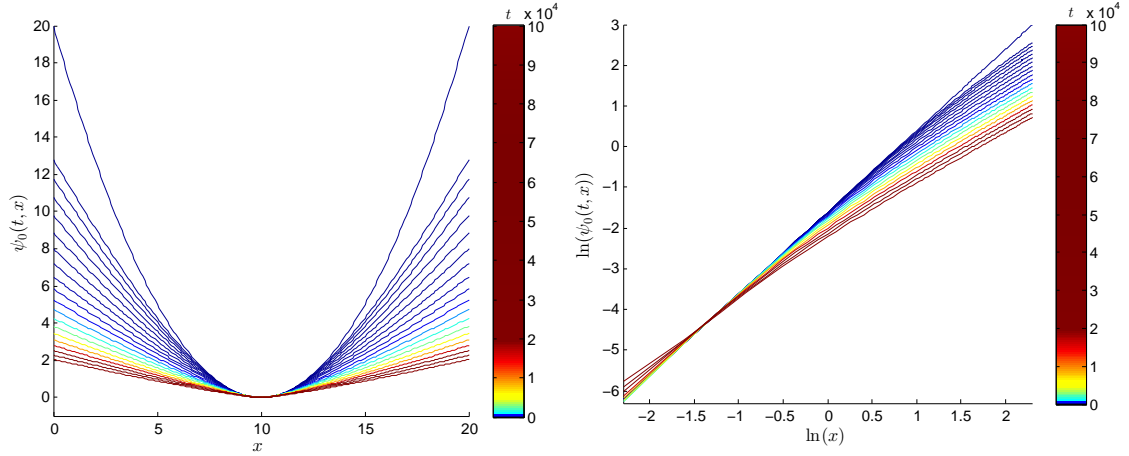
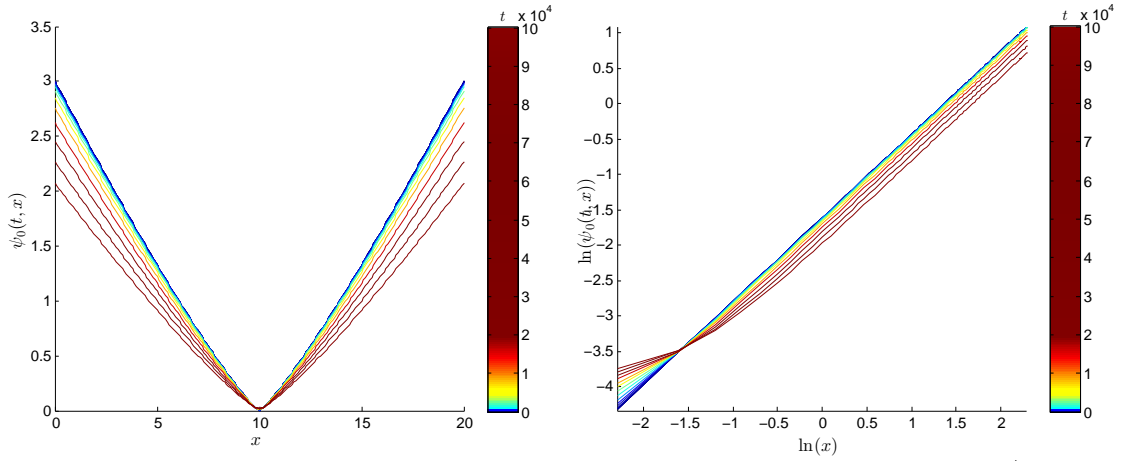


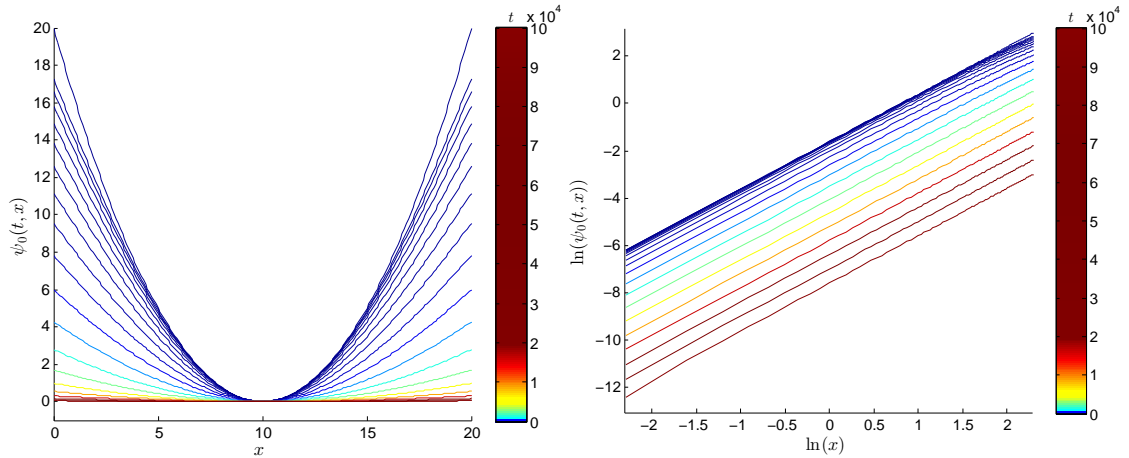
Figure 2: Comparison of  $H$  and its asymptotic behaviour near 0 for  $\mu = 0.3$ . Far from 0, they do not match. Here  $K = (2\Gamma(1 - \mu))^{-1/\mu}$ .



Subdiffusive case with  $\mu = 0.3$  and  $\beta(a) = \mu/(1+a)$ .  $\psi_0(0, x) = 0.2(x-10)^2$ .



Subdiffusive case with  $\mu = 0.3$  and  $\beta(a) = \mu/(1+a)$ .  $\psi_0(0, x) = 0.2(x-10)^{2/2-\mu}$ .



Diffusive case with  $D = 0.01$  and  $\beta(a) = D$ .  $\psi_0(0, x) = 0.2(x-10)^2$ .

Figure 3: Decay of  $\psi_0(t, \cdot)$  (left) and  $\ln(\psi_0(t, \cdot))$  (right) for  $\sigma = 1$ ,  $t \in [0, 100000]$  (shown in the color bar) and  $x \in [0, 20]$  with periodic boundary conditions. The presented plots are taken at 20 regular intervals in  $\ln(t)$ . 7

### 1.3 Main hypotheses and results

Throughout the article, we will work over the set  $[0, T] \times \mathbb{R}$  for some  $T > 0$ , and we will denote by  $C$  any real constant whose value is irrelevant.

**Hypothesis 1** (Space-jump kernel  $\omega$ ).

$\omega$  is a Gaussian probability distribution of mean 0 and variance  $\sigma^2$ .

*Remark.* This assumption could be weakened, but for the sake of clarity we will not do so.

**Hypothesis 2** (Initial condition  $\phi_\varepsilon^0$ ).

Throughout the article, we will consider an initial condition of the form:

$$\phi_\varepsilon^0(x, a) = v(x) + \varepsilon\eta(x, a) \quad (14)$$

where:

1.  $v$  is bounded.
2. The initial condition perturbation term is integrable in age uniformly in space, i.e.  $\eta$  is sufficiently large for large age; quantitatively:

$$\exp\left(-\inf_x \eta(x, a)\right) \in L^1. \quad (15)$$

3. At time 0 the initial condition perturbation  $\eta$  is redistributed so that at time  $0^+$  it contributes in a positive way uniformly in  $x$  to  $\psi_\varepsilon^0$ , depending on  $\varepsilon$  at worst as follows:

$$\exists C > 0 \mid \forall \varepsilon > 0, \forall x \in \mathbb{R} \int_0^1 \int_{\mathbb{R}} \Phi(a)\omega(z)e^{\int_0^a \beta} e^{-\eta(x-\varepsilon z, a)} dz da > \exp(-C/\varepsilon). \quad (16)$$

4.  $\phi_\varepsilon^0$  is Lipschitz in  $x$  uniformly over  $0 < \varepsilon < 1$ .
5. Semi-concavity: there exists some  $\mathfrak{C}_{xx} \in \mathbb{R}$  such that

$$\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx} \quad (17)$$

in the sense of distributions.

*Remark* (Interpretation in terms of  $n$ ). This means

$$n_\varepsilon^0(x, a) = \tilde{n}(x, a) \exp(-v(x)/\varepsilon),$$

which allows for simpler interpretations of hypotheses 2.2 and 2.3, respectively:

$$\sup_x \tilde{n}(x, a) \in L^1(da)$$

and

$$\int_0^1 \int_{\mathbb{R}} \beta(a)\omega(z)\tilde{n}(x - \varepsilon z, a) dz da \geq C.$$



*Remark* (Compatibility of the initial condition). We will take a smooth enough initial condition. However, we do not impose for it to be compatible in the sense that the influx relation at age  $a = 0$  is not necessarily satisfied at time  $t = 0$  in (1). As a consequence, we allow discontinuities along  $t = \varepsilon a$ . This means that in general, we have:

$$\exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(0, x)\right) = \int_0^1 \int_{\mathbb{R}} \omega(z)\Phi(a) \exp\left(-\frac{1}{\varepsilon}\phi_\varepsilon^0(x - \varepsilon z, a)\right) dz da \neq \exp\left(-\frac{1}{\varepsilon}\phi_\varepsilon^0(x, 0)\right). \quad (18)$$

Such compatibility is assumed in [25] so as to use a comparison principle when dealing with regularity results for solutions of the renewal equation in a space-homogenous setting (see chapter 3.4). In our case, assuming both compatibility and regularity of the initial condition restricts the scope of our results. Taking a smooth initial condition is the reasonable choice, for instance because the Lipschitz constants of  $\phi_\varepsilon^0$  play a role in our proofs of the a priori estimates of Propositions 5, 9 and 13 (hence of Theorem 4).

*Remark* (Initial age profile). We make the ansatz  $\partial_a \phi_\varepsilon^0 = O(\varepsilon)$ , since  $\partial_a \phi_\varepsilon^0 = O(1)$  would introduce boundary layer phenomena. We would then expect some rapid age dynamic at short time, which is not our focus.

The following Theorem is the main result on the paper.

**Theorem 3.** *Under hypotheses 1 and 2,  $\psi_\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{L_{\text{loc}}^\infty} \psi_0$ , which is the viscosity solution of the limiting Hamilton-Jacobi equation (10) with initial condition  $v(x)$ .*

A crucial step in the proof of Theorem 3 is the proof of the uniform bounds of Theorem 4.

The outline of our paper is the following: section 2 deals with the proof of Theorem 4, namely, a uniform bound on  $\psi_\varepsilon$  in  $W^{1,\infty}([0, T] \times \mathbb{R})$ , each subsection corresponding to the bounds on  $\psi_\varepsilon$  and its first partial derivatives. The proofs mainly involve comparison principles. Section 3 proves that  $\psi_0$  is a solution of the limiting Hamilton-Jacobi equation (10), which is the result of Theorem 3. The first subsection shows it is a subsolution, the second, a supersolution.

## 2 $\psi_\varepsilon$ is uniformly bounded in $W^{1,\infty}([0, T] \times \mathbb{R})$

*This whole section deals with the proof of the following*

**Theorem 4.** *Let  $T > 0$  and  $0 < \varepsilon < 1$ . Under hypotheses 1 and 2,  $\psi_\varepsilon$  is bounded in  $W^{1,\infty}([0, T] \times \mathbb{R})$  uniformly in  $\varepsilon$ , with the following quantitative bounds:*

1.

$$\psi_\varepsilon(t, x) \geq \inf v - \left| \ln \left\| \exp\left(-\inf_x \eta(x, \cdot)\right) \right\|_{L^1} \right| \quad (19)$$

$$\psi_\varepsilon(t, x) \leq \sup v + (1 + \mu)T + C \quad (20)$$

2.

$$\partial_x \psi_\varepsilon \geq \inf_{\varepsilon \in (0,1)} \inf_{(x,a) \in \mathbb{R} \times [0,1]} \partial_x \phi_\varepsilon^0(x, a), \quad (21)$$

$$\partial_x \psi_\varepsilon \leq \sup_{\varepsilon \in (0,1)} \sup_{(x,a) \in \mathbb{R} \times [0,1]} \partial_x \phi_\varepsilon^0(x, a). \quad (22)$$

3.

$$\partial_t \psi_\varepsilon(t, x) \geq C \quad (23)$$

$$\partial_t \psi_\varepsilon(t, x) \leq \mu(1 + \mu). \quad (24)$$

Each subsection proves the more accurate  $\varepsilon$ -dependant bounds of Propositions 5, 9 and 13, from which the respective uniform bounds of Theorem 4 can be deduced as follows:

*Proof.*

1. Consider the bounds in Proposition 5. For the lower bound on  $\psi_\varepsilon$ , we remark that  $\beta(0) = \mu \in (0, 1)$  and conclude thanks to hypothesis 2.2. For the upper bound, first we compute:

$$\begin{aligned} & -\varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a + T/\varepsilon) \omega(z) \exp \left( \int_0^a \beta \right) \exp(-\eta(x - \varepsilon z, a)) \, dz \, da \right) \\ &= -\varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \left( \frac{1+a}{1+a+T/\varepsilon} \right)^{1+\mu} \Phi(a) \omega(z) \exp \left( \int_0^a \beta \right) \exp(-\eta(x - \varepsilon z, a)) \, dz \, da \right) \\ &\leq \varepsilon(1+\mu) \ln(1+T/\varepsilon) - \varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a) \omega(z) \exp \left( \int_0^a \beta \right) \exp(-\eta(x - \varepsilon z, a)) \, dz \, da \right) \\ &\leq (1+\mu)T + C, \end{aligned}$$

which is uniform in  $\varepsilon \in (0, 1)$  thanks to hypothesis 2.3.

2. The bounds on  $\partial_x \psi_\varepsilon$  follow from Proposition 9. They are uniform in  $\varepsilon \in (0, 1)$  since  $\phi_\varepsilon^0$  is Lipschitz continuous in  $x$  uniformly in  $\varepsilon \in (0, 1)$  (hypothesis 2.4).
3. The bounds on  $\partial_t \psi_\varepsilon$  follow from Proposition 13. The constant for the lower bound is uniform in  $\varepsilon \in (0, 1)$  thanks to hypothesis 2.4.

□

*Remark* (Lack of integrability of the invariant measure).  $F(a) = \exp(-\int_0^a \beta)$  is an invariant measure for the homogeneous problem (though not a probability measure). Thanks to a maximum principle, there exist constants  $0 < c < C$  such that:

$$cF \leq n^0 \leq CF \implies cF \leq n(t, \cdot) \leq CF. \quad (25)$$

If  $F \in L^1$ , the mean waiting time equals  $\|F\|_1$  and we recover age-integrable estimates on  $n$ . If  $F \notin L^1$ , which is our case, integrability of  $n$  with respect to age would be violated according to such estimate. However, in agreement with the self-similar decay with an invariant profile in the space-homogenous setting of our equation proved in [3], we expect time-dependent corrections in the estimates we will give. This forces us to work on a bounded time interval  $[0, T]$  for any  $T$ . It is the first main complication tackled in this paper.

## 2.1 $\psi_\varepsilon$ is bounded over $[0, T] \times \mathbb{R}$ .

*This subsection deals with the proof of the following*

**Proposition 5.** *Let  $T > 0$  and  $\varepsilon > 0$ . Under hypotheses 1 and 2,  $\psi_\varepsilon \in L^\infty([0, T] \times \mathbb{R})$ , with the quantitative bounds stated below:*

$$\psi_\varepsilon(t, x) \geq \mathbf{c} = \inf v - \varepsilon \ln \beta(0) - \varepsilon \ln \left\| \exp \left( -\inf_x \eta(x, a) \right) \right\|_{L^1} \quad (26)$$

$$\psi_\varepsilon(t, x) \leq \mathfrak{C}_T = \sup v - \varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a + T/\varepsilon) e^{\int_0^a \beta} \omega(z) e^{-\eta(x - \varepsilon z, a)} dz da \right). \quad (27)$$

Let us start by bounding  $\psi_\varepsilon$  from below, proving the first (easier) half of the proposition.

*Proof.* From (6) and hypothesis 2 we gather:

$$n^0(x, a) = n_\varepsilon^0(\varepsilon x, a) = \exp \left( -\frac{v(\varepsilon x)}{\varepsilon} - \eta(\varepsilon x, a) \right).$$

Let us define  $\bar{n} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  as the solution of the following equation:

$$\begin{cases} \partial_t \bar{n}(t, a) + \partial_a \bar{n}(t, a) + \beta(a) \bar{n}(t, a) = 0 \\ \bar{n}(t, 0) = \int_0^\infty \beta(a) \bar{n}(t, a) da \\ \bar{n}^0(a) = \exp \left( -\inf_x v(\varepsilon x)/\varepsilon \right) \exp \left( -\inf_x \eta(\varepsilon x, a) \right) > \sup_x n^0(x, a). \end{cases} \quad (28)$$

Since  $\beta$  is a non-increasing function,  $\omega$  is a probability measure,  $n$  satisfies equation (1) and  $\bar{n}^0(a) \geq \sup_x n^0(x, a)$ , it follows that for any  $t, a \geq 0$ :

$$\bar{n}(t/\varepsilon, a) \geq \sup_x n(t/\varepsilon, x/\varepsilon, a) = \sup_x n_\varepsilon(t, x, a).$$

Moreover, since  $\beta$  is non-increasing and the  $L^1$  norm of  $\bar{n}$  is preserved,

$$\bar{n}(t/\varepsilon, 0) = \int_0^\infty \beta(a) \bar{n}(t/\varepsilon, a) da \leq \beta(0) \int_0^\infty \bar{n}(t/\varepsilon, a) da = \beta(0) \|\bar{n}^0\|_{L^1}.$$

It follows that:

$$\psi_\varepsilon(t, x) = -\varepsilon \ln n_\varepsilon(t, x, 0) \geq -\varepsilon \ln(\beta(0) \|\bar{n}^0\|_{L^1}).$$

After computing the  $L^1$  norm of  $\bar{n}^0$ :

$$\|\bar{n}^0\|_{L^1} = \exp(-\inf_x v/\varepsilon) \int_0^\infty \exp(-\inf_x \eta(x, a)) da,$$

and taking into account (15), we obtain the claimed result (26).  $\square$

The rest of this subsection is devoted to proving the upper bound (27) in Proposition 5.

*Remark.* As mentioned previously in a remark page 10, the space-homogenous problem does not admit an integrable stationary measure. Moreover, we have deciphered the self-similar behaviour in [3]. The side effect in our context is a time-dependent correction term of the form:

$$-\varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a + T/\varepsilon) e^{\int_0^a \beta} \omega(z) \exp(-\eta(x - \varepsilon z, a)) dz da \right).$$

*Proof.* This proof relies on a maximum principle. Since  $\psi_\varepsilon$  does not necessarily reach a maximum <sup>1</sup> over  $[0, T] \times \mathbb{R}$ , we will introduce penalising terms.

<sup>1</sup>For the sake of clarity, the reader may at first assume that  $\psi_\varepsilon$  does reach its maximum and  $\delta_1 = \delta_2 = 0$ . This allows to focus on the application of the maximum principle instead of the technical details.

**Definition 6.** Let  $\bar{\delta} > 0$ . For  $0 < \delta_1, \delta_2 < \bar{\delta}$ , we define:

$$\tilde{\psi}(t, x) = \psi_\varepsilon(t, x) - \delta_1 t - \delta_2 x^2. \quad (29)$$

Let  $(t_0, x_0)$  be a maximum point for  $\tilde{\psi}$  over  $[0, T] \times \mathbb{R}$ .

Indeed,  $\tilde{\psi}$  reaches its maximum: since  $\Phi$  is non increasing, from hypothesis 2.3 we gather, for  $x \in \mathbb{R}$  and  $0 \leq t \leq T$ :

$$\int_0^1 \int_{\mathbb{R}} \Phi(a + t/\varepsilon) \omega(z) e^{\int_0^a \beta} e^{-\eta(x - \varepsilon z, a)} dz da > e^{-C/\varepsilon} \Phi(1 + T/\varepsilon) / \Phi(0) > 0.$$

Since  $v$  is bounded, this gives a (very) suboptimal upper bound at fixed  $\varepsilon$  for  $\psi_\varepsilon$  that allows us to conclude that  $\tilde{\psi}$  reaches its maximum.

*Remark.* If  $(t_0, x_0) = (0, 0)$ , the upper bound in Proposition 5 is trivial. We will therefore exclude this case in the computations that follow.

Let  $0 < \delta_1, \delta_2 < \bar{\delta}$ . Since  $\Phi$  and  $\omega$  are probability measures, we rewrite the left hand side of equation (9) as:

$$1 = \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) dz da + \int_{t/\varepsilon}^\infty \Phi(a) da.$$

By injecting the expression (29) of  $\tilde{\psi}$ , we obtain the following equation:

$$\begin{aligned} 0 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ \exp \left( \frac{1}{\varepsilon} \left[ \tilde{\psi}(t, x) - \tilde{\psi}(t - \varepsilon a, x - \varepsilon z) + \delta_1 \varepsilon a + 2\varepsilon \delta_2 x z - \varepsilon^2 \delta_2 z^2 \right] \right) - 1 \right] dz da \\ &\quad + \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi \omega \exp \left( \frac{1}{\varepsilon} \left[ \tilde{\psi}(t, x) + \delta_1 t + \delta_2 x^2 - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon) \right] \right) \exp \left( \int_0^{a-t/\varepsilon} \beta \right) dz da \\ &\quad - \int_{t/\varepsilon}^\infty \Phi(a) da, \end{aligned} \quad (30)$$

**Lemma 7.** If  $\tilde{\psi}$  satisfies at  $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R} \setminus (0, 0)$ :

$$\tilde{\psi}(t_0, x_0) > \sup v - \varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) e^{\int_0^a \beta} \omega(z) e^{-\eta(x_0 - \varepsilon z, a)} dz da \right) \quad (31)$$

then:

$$0 > \underbrace{\int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \left( e^{\delta_1 a - \delta_2 \varepsilon z^2} - 1 \right) dz da}_{\mathfrak{A}} + \underbrace{e^{\delta_1 \frac{t_0}{\varepsilon} + \delta_2 \frac{x_0^2}{\varepsilon}} \int_0^{t_0/\varepsilon} \Phi(a) da + \frac{1}{2} \left[ e^{\delta_1 t_0/\varepsilon + \delta_2 x_0^2/\varepsilon} - 1 \right] \int_{t_0/\varepsilon}^\infty \Phi(a) da}_{\mathfrak{B}}, \quad (32)$$

with equality if  $(t_0, x_0) = (0, 0)$ .

*Proof.* At  $(t_0, x_0)$ , equation (30) becomes:

$$\begin{aligned} 0 &\geq \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ e^{\delta_1 a - \varepsilon \delta_2 z^2 + 2\delta_2 x_0 z} - 1 \right] dz da \\ &\quad + \left( e^{\delta_1 t_0/\varepsilon + \delta_2 x_0^2/\varepsilon} \int_{t_0/\varepsilon}^{1+t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega \exp \left( \frac{1}{\varepsilon} \left[ \tilde{\psi}(t_0, x_0) - v(x_0 - \varepsilon z) - \varepsilon \eta(x_0 - \varepsilon z, a - t_0/\varepsilon) \right] \right) + \int_0^{a-t_0/\varepsilon} \beta \right) dz da \\ &\quad - \int_{t_0/\varepsilon}^\infty \Phi(a) da \end{aligned}$$

Since  $\omega$  is an even function, the first right-hand-side term satisfies:

$$\begin{aligned} \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ e^{\delta_1 a - \varepsilon \delta_2 z^2 + 2\delta_2 x_0 z} - 1 \right] dz da &= \int_0^{t_0/\varepsilon} \int_{\mathbb{R}_+} \Phi \omega \left[ e^{\delta_1 a - \varepsilon \delta_2 z^2} \underbrace{\left( e^{2\delta_2 x_0 z} + e^{-2\delta_2 x_0 z} \right)}_{=2 \cosh(2\delta_2 x_0 z) \geq 2} - 2 \right] dz da \\ &\geq \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ e^{\delta_1 a - \varepsilon \delta_2 z^2} - 1 \right] dz da = \mathfrak{A}. \end{aligned}$$

We now use the strict lower bound on  $\tilde{\psi}$  (31) and obtain:

$$\begin{aligned} 0 &\geq \mathfrak{A} + \left( e^{\delta_1 t_0/\varepsilon + \delta_2 x_0^2/\varepsilon} \right) \frac{\int_0^1 \int_{\mathbb{R}} \Phi \left( a + \frac{t_0}{\varepsilon} \right) \omega(z) e^{-\eta(x_0 - \varepsilon z, a)} e^{\int_0^a \beta} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi \left( a + \frac{t_0}{\varepsilon} \right) \omega(z) e^{-\eta(x_0 - \varepsilon z, a)} e^{\int_0^a \beta} dz da} - \int_{t_0/\varepsilon}^{\infty} \Phi(a) da \\ &\geq \mathfrak{A} + \left( e^{\delta_1 t_0/\varepsilon + \delta_2 x_0^2/\varepsilon} \right) \int_0^{t_0/\varepsilon} \Phi(a) da + \left[ e^{\delta_1 t_0/\varepsilon + \delta_2 x_0^2/\varepsilon} - 1 \right] \int_{t_0/\varepsilon}^{\infty} \Phi(a) da \end{aligned} \quad (33)$$

the third right hand side term being non-negative, and only equal to 0 if  $t_0 = 0$  and  $x_0 = 0$ . This gives the claimed inequality (32), the equality case requiring  $t_0 = 0$  and  $x_0 = 0$ .  $\square$

There remains to find a sequence of  $(\delta_1, \delta_2)$  converging to 0 such that for any such couple, (32) is contradicted. However, we must bear in mind that  $(t_0, x_0) = \arg \max_{[0, T] \times \mathbb{R}} \tilde{\psi}$  depends on  $(\delta_1, \delta_2)$ . For that reason, we will prove (32) is contradicted for a suitable set of couples  $(\delta_1, \delta_2)$  uniformly over  $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}$ .

**Lemma 8.** *There exists a positive  $\bar{\delta}_2 < \bar{\delta}$  such that for any couple  $(\delta_1, \delta_2) \in (0, \bar{\delta}) \times (0, \bar{\delta}_2)$ , whatever the value of  $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R} \setminus (0, 0)$ ,*

$$\mathfrak{A} + \mathfrak{B} > 0. \quad (34)$$

*Proof.* We distinguish between two cases:

1.  $t_0 = 0$ . For  $x_0 \neq 0$ , (34) clearly holds.
2.  $t_0 > 0$ . In this case,

$$\begin{aligned} \mathfrak{A} &= \int_0^{t_0/\varepsilon} \Phi(a) e^{\delta_1 a} da \int_{\mathbb{R}} \omega(z) e^{-\varepsilon \delta_2 z^2} dz - \int_0^{t_0/\varepsilon} \Phi(a) da \\ &\geq \int_0^{t_0/\varepsilon} \Phi(a) da \underbrace{\left( \int_{\mathbb{R}} \omega(z) e^{-\varepsilon \delta_2 z^2} dz - 1 \right)}_{f(\delta_2)} \end{aligned}$$

where  $f(0) = 0$ . Since  $f$  is continuous, there exists some positive  $\bar{\delta}_2$  small enough such that for all  $0 < \delta_2 < \bar{\delta}_2$ ,  $f(\delta_2) > -1/2$ . Since  $t_0$  and  $\delta_1$  are positive, we also have:

$$\mathfrak{B} \geq e^{\delta_1 \frac{t_0}{\varepsilon}} \int_0^{t_0/\varepsilon} \Phi(a) da > \int_0^{t_0/\varepsilon} \Phi(a) da.$$

For  $\delta_2 < \bar{\delta}_2$ , it follows that:

$$\mathfrak{A} + \mathfrak{B} > \frac{1}{2} \int_0^{t_0/\varepsilon} \Phi(a) da > 0.$$

□

We will now use the result of this lemma for  $t < T$  rather than for  $t \in \mathbb{R}_+$  since the upper bound we are proving depends on  $T$ .

*End of the proof of Proposition 5:*

The hypothesis of Lemma 8 leads to the contradiction  $0 \geq \mathfrak{A} + \mathfrak{B} > 0$ . Therefore,

$$\tilde{\psi}(t_0, x_0) \leq \mathfrak{C}_{t_0} = \sup v - \varepsilon \ln \left( \int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) e^{\int_0^a \beta} \omega(z) e^{-\eta(x_0 - \varepsilon z, a)} dz da \right),$$

hence:  $\forall (t, x) \in [0, T] \times \mathbb{R} \setminus (0, 0) \forall (\delta_1, \delta_2) \in (0, \bar{\delta})^2 \mid 0 < \delta_2 < \bar{\delta}_2$ ,

$$\psi_\varepsilon(t, x) \leq \mathfrak{C}_{t_0} + \delta_1 t + \delta_2 x^2 \leq \mathfrak{C}_T + \delta_1 t + \delta_2 x^2$$

and this is clearly also true for  $(t, x) = (0, 0)$ . Passing to the limit  $(\delta_1, \delta_2) \rightarrow (0, 0)$  yields the upper bound (27). □

## 2.2 $\psi_\varepsilon$ is Lipschitz-continuous with respect to $x$ over $[0, T] \times \mathbb{R}$ .

*This subsection will deal with the proof of the following*

**Proposition 9.** *Let  $T > 0$  and  $\varepsilon > 0$ . Under hypotheses 1 and 2,  $\psi_\varepsilon$  is Lipschitz continuous in  $x$  over  $(t, x) \in [0, T] \times \mathbb{R}$ , with the quantitative bounds stated below:*

$$\partial_x \psi_\varepsilon \geq \inf_{(t, x) \in [0, T] \times \mathbb{R}} \frac{\int_0^1 \int_{\mathbb{R}} \Phi(a + t/\varepsilon) \omega(z) \partial_x \phi_\varepsilon^0(x - \varepsilon z, a) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi(a + t/\varepsilon) \omega(z) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}, \quad (35)$$

$$\partial_x \psi_\varepsilon \leq \sup_{(t, x) \in [0, T] \times \mathbb{R}} \frac{\int_0^1 \int_{\mathbb{R}} \Phi(a + t/\varepsilon) \omega(z) \partial_x \phi_\varepsilon^0(x - \varepsilon z, a) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi(a + t/\varepsilon) \omega(z) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}. \quad (36)$$

Let us start by giving a non optimal bound on  $\partial_x \psi_\varepsilon$  that will guarantee that some auxiliary function along the lines of  $\partial_x \psi_\varepsilon - \delta_1 t - \delta_2 x^2$  reaches its extrema.

**Lemma 10** (Non optimal bound).

$$|\partial_x \psi_\varepsilon(t, x)| \leq \mathfrak{G}(\varepsilon) < \infty \quad (37)$$

where, denoting by  $L$  is the Lipschitz constant in  $x$  of  $\phi_\varepsilon^0$  (see hypothesis 2.4),

$$\mathfrak{G}(\varepsilon) = L\beta(0) \exp \left( \frac{1}{\varepsilon} \left[ \sup_x \psi_\varepsilon(t, x) - \inf_x v(x) \right] \right) \int_0^\infty \exp \left( - \inf_x \eta(x, a) \right) da.$$

*Proof.* By definition of  $\psi_\varepsilon$  and  $n_\varepsilon$ ,

$$\begin{aligned} \partial_x \psi_\varepsilon(t, x) &= -\varepsilon \exp(\psi_\varepsilon(t, x)/\varepsilon) \partial_x n_\varepsilon(t, x, 0) \\ &= -\varepsilon \exp(\psi_\varepsilon(t, x)/\varepsilon) (\partial_x n)(t/\varepsilon, x/\varepsilon, 0). \end{aligned}$$

$n$  satisfies equation (1) and so does  $\partial_x n$ , with initial condition:

$$\partial_x n(0, x, a) = -(\partial_x \phi_\varepsilon^0)(\varepsilon x, a) \exp \left( -\frac{1}{\varepsilon} \phi_\varepsilon^0(\varepsilon x, a) \right),$$

so:

$$|\partial_x n(0, x, a)| \leq L \exp\left(-\frac{1}{\varepsilon} \inf_x v(x)\right) \exp\left(-\inf_x \eta(x, a)\right)$$

Equation (1) preserves positivity and  $L^1$  norm, and  $\omega$  is a probability distribution. It follows that for all positive  $t, a$  and  $x \in \mathbb{R}$ ,

$$\underline{n}_x(t, a) \leq \partial_x n(t, x, a) \leq \bar{n}_x(t, a),$$

where  $\underline{n}_x$  and  $\bar{n}_x$  are defined as solutions of (1) for respective initial conditions:

$$\begin{cases} \underline{n}_x(0, a) = -L \exp\left(-\frac{1}{\varepsilon} \inf_x v(x)\right) \exp\left(-\inf_x \eta(x, a)\right) & \leq 0 \text{ for all } a \\ \bar{n}_x(0, a) = L \exp\left(-\frac{1}{\varepsilon} \inf_x v(x)\right) \exp\left(-\inf_x \eta(x, a)\right) & \geq 0 \text{ for all } a. \end{cases} \quad (38)$$

Since (1) preserves positivity and  $L^1$  norm,

$$\begin{aligned} \bar{n}_x\left(\frac{t}{\varepsilon}, 0\right) &= \int_0^\infty \beta(a) \bar{n}_x\left(\frac{t}{\varepsilon}, a\right) da \leq \|\beta\|_\infty \int_0^\infty \bar{n}_x\left(\frac{t}{\varepsilon}, a\right) da = \|\beta\|_\infty \int_0^\infty \bar{n}_x(0, a) da \\ &\leq L \|\beta\|_\infty \exp\left(-\frac{1}{\varepsilon} \inf_x v(x)\right) \int_0^\infty \exp\left(-\inf_x \eta(x, a)\right) da, \end{aligned}$$

and

$$\begin{aligned} \underline{n}_x\left(\frac{t}{\varepsilon}, 0\right) &= \int_0^\infty \beta(a) \underline{n}_x\left(\frac{t}{\varepsilon}, a\right) da \geq \|\beta\|_\infty \int_0^\infty \underline{n}_x\left(\frac{t}{\varepsilon}, a\right) da = \|\beta\|_\infty \int_0^\infty \underline{n}_x(0, a) da \\ &\geq -L \|\beta\|_\infty \exp\left(-\frac{1}{\varepsilon} \inf_x v(x)\right) \int_0^\infty \exp\left(-\inf_x \eta(x, a)\right) da. \end{aligned}$$

$\beta$  is non-increasing so  $\|\beta\|_\infty = \beta(0)$ . Thanks to Proposition 5 and hypotheses 2.1 and 2.2,  $\mathfrak{G}(\varepsilon)$  is finite, which concludes the proof of the Lemma.  $\square$

We argue in a similar way to the previous subsection. Since  $\partial_x \psi_\varepsilon$  does not necessarily reach its bounds over  $[0, T] \times \mathbb{R}$ , let us define a family of modified functions that do, thanks to the result of Lemma 10.

**Definition 11.** For  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ , for  $0 < \delta_1, \delta_2 < \bar{\delta}$ , let

$$\tilde{\chi}(t, x) = \partial_x \psi_\varepsilon(t, x) - \delta_1 t - \delta_2 x^2. \quad (39)$$

Let

$$(t_0, x_0) = \arg \max_{[0, T] \times \mathbb{R}} \tilde{\chi}. \quad (40)$$

We may now prove the upper bound of Proposition 9 by contradiction using  $\tilde{\chi}$ . The proof of the lower bound is analogous, using  $\partial_x \psi_\varepsilon(t, x) + \delta_1 t + \delta_2 x^2$ .

*Proof of Proposition 9:*

By differentiating (9) with respect to  $x$ , we obtain:

$$\begin{aligned} 0 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\tilde{\chi}(t, x) - \tilde{\chi}(t - \varepsilon a, x - \varepsilon z) + \delta_1 \varepsilon a - \delta_2 \varepsilon (\varepsilon z^2 - 2xz)] e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]} dz da \\ &+ \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\tilde{\chi}(t, x) - \partial_x \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon) + \delta_1 t + \delta_2 x^2] e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta} dz da. \end{aligned} \quad (41)$$

Suppose  $\tilde{\chi}$  does not satisfy the upper bound (36). This implies:

$$\tilde{\chi}(t_0, x_0) > \frac{\int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) \omega(z) \partial_x \phi_\varepsilon^0(x_0 - \varepsilon z, a) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \phi_\varepsilon^0(x_0 - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi(a + t_0/\varepsilon) \omega(z) e^{\frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \phi_\varepsilon^0(x_0 - \varepsilon z, a)]} e^{\int_0^a \beta} dz da}.$$

At  $(t_0, x_0)$ , equation (41) now gives us:

$$0 > \mathfrak{A}_x + \mathfrak{B}_x$$

where  $\mathfrak{A}_x$  and  $\mathfrak{B}_x$  play roles analogous to that of  $\mathfrak{A}$  and  $\mathfrak{B}$  in the previous subsection:

$$\begin{aligned} \mathfrak{A}_x &= \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\delta_1 \varepsilon a - \delta_2 \varepsilon (\varepsilon z^2 - 2x_0 z)] \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \psi_\varepsilon(t_0 - \varepsilon a, x_0 - \varepsilon z)] \right) dz da \\ \mathfrak{B}_x &= \int_{t_0/\varepsilon}^{1+t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\delta_1 t_0 + \delta_2 x_0^2] \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \phi_\varepsilon^0(x_0 - \varepsilon z, a - t_0/\varepsilon)] \right) \exp \left( \int_0^{a-t_0/\varepsilon} \beta \right) dz da. \end{aligned}$$

The following estimate allows us to bound  $x_0$ :

$$\partial_x \psi_\varepsilon(t_0, x_0) - \delta_1 t_0 - \delta_2 x_0^2 = \tilde{\chi}(t_0, x_0) \geq \tilde{\chi}(0, 0) = \partial_x \psi_\varepsilon(0, 0)$$

hence

$$\delta_2 x_0^2 \leq \partial_x \psi_\varepsilon(t_0, x_0) - \partial_x \psi_\varepsilon(0, 0) - \delta_1 t_0 \leq 2\mathfrak{G}(\varepsilon)$$

(with  $\mathfrak{G}$  defined in Lemma 10), which implies:

$$|x_0| \leq \delta_2^{-1/2} \sqrt{2\mathfrak{G}(\varepsilon)} \quad (42)$$

We may now bound  $\mathfrak{A}_x$  and  $\mathfrak{B}_x$  below in a rough yet sufficiently accurate way. Let us start with  $\mathfrak{A}_x$ .

$$\begin{aligned} \mathfrak{A}_x &\geq \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\delta_1 \varepsilon a] \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \psi_\varepsilon(t_0 - \varepsilon a, x_0 - \varepsilon z)] \right) dz da \\ &\quad - \int_0^{t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ 2\sqrt{\delta_2} \sqrt{2\mathfrak{G}(\varepsilon)} |z| + \delta_2 \varepsilon^2 z^2 \right] \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \psi_\varepsilon(t_0 - \varepsilon a, x_0 - \varepsilon z)] \right) dz da. \end{aligned}$$

Hence, respectively denoting by  $\mathfrak{c}$  and  $\mathfrak{C}_T$  the lower and upper bounds of Proposition 5:

$$\begin{aligned} \mathfrak{A}_x &\geq \delta_1 \left( \varepsilon e^{\frac{1}{\varepsilon}[\mathfrak{c} - \mathfrak{C}_T]} \int_0^{t_0/\varepsilon} a \Phi(a) da \right) \\ &\quad - \sqrt{\delta_2} \left( e^{\frac{1}{\varepsilon}[\mathfrak{C}_T - \mathfrak{c}]} \int_{\mathbb{R}} \omega(z) \left[ 2\sqrt{2\mathfrak{G}(\varepsilon)} |z| + \sqrt{\delta_2} \varepsilon^2 z^2 \right] dz \int_0^{t_0/\varepsilon} \Phi(a) da \right). \quad (43) \end{aligned}$$

As for  $\mathfrak{B}_x$ :

$$\mathfrak{B}_x \geq \int_{t_0/\varepsilon}^{1+t_0/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\delta_1 t_0 + \delta_2 x_0^2] \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t_0, x_0) - \phi_\varepsilon^0(x_0 - \varepsilon z, a - t_0/\varepsilon)] \right) \exp \left( \int_0^{a-t_0/\varepsilon} \beta \right) dz da.$$

Hence:

$$\mathfrak{B}_x \geq [\delta_1 t_0 + \delta_2 x_0^2] \left( e^{\frac{1}{\varepsilon}[\mathfrak{c} - \mathfrak{C}_T]} \int_{t_0/\varepsilon}^{1+t_0/\varepsilon} \Phi(a) e^{\int_0^{a-t_0/\varepsilon} \beta} da \right). \quad (44)$$

The following lemma allows us to conclude:



**Lemma 12.** *There exists a positive function  $h : (0, \bar{\delta}) \rightarrow \mathbb{R}_+^*$  such that for  $\delta_1, \delta_2 \in (0, \bar{\delta})$  satisfying  $\delta_2 < h(\delta_1)$ , whatever value  $(t, x)$  may take in  $\mathbb{R}_+ \times \mathbb{R} \setminus (0, 0)$ , we have:*

$$(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t, x) > 0$$

(with equality for  $(t, x) = (0, 0)$ ).

*Proof.*

If  $t = 0$ , it is clear that  $\mathfrak{A}_x + \mathfrak{B}_x \geq 0$  with equality if and only if  $x = 0$ .

If  $t \neq 0$ , let us consider the two following cases:

1.  $t < T_1$  small enough.

We recall  $\Phi(a) = \mu(1+a)^{-1-\mu}$ . The following bound holds:

$$\int_0^{t/\varepsilon} \Phi(a) da \leq \mu \frac{t}{\varepsilon}.$$

Plugging this expression into (43) and dropping the positive term yields:

$$\mathfrak{A}_x \geq - \left[ e^{\frac{1}{\varepsilon}[\mathfrak{C}_T - c]} \int_{\mathbb{R}} \omega(z) \left( 2\sqrt{2\mathfrak{G}(\varepsilon)}|z| + \sqrt{\delta_2}\varepsilon^2 z^2 \right) dz \right] \left( \frac{\mu}{\varepsilon} \right) t \sqrt{\delta_2}$$

$$\mathfrak{B}_x \geq e^{\frac{1}{\varepsilon}[\mathfrak{C} - \mathfrak{C}_T]} \left[ \int_{t/\varepsilon}^{1+t_0/\varepsilon} \Phi(a) e^{\int_0^{a-t/\varepsilon} \beta da} da \right] t \delta_1 \geq e^{\frac{1}{\varepsilon}[\mathfrak{C} - \mathfrak{C}_T]} \left[ \int_{T_1/\varepsilon}^{1+T_1/\varepsilon} \Phi(a) e^{\int_0^{a-t/\varepsilon} \beta da} da \right] t \delta_1,$$

since  $\Phi$  is non-increasing. Since  $\omega$  has bounded first and second moments, it follows that, for  $\sqrt{\delta_2}/\delta_1$  small enough,  $\mathfrak{A}_x + \mathfrak{B}_x > 0$ : there exists a positive function  $h^0$  defined over  $(0, \bar{\delta})$  such that for  $\delta_1, \delta_2 \in (0, \bar{\delta})$  satisfying  $\delta_2 < h^0(\delta_1)$ , for any  $(t, x) \in (0, T_1) \times \mathbb{R}$ , we have:

$$(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t, x) > 0.$$

2.  $t \geq T_1$ .

If  $t_0$  is greater than some  $T_1 > 0$ , (44) shows us  $\mathfrak{B}_x > 0$ , and (43) allows us to see  $\mathfrak{A}_x$  is positive for  $\sqrt{\delta_2}/\delta_1$  small enough since  $\omega$  has bounded first and second moments.

Since  $0 < \delta_2 < \bar{\delta}$ , there exists a positive function  $h^1$  defined over  $(0, \bar{\delta})$  such that for  $\delta_1, \delta_2 \in (0, \bar{\delta})$  satisfying  $\delta_2 < h^1(\delta_1)$ , for any  $(t, x) \in [T_1, \infty) \times \mathbb{R}$ , we have:

$$(\mathfrak{A}_x + \mathfrak{B}_x)(\delta_1, \delta_2, t, x) > 0.$$

Setting  $h = \min \{h^0, h^1\}$  ends the proof of the Lemma. □

*End of the proof of the Proposition:*

We now use the result of the previous Lemma for  $t < T$  since the upper bound we are proving depends on  $T$ . We have proved that, provided  $\tilde{\chi}$  does not satisfy the upper bound (36), for  $\delta_1, \delta_2 \in (0, \bar{\delta})$  satisfying  $\delta_2 < h(\delta_1)$ , with  $h > 0$ , for any  $(t, x) \in [0, T] \times \mathbb{R}$ , we have the contradiction  $0 > \mathfrak{A}_x + \mathfrak{B}_x \geq 0$ . This shows  $\tilde{\chi}$  satisfies the upper bound (36). Since  $h > 0$ , we may pass to the limit as in the previous subsection, which proves the upper bound of Proposition 9.

*Remark.* The lower bound (35) can be proved in an analogous way by introducing  $\partial_x \psi_\varepsilon(t, x) + \delta_1 t + \delta_2 x^2$  instead of  $\tilde{\chi}$ . □

### 2.3 $\psi_\varepsilon$ is Lipschitz-continuous with respect to $t$ over $[0, T] \times \mathbb{R}$ .

This subsection will deal with the proof of the following

**Proposition 13.** *Let  $T > 0$  and  $\varepsilon > 0$ . Under hypotheses 1 and 2,  $\psi_\varepsilon$  is Lipschitz continuous in  $t$  over  $(t, x) \in [0, T] \times \mathbb{R}$ , with the quantitative bounds stated below:*

$$\partial_t \psi_\varepsilon(t, x) \geq -C(\omega, \|\partial_x \phi_\varepsilon^0\|_\infty) \quad (45)$$

$$\partial_t \psi_\varepsilon(t, x) \leq \mu(1 + \mu). \quad (46)$$

The structure of the proof is the same as in the previous subsection. The following rough estimate of  $|\psi_\varepsilon(t + h, x) - \psi_\varepsilon(t, x)|$  plays the role of Lemma 10:

**Lemma 14.** *There exists a positive constant  $C$  such that for all  $h \in [0, 1]$ ,*

$$|\psi_\varepsilon(t + h, x) - \psi_\varepsilon(t, x)| \leq C \frac{h}{\varepsilon}.$$

*Proof.* From equation (9) we gather:

$$\begin{aligned} & \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t + h, x)\right) - \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t, x)\right) = \\ & \int_0^{t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)\right) dz da \\ & + \int_0^{h/\varepsilon} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t + h - \varepsilon a, x - \varepsilon z)\right) dz da \\ & + \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp\left(-\frac{1}{\varepsilon}\phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)\right) \exp\left(\int_0^{a-t/\varepsilon} \beta\right) dz da. \end{aligned}$$

We recall that  $t \in [0, T]$ ,  $\varepsilon$  is fixed, and  $h \in [0, 1]$ .  $\Phi$  is bounded and non-negative, and  $\psi_\varepsilon$  is bounded over  $[0, T + h/\varepsilon] \times \mathbb{R}$  thanks to Proposition 5. It follows that for some  $C > 0$ ,

$$\left| \frac{\Phi(a + h/\varepsilon)}{\Phi(a)} - 1 \right| \leq C \frac{h}{\varepsilon},$$

which allows us to bound the first and third right-hand-side terms as follows:

$$\begin{aligned} & \left| \int_0^{t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)\right) dz da \right| \\ & \leq C \frac{h}{\varepsilon} \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon}\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)\right) dz da \end{aligned}$$

and

$$\begin{aligned} & \left| \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} [\Phi(a + h/\varepsilon) - \Phi(a)] \omega(z) \exp\left(-\frac{1}{\varepsilon}\phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)\right) \exp\left(\int_0^{a-t/\varepsilon} \beta\right) dz da \right| \\ & \leq C \frac{h}{\varepsilon} \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon}\phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)\right) \exp\left(\int_0^{a-t/\varepsilon} \beta\right) dz da. \end{aligned}$$

The sum of the two previous upper bounds is  $Ch/\varepsilon$ , thanks to equation (9). We can also bound the second right-hand-side term:

$$\left| \int_0^{h/\varepsilon} \int_{\mathbb{R}} \Phi(a) \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t+h-\varepsilon a \cdot x - \varepsilon z)\right) dz da \right| \leq C' \frac{h}{\varepsilon}$$

with  $C' = \|\Phi\|_\infty \|\omega\|_\infty \exp\left(-\frac{1}{\varepsilon} \|\psi_\varepsilon\|_{L^\infty([0, T+h/\varepsilon] \times \mathbb{R})}\right)$ . It follows that:

$$\left| \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t+h, x)\right) - \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t, x)\right) \right| \leq \frac{h}{\varepsilon} [C + C'].$$

Since  $\ln$  is Lipschitz-continuous over  $[\exp(-\sup_{[0, T+h] \times \mathbb{R}} \psi_\varepsilon/\varepsilon), \exp(-\inf_{[0, T+h] \times \mathbb{R}} \psi_\varepsilon/\varepsilon)]$  with some Lipschitz constant  $L$ , we have:

$$|\psi_\varepsilon(t+h, x) - \psi_\varepsilon(t, x)| \leq \left| \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t+h, x)\right) - \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(t, x)\right) \right| \leq \frac{h}{\varepsilon} [C + C'].$$

□

The suboptimal,  $\varepsilon$ -dependent bound for  $\partial_t \psi_\varepsilon$  over  $[0, T] \times \mathbb{R}$  that we have just recovered allows us to define modified functions reaching their extrema over that set. Applying a maximum principle would deliver the bound given in (49) (and the corresponding lower bound). Such bounds correspond to those given in Proposition 13, as shown. We would have to deal with parasitic terms  $\mathfrak{A}_t$  and  $\mathfrak{B}_t$ , which are (not surprisingly) equal to  $\mathfrak{A}_x$  and  $\mathfrak{B}_x$ . This would allow us to follow nearly to the letter the computations preceding Lemma 12. A similar Lemma would allow us to conclude. Therefore, without loss of generality and with some gain of clarity, we may assume that  $\partial_t \psi_\varepsilon$  reaches its extrema, which we do throughout the rest of this subsection.

*Proof of the upper bound.*

Differentiating (9) with respect to  $t$  and multiplying by  $e^{\frac{1}{\varepsilon} \psi_\varepsilon(t, x)} \neq 0$  yields:

$$\begin{aligned} 0 &= \Phi\left(\frac{t}{\varepsilon}\right) \left[ \int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \psi(0, x - \varepsilon z)} dz - \int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \phi_\varepsilon^0(x - \varepsilon z, 0)} dz + \int_{\mathbb{R}} \omega(z) e^{-\frac{1}{\varepsilon} \phi_\varepsilon^0(x - \varepsilon z, 1)} e^{\int_0^1 \beta} dz \right] \\ &+ \int_0^{t/\varepsilon} \int_{\mathbb{R}} \Phi \omega [\partial_t \psi_\varepsilon(t, x) - \partial_t \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)] e^{-\frac{1}{\varepsilon} \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)} dz da \\ &+ \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \Phi \omega \left[ \partial_t \psi_\varepsilon(t, x) - \left[ \beta\left(a - \frac{t}{\varepsilon}\right) - \frac{1}{\varepsilon} \partial_a \phi_\varepsilon^0\left(x - \varepsilon z, a - \frac{t}{\varepsilon}\right) \right] \right] e^{-\frac{1}{\varepsilon} \psi_\varepsilon^0(x - \varepsilon z, a - \frac{t}{\varepsilon})} e^{\int_0^{a-t/\varepsilon} \beta} dz da. \end{aligned} \tag{47}$$

where  $\int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x - \varepsilon z, 1)\right) \exp\left(\int_0^1 \beta\right) dz = 0$  since  $n^0(x - \varepsilon z, 1) = 0$ .

At  $(t_0, x_0) = \arg \max_{[0, T] \times \mathbb{R}} \partial_t \psi_\varepsilon$ , the previous equation gives us:

$$\begin{aligned} 0 &\geq \Phi\left(\frac{t_0}{\varepsilon}\right) \left[ \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi(0, x_0 - \varepsilon z)\right) dz - \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, 0)\right) dz \right] \\ &+ \int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega \left[ \partial_t \psi_\varepsilon(t_0, x_0) - \left[ \beta(a) - \frac{1}{\varepsilon} \partial_a \phi_\varepsilon^0(x_0 - \varepsilon z, a) \right] \right] \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon^0(x_0 - \varepsilon z, a)\right) \exp\left(\int_0^a \beta\right) dz da. \end{aligned} \tag{48}$$

It follows that:

$$\begin{aligned} \partial_t \psi_\varepsilon(t_0, x_0) \leq & \frac{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) [\beta(a) - \partial_a \phi_\varepsilon^0(x_0 - \varepsilon z, a)/\varepsilon] \exp\left(\int_0^a \beta(s) ds - \frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da} \\ & + \frac{\Phi\left(\frac{t_0}{\varepsilon}\right) \int_{\mathbb{R}} \omega(z) [\exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, 0)\right) - \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(0, x_0 - \varepsilon z)\right)] dz}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da}. \end{aligned} \quad (49)$$

Integrating the numerator of the first term by parts yields:

$$\partial_t \psi_\varepsilon(t_0, x_0) \leq \mathfrak{A}_t^+ + \mathfrak{B}_t^+, \quad (50)$$

with:

$$\mathfrak{A}_t^+ = -\frac{\Phi\left(\frac{t_0}{\varepsilon}\right) \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(0, x_0 - \varepsilon z)\right) dz}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da}$$

and

$$\mathfrak{B}_t^+ = -\frac{\int_0^1 \int_{\mathbb{R}} \frac{\Phi'(a + \frac{t_0}{\varepsilon})}{\Phi(a + \frac{t_0}{\varepsilon})} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_0}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_0 - \varepsilon z, a)\right) dz da},$$

where  $\mathfrak{A}_t^+ \leq 0$  and since  $-\frac{\Phi'(a + \frac{t_0}{\varepsilon})}{\Phi(a + \frac{t_0}{\varepsilon})} = \frac{\mu(1+\mu)}{1+a+\frac{t_0}{\varepsilon}}$ , we have  $\mathfrak{B}_t^+ \leq \mu(1+\mu)$ , which ends the proof.  $\square$

*Proof of the lower bound.*

We set  $(t_1, x_1) = \arg \min_{[0, T] \times \mathbb{R}} \psi_\varepsilon$ . Calculations analogous to those above lead to the following inequality (after the same integration by parts):

$$\partial_t \psi_\varepsilon(t_1, x_1) \geq \mathfrak{A}_t^- + \mathfrak{B}_t^-, \quad (51)$$

with:

$$\mathfrak{A}_t^- = -\frac{\Phi\left(\frac{t_1}{\varepsilon}\right) \int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(0, x_1 - \varepsilon z)\right) dz}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z, a)\right) dz da}$$

and

$$\mathfrak{B}_t^- = -\frac{\int_0^1 \int_{\mathbb{R}} \frac{\Phi'(a + \frac{t_1}{\varepsilon})}{\Phi(a + \frac{t_1}{\varepsilon})} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z, a)} dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) \exp\left(\int_0^a \beta\right) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z, a)\right) dz da}.$$

$\Phi$  is decreasing, so  $\mathfrak{B}_t^- \geq 0$ . In order to bound  $\mathfrak{A}_t^-$ , we may notice:

$$\Phi(a) \Phi\left(\frac{t}{\varepsilon}\right) = \Phi\left(a + \frac{t}{\varepsilon}\right) \mu \left(\frac{1+a+\frac{t}{\varepsilon}}{1+a+\frac{t}{\varepsilon}+a\frac{t}{\varepsilon}}\right)^{1+\mu} \leq \Phi\left(a + \frac{t}{\varepsilon}\right).$$

Since at  $t = 0$  we have

$$\exp\left(-\frac{1}{\varepsilon} \psi_\varepsilon(0, x_1 - \varepsilon z)\right) = \int_0^1 \int_{\mathbb{R}} \Phi(a) \omega(y) \exp\left(-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z - \varepsilon y, a)\right) \exp\left(\int_0^a \beta\right) dy da,$$

we recover:

$$\mathfrak{A}_t^- \geq -\frac{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z, a)} \left(\int_{\mathbb{R}} \omega(y) e^{\frac{1}{\varepsilon} [\phi_\varepsilon^0(x_1 - \varepsilon z, a) - \phi_\varepsilon^0(x_1 - \varepsilon z - \varepsilon y, a)]} dy\right) dz da}{\int_0^1 \int_{\mathbb{R}} \Phi\left(a + \frac{t_1}{\varepsilon}\right) \omega(z) e^{\int_0^a \beta} e^{-\frac{1}{\varepsilon} \phi_\varepsilon^0(x_1 - \varepsilon z, a)} dz da}.$$

Under Hypothesis 2,  $\phi_0$  is Lipschitz in  $x$ . Hence for some positive  $C$  depending on  $\omega$  and on the Lipschitz constant in  $x$  of  $\phi^0$ ,

$$\int_{\mathbb{R}} \omega(y) \exp(y \|\partial_x \phi_\varepsilon^0\|_\infty) dy \leq C.$$

It follows that

$$\int_{\mathbb{R}} \omega(y) \exp\left(\frac{1}{\varepsilon} [\phi_\varepsilon^0(x_1 - \varepsilon z, a) - \phi_\varepsilon^0(x_1 - \varepsilon z - \varepsilon y, a)]\right) dy \leq C,$$

which implies:

$$\mathfrak{A}_t^- \geq -C(\omega, \|\partial_x \phi_\varepsilon^0\|_\infty).$$

□

### 3 Viscosity limit procedure

In this section, we continue to work over  $[0, T] \times \mathbb{R}$ . Our results hold for  $T \rightarrow \infty$ , which allows us to close the proof of Theorem 3.

Suppose the a priori estimates hold. By the Arzelà-Ascoli theorem, since  $W^{1,\infty}([0, T] \times \mathbb{R})$  is a Banach space, there exists a subsequence  $(\varepsilon_n)_{n \in \mathbb{N}}$  with  $\varepsilon_n \xrightarrow[n \rightarrow \infty]{} 0$  such that  $\psi_{\varepsilon_n} \xrightarrow[n \rightarrow \infty]{L_{\text{loc}}^\infty} \psi_0$ . We will proceed to prove that  $\psi_0$  so defined is the unique viscosity solution of the Hamilton-Jacobi equation

$$1 = \int_0^\infty \Phi(a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz \quad (\text{HJ})$$

with initial condition  $v$ .

Equation (9) is equivalent to the following, which allows us to define  $\mathfrak{A}_\varepsilon$  and  $\mathfrak{B}_\varepsilon$  and is better suited for the following proofs:

$$1 = (\mathfrak{A}_\varepsilon + \mathfrak{B}_\varepsilon)(\psi_\varepsilon)(t, x), \quad \text{where:}$$

$$\begin{aligned} \mathfrak{A}_\varepsilon(\psi_\varepsilon)(t, x) &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da, \\ \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) &= \int_0^1 \int_{\mathbb{R}} \omega(z) \frac{\Phi(a + t/\varepsilon)}{\Phi(a)} \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]\right) \exp\left(\int_0^a \beta\right) dz da. \end{aligned} \quad (52)$$

#### 3.1 Viscosity subsolution

**Proposition 15.** *Under hypotheses 1 and 2,  $\psi_0$  is a viscosity subsolution of (HJ).*

*Proof.* Let  $\Psi \in C^2(\mathbb{R}^+ \times \mathbb{R})$  be a test function such that  $\psi_0 - \Psi$  admits a maximum at  $(t_0, x_0)$ . By compactness in  $W_{\text{loc}}^{1,\infty}([0, T] \times \mathbb{R})$ , thanks to the a priori estimates, we obtain for a subsequence of  $\varepsilon \rightarrow 0$  which we will not rename:  $(t_\varepsilon, x_\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} (t_0, x_0)$ , where  $(t_\varepsilon, x_\varepsilon)$  is a point at which  $\psi_\varepsilon - \Psi$  reaches its maximum. We have then:

$$\forall \varepsilon > 0 \forall (z, a) \in \mathbb{R} \times [0, \frac{t_\varepsilon}{\varepsilon}],$$

$$\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon, x_\varepsilon) \geq \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z).$$

Since  $\mathfrak{B}_\varepsilon$  is non-negative, it follows that:

$$1 \geq \mathfrak{A}_\varepsilon(\psi_\varepsilon)(t_\varepsilon, x_\varepsilon) \geq \mathfrak{A}_\varepsilon(\Psi)(t_\varepsilon, x_\varepsilon).$$

However:

$$\begin{aligned} \Psi(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) &= \varepsilon a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + \varepsilon z \partial_x \Psi(t_\varepsilon, x_\varepsilon) \\ &+ \frac{1}{2} \varepsilon^2 \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2az \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\ &+ z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds. \end{aligned} \quad (53)$$

Therefore we have, for all  $A > 0$ :

$$\begin{aligned} 1 &\geq \int_0^A \int_{-A}^A \Phi(a) \omega(z) \exp \left\{ a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + z \partial_x \Psi(t_\varepsilon, x_\varepsilon) \right. \\ &+ \frac{1}{2} \varepsilon \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2az \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\ &\left. + z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds \right\} dz da. \end{aligned}$$

Since  $\Psi$  is  $C^2$ , the previous expression tends, for fixed  $A$ , when  $\varepsilon \rightarrow 0$ , to:

$$1 \geq \int_0^A \int_{-A}^A \Phi(a) \omega(z) \exp [a \partial_t \Psi(t_0, x_0) + z \partial_x \Psi(t_0, x_0)] dz da.$$

It follows that:

$$1 \geq \int_0^\infty \int_{\mathbb{R}} \Phi(a) \omega(z) \exp [a \partial_t \Psi(t_0, x_0) + z \partial_x \Psi(t_0, x_0)] dz da.$$

Therefore  $\psi_0$  is a viscosity subsolution of (HJ). □

### 3.2 Viscosity supersolution

In order to prove that  $\psi_0$  is a viscosity supersolution of (HJ), we need to control the  $\mathfrak{B}_\varepsilon$  term in equation (52), whose positivity sufficed in the previous subsection. This is tantamount to controlling the fate of the aging particles that come from the initial data and have never jumped. We show accurate bounds for  $\mathfrak{A}_\varepsilon$  and  $\mathfrak{B}_\varepsilon$  and a maximum principle for  $\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)$ . We recover an anomalous scaling  $\mathfrak{B}_\varepsilon \lesssim \varepsilon^\mu$  locally uniformly in time: this is the second main difficulty tackled in this paper.

**Lemma 16** (Bounds for  $\mathfrak{B}_\varepsilon$ ).

$$\frac{\Phi(t/\varepsilon)}{\Phi(0)} \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)] \right) \leq \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) \leq \frac{\Phi(1+t/\varepsilon)}{\Phi(1)} \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)] \right) \quad (54)$$

*Proof.* Claim: for all  $h > 0$ ,

$$a \mapsto \Phi(a+h)/\Phi(a)$$

is an increasing function. Indeed,

$$\begin{aligned} \frac{d}{da} \frac{\Phi(a+h)}{\Phi(a)} &= \frac{\Phi'(a+h)\Phi(a) - \Phi'(a)\Phi(a+h)}{(\Phi(a))^2} \\ &= \frac{\exp\left(\int_0^a \beta\right) \exp\left(\int_0^{a+h} \beta\right)}{(\Phi(a))^2} \left( \beta'(a+h)\beta(a) - \beta'(a)\beta(a+h) + \underbrace{\beta(a)\beta(a+h)[\beta(a) - \beta(a+h)]}_{\geq 0} \right), \end{aligned}$$

which is positive since,  $\beta(a) = \mu/(1+a)$  being non-increasing and convex,  $\beta'(a+h) \geq \beta'(a)$  by convexity and  $\beta(a) \geq \beta(a+h) \geq 0$ . This proves the claim.

We now write  $\mathfrak{B}_\varepsilon$  as follows:

$$\mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) = \int_0^1 \frac{\Phi(a+t/\varepsilon)}{\Phi(a)} \Phi(a) \int_{\mathbb{R}} \omega(z) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]\right) \exp\left(\int_0^a \beta\right) dz da$$

and recover the lower and upper bounds by monotonicity and thanks to the  $a = 0$  boundary condition in (9).  $\square$

The next lemma gives a lower bound for  $\mathfrak{A}_\varepsilon$ . Its proof relies on an upper bound of  $\partial_x^2 \psi_\varepsilon$  obtained from equation (9) thanks to a maximum principle. This result is in agreement with classical semi-concavity preservation results for Hamilton-Jacobi equations.

*Remark.* The Hamiltonian  $H$  is not strictly convex (see proposition 1). This forbids a natural approach to bounding  $\partial_x^2 \psi_\varepsilon$  in one space dimension for a strictly convex Hamiltonian, namely the use of the Oleinik Lip+ property (see [24]):

Since the Hamiltonian  $H$  is not uniformly strictly convex (see proposition 1), boundedness and regularity hypotheses on the initial condition are required. A natural approach to bounding  $\partial_x^2 \psi_\varepsilon$  in one space dimension for a strictly convex Hamiltonian would have been the Oleinik Lip+ property (see [24]):

$$\partial_x^2 \psi_\varepsilon(t, \cdot) \leq \frac{1}{t \inf H''}.$$

**Lemma 17** (Lower bound for  $\mathfrak{A}_\varepsilon$ ). *For  $\varepsilon$  small enough and  $(t, x) \in [0, T] \times \mathbb{R}$ ,*

$$\mathfrak{A}_\varepsilon(\psi)(t, x) \geq \left[1 - \varepsilon \frac{\mathfrak{C}_{xx}}{2} \int_{\mathbb{R}} \omega(z) z^2 dz\right] \int_0^{t/\varepsilon} \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x)]\right) da \quad (55)$$

where  $\mathfrak{C}_{xx}$  is the upper bound of  $\partial_x^2 \phi_\varepsilon^0$  from hypothesis 2.5.

*Proof.* Let us differentiate (9) twice with respect to  $x$ . We obtain:

$$\begin{aligned} 0 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(a) \left[ (\partial_x \psi_\varepsilon(t, x) - \partial_x \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z))^2 + \right. \\ &\quad \left. \frac{1}{\varepsilon} (\partial_x^2 \psi_\varepsilon(t, x) - \partial_x^2 \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)) \right] \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)]\right) dz da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \omega(z) \Phi(a+t/\varepsilon) \left[ (\partial_x \psi_\varepsilon(t, x) - \partial_x \phi_\varepsilon^0(x - \varepsilon z, a))^2 + \right. \\ &\quad \left. \frac{1}{\varepsilon} (\partial_x^2 \psi_\varepsilon(t, x) - \partial_x^2 \phi_\varepsilon^0(x - \varepsilon z, a)) \right] \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a)]\right) \exp\left(\int_0^a \beta\right) dz da \end{aligned} \quad (56)$$

The squared terms are obviously non-negative, and hypothesis 2.5 gives us  $\partial_x^2 \phi_\varepsilon^0 \leq \mathfrak{C}_{xx}$ . We recover an upper bound for  $\partial_x^2 \psi_\varepsilon$ . Indeed, at  $(t_0, x_0) = \arg \max \partial_x^2 \psi_\varepsilon^2$ , a maximum principle gives us directly:

$$\partial_x^2 \psi_\varepsilon(t_0, x_0) \leq \mathfrak{C}_{xx}. \quad (57)$$

This gives us:

$$\begin{aligned} \exp\left(-\frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)]\right) &\geq 1 - \frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)] \\ &\geq 1 - \frac{1}{\varepsilon} \left[-\varepsilon z \partial_x \psi_\varepsilon(t - \varepsilon a, x) + \frac{\mathfrak{C}_{xx}}{2} \varepsilon^2 z^2\right]. \end{aligned}$$

Then, since  $\int_{\mathbb{R}} z \omega(w) dz = 0$ ,

$$\int_{\mathbb{R}} \omega(z) \exp\left(-\frac{1}{\varepsilon} [\psi_\varepsilon(t - \varepsilon a, x - \varepsilon z) - \psi_\varepsilon(t - \varepsilon a, x)]\right) dz \geq 1 - \varepsilon \frac{\mathfrak{C}_{xx}}{2} \int_{\mathbb{R}} \omega(z) z^2 dz. \quad (58)$$

The result of the lemma follows.  $\square$

The following Lemma proves a maximum principle for  $\psi_\varepsilon(\cdot, x)$ . For the sake of conciseness, we will assume the maximum is reached: if it is not, we may yet again define a modified function as we have done in the previous sections and recover the same bound.

**Lemma 18** (Maximum principle). *Let us fix  $x \in \mathbb{R}$ . Let  $m$  be the maximum over  $t \in [0, T]$  of  $\psi_\varepsilon(\cdot, x) - \psi_\varepsilon(0, x)$ . For  $K = (\mathfrak{C}_{xx}/2) \int_{\mathbb{R}} \omega(z) z^2 dz > 0$ , we have:*

$$e^{m/\varepsilon} \leq 1 + \frac{T}{\varepsilon} + K\varepsilon \left(1 + \frac{T}{\varepsilon}\right)^{1+\mu}. \quad (59)$$

*Proof.* Lemma 17 and the lower bound in Lemma 16 give us, for  $K$  defined above,

$$1 \geq [1 - K\varepsilon] \int_0^{t/\varepsilon} \Phi(a) \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x)]\right) da + \frac{\Phi(t/\varepsilon)}{\Phi(0)} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)]\right).$$

Applying the maximum principle and denoting  $t_0 = \arg \max \psi_\varepsilon(\cdot, x) - \psi_\varepsilon(0, x)$  results in

$$1 \geq [1 - K\varepsilon] \int_0^{t_0/\varepsilon} \Phi(a) da + \frac{\Phi(t_0/\varepsilon)}{\Phi(0)} e^{m/\varepsilon},$$

hence:

$$\begin{aligned} e^{m/\varepsilon} &\leq \frac{\Phi(0)}{\Phi(t_0/\varepsilon)} \left[1 - (1 - K\varepsilon) \int_0^{t_0/\varepsilon} \Phi(a) da\right] \\ &\leq \left(1 + \frac{t_0}{\varepsilon}\right)^{1+\mu} \left[\int_{t_0/\varepsilon}^\infty \Phi(a) da + K\varepsilon \int_0^{t_0/\varepsilon} \Phi(a) da\right] \\ &\leq 1 + \frac{t_0}{\varepsilon} + K\varepsilon \left[\left(1 + \frac{t_0}{\varepsilon}\right)^{1+\mu} - \left(1 + \frac{t_0}{\varepsilon}\right)\right], \end{aligned}$$

hence the result.  $\square$

---

<sup>2</sup> $\partial_x^2 \psi_\varepsilon$  may not reach its maximum. The bounds over  $\partial_x \psi_\varepsilon$  of Proposition 9 allow us to define a modified function that does reach its maximum and to proceed as in subsection 2.2.



**Proposition 19** (Upper bound for  $\mathfrak{B}_\varepsilon$ ). *Under hypotheses 1 and 2, for any  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $\mathfrak{B}_\varepsilon$  decays in the following way as  $\varepsilon \rightarrow 0$ :*

$$\mathfrak{B}_\varepsilon \leq \varepsilon^\mu \frac{2^{1+\mu}}{t^{1+\mu}} [T + \varepsilon^{1-\mu} K T^{1+\mu}] \quad (60)$$

where  $K = (\mathfrak{C}_{xx}/2) \int_{\mathbb{R}} \omega(z) z^2 dz$ , as in the previous Lemma.

*Proof.* Lemma 18 and the upper bound in Lemma 16 give us:

$$\begin{aligned} \mathfrak{B}_\varepsilon(\psi_\varepsilon)(t, x) &\leq \frac{\Phi(1+t/\varepsilon)}{\Phi(1)} \exp\left(\frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(0, x)]\right) \\ &\leq \frac{2^{1+\mu}}{(2+t/\varepsilon)^{1+\mu}} e^{m/\varepsilon} \\ &\leq \frac{2^{1+\mu}}{(2+t/\varepsilon)^{1+\mu}} \left(1 + \frac{T}{\varepsilon} + K\varepsilon \left(1 + \frac{T}{\varepsilon}\right)^{1+\mu}\right) \\ &\leq \varepsilon^\mu 2^{1+\mu} \frac{\varepsilon + T}{(2\varepsilon + t)^{1+\mu}} + \varepsilon 2^{1+\mu} K \left(\frac{\varepsilon + T}{2\varepsilon + t}\right)^{1+\mu} \\ &\leq \frac{2^{1+\mu}}{t^{1+\mu}} [\varepsilon^\mu T + \varepsilon K T^{1+\mu}]. \end{aligned}$$

□

We may now prove the following

**Proposition 20.** *Under hypotheses 1 and 2,  $\psi_0$  is a viscosity supersolution of (HJ).*

*Proof.* Let  $\Psi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R})$  be a test function such that  $\psi_0 - \Psi$  admits a minimum at  $(t_0, x_0)$ , with  $t_0 > 0$ . By compactness in  $W_{\text{loc}}^{1,\infty}([0, T] \times \mathbb{R})$ , thanks to the a priori estimates, we obtain for a subsequence of  $\varepsilon \rightarrow 0$  which we will not rename:  $(t_\varepsilon, x_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} (t_0, x_0)$ , where  $(t_\varepsilon, x_\varepsilon)$  is a point at which  $\psi_\varepsilon - \Psi$  reaches its minimum. We have then:

$\forall \varepsilon > 0 \forall (z, a) \in \mathbb{R} \times [0, \frac{t_\varepsilon}{\varepsilon}]$ ,

$$\psi_\varepsilon(t_\varepsilon, x_\varepsilon) - \Psi(t_\varepsilon, x_\varepsilon) \leq \psi_\varepsilon(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z) - \Psi(t_\varepsilon - \varepsilon a, x_\varepsilon - \varepsilon z).$$

Proposition 19 assures  $\mathfrak{B}_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0$  uniformly in  $t \in [t_0/2, T)$ . Therefore:

$$\forall \delta > 0, \exists \varepsilon_\delta > 0 \mid \forall \varepsilon \in (0, \varepsilon_\delta), \mathfrak{B}_\varepsilon(\Psi)(t_\varepsilon, x_\varepsilon) < \delta.$$

hence, for  $\varepsilon \in (0, \varepsilon_\delta)$ :

$$1 - \delta \leq \mathfrak{A}_\varepsilon(\psi_\varepsilon)(t_\varepsilon, x_\varepsilon) \leq \mathfrak{A}_\varepsilon(\Psi)(t_\varepsilon, x_\varepsilon).$$

The test function  $\Psi$  is  $\mathcal{C}^2$ . This allows us to take the same Taylor expansion as in (53), and ensures that for all  $\delta > 0$  there exists  $L_\delta$  such that, uniformly in  $\varepsilon \in (0, \varepsilon_\delta)$ :

$$\begin{aligned} 1 - 2\delta &\leq \int_0^{L_\delta} \int_{-L_\delta}^{L_\delta} \Phi(a) \omega(z) \exp\left\{a \partial_t \Psi(t_\varepsilon, x_\varepsilon) + z \partial_x \Psi(t_\varepsilon, x_\varepsilon)\right. \\ &\quad + \frac{1}{2} \varepsilon \int_0^1 (1-s)^2 [a^2 \partial_t^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) + 2a z \cdot \partial_t \partial_x \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z) \\ &\quad \left. + z^2 \partial_x^2 \Psi(t_\varepsilon - \varepsilon s a, x_\varepsilon - \varepsilon s z)] ds\right\} dz da. \end{aligned}$$

Since the previous inequality holds uniformly in  $\varepsilon \in (0, \varepsilon_\delta)$ , we take the limit  $\varepsilon \rightarrow 0$  at fixed  $\delta$  and  $L_\delta$  and obtain:

$$1 - 2\delta \leq \int_0^{L_\delta} \int_{-L_\delta}^{L_\delta} \Phi(a)\omega(z) \exp([a\partial_t\Psi + z\partial_x\Psi](t_0, x_0)) \, dz \, da,$$

hence:

$$1 - 2\delta \leq \int_0^\infty \int_{\mathbb{R}} \Phi(a)\omega(z) \exp([a\partial_t\Psi + z\partial_x\Psi](t_0, x_0)) \, dz \, da.$$

By taking the limit when  $\delta \rightarrow 0$  we recover:

$$1 \leq \int_0^\infty \Phi(a) \exp(a\partial_t\Psi(t_0, x_0)) \, da \int_{\mathbb{R}} \omega(z) \exp(z\partial_x\Psi(t_0, x_0)) \, dz.$$

Therefore,  $\psi_0$  is a viscosity supersolution of (HJ).  $\square$

*Proof of Theorem 3.*

Propositions 15 and 20 prove  $\psi_0$  is a viscosity solution of the Hamilton-Jacobi equation (10). For a proof of the uniqueness of the viscosity solution we refer to Barles' book [2]. This concludes the proof of the Theorem.  $\square$

## 4 Discussion and Perspectives

There are two main aspects we would like to discuss in this section. First, we will support and elaborate on our claim at page 4 that the limiting Hamilton-Jacobi equation derived after renormalising  $n$  by an instationary measure inspired of [3] is the same as (10). Second, we will discuss a setting in which the jump rate  $\beta$  depends not only on age but also on space.

### 4.1 Renormalising by an instationary measure

The idea of renormalising by a stationary measure is classical. However, as has been shown in [3], it does not work here because, were a steady state to exist in self-similar variables for our equation, it would be infinite at age 0, rendering the boundary condition a meaningless “ $\infty = \infty$ ” equality. We will therefore use a function corresponding to the pseudo-equilibrium of [3]:

**Definition 21.** For any  $t > 0$  and  $0 < a < 1 + t$ , let

$$N(t, a) = (1 + a)^{-\mu}(1 + t - a)^{\mu-1}. \quad (61)$$

We also set, for any  $x \in \mathbb{R}$ ,  $t > 0$  and  $0 < a < 1 + t$ :

$$u(t, x, a) = \frac{n(t, x, a)}{N(t, a)}. \quad (62)$$

**Definition 22.** We define the following measure, for  $t > 0$  and  $0 < a < 1 + t$ :

$$\nu_t(a) = \beta(a) \frac{N(t, a)}{N(t, 0)} = \frac{\mu(1 + t)^{1-\mu}}{(1 + a)^{1+\mu}(1 + t - a)^{1-\mu}}. \quad (63)$$

Direct computation gives us

$$\partial_t \ln N + \partial_a \ln N + \beta(a) = 0,$$

which is also satisfied by  $n$ . Hence,  $u$  satisfies:

$$\begin{cases} \partial_t u(t, x, a) + \partial_a u(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ u(t, x, 0) = \int_0^{1+t} \int_{\mathbb{R}} \nu_t(a) \omega(x - x') u(t, x', a) dx' da' \\ u(0, x, a) = u^0(x, a) = n^0(x, a) N(0, a) \quad \text{with} \quad \text{supp}(u^0(x, \cdot)) = [0, 1]. \end{cases} \quad (64)$$

Let us take a hyperbolic time - space scaling and a Hopf-Cole transform:

**Definition 23.**

$$u_\varepsilon(t, x, a) = u\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, a\right) = \exp\left(-\frac{1}{\varepsilon} \tilde{\varphi}_\varepsilon^0(t, x, a)\right). \quad (65)$$

Characteristic flow of (64) leads us to define:

$$\tilde{\varphi}_\varepsilon(t, x, a) = \begin{cases} \tilde{\varphi}_\varepsilon^0(x, a - t/\varepsilon), & a > t/\varepsilon \\ \tilde{\psi}_\varepsilon(t - \varepsilon a, x), & a \leq t/\varepsilon. \end{cases} \quad (66)$$

Let us also set, in agreement with hypothesis 2:

$$\begin{aligned} \tilde{\varphi}_\varepsilon^0(x, a) &= v(x) + \varepsilon \xi(x, a) \\ &= v(x) + \varepsilon [\eta(x, a) - (1 + \mu) \ln(1 + a) - (1 - \mu) \ln(1 - a)] \end{aligned} \quad (67)$$

With the previous definitions,  $\tilde{\varphi}_\varepsilon$  satisfies the following equation, which is analogous to (9):

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \exp\left(\frac{1}{\varepsilon} \left[ \tilde{\psi}_\varepsilon(t, x) - \tilde{\psi}_\varepsilon(t - \varepsilon a, x - \varepsilon z) \right]\right) \nu_{t/\varepsilon}(a) \omega(z) dz da \\ &\quad + \int_0^1 \int_{\mathbb{R}} \exp\left(\frac{1}{\varepsilon} \left[ \tilde{\psi}_\varepsilon(t, x) - \tilde{\varphi}_\varepsilon^0(x - \varepsilon z, a) \right]\right) \nu_{t/\varepsilon}(a + t/\varepsilon) \omega(z) dz da. \end{aligned} \quad (68)$$

*Remark.* For any positive  $t$ ,

$$\int_0^{1+t/\varepsilon} \nu_{t/\varepsilon}(a) da = \frac{1}{1 + \frac{1}{1+t/\varepsilon}} \xrightarrow{\varepsilon \rightarrow 0} 1.$$

Assuming sufficient regularity, (68) gives us:

$$1 = \int_0^{t/\varepsilon} \int_{\mathbb{R}} \exp\left(a \partial_t \tilde{\psi}_\varepsilon(t, x)\right) \exp\left(z \partial_x \tilde{\psi}_\varepsilon(t, x)\right) \exp(o(1)) \omega(z) \Phi(a) \left(\frac{1 + t/\varepsilon}{1 - a + t/\varepsilon}\right)^{1-\mu} dz da.$$

Hence the formal limit of (68) is the same Hamilton-Jacobi equation as (10):

$$1 = \int_0^\infty \Phi(a) \exp\left(a \partial_t \tilde{\psi}_0(t, x)\right) da \int_{\mathbb{R}} \omega(z) \exp\left(z \partial_x \tilde{\psi}_0(t, x)\right) dz,$$

with the same initial condition  $v$ .

*Remark.* In order to prove convergence of this newly defined  $\tilde{\psi}_\varepsilon$  to  $\tilde{\psi}_0$ , solution of the limiting Hamilton-Jacobi equation, the computations required are more or less the same as those presented in the article, with a parasite term:

$$\begin{aligned} \frac{1}{2+t/\varepsilon} &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \left[ \exp \left( \frac{1}{\varepsilon} [\tilde{\psi}_\varepsilon(t, x) - \tilde{\psi}_\varepsilon(t - \varepsilon a, x - \varepsilon z)] \right) - 1 \right] \nu_{t/\varepsilon}(a) \omega(z) dz da \\ &+ \int_0^1 \int_{\mathbb{R}} \left[ \exp \left( \frac{1}{\varepsilon} [\tilde{\psi}_\varepsilon(t, x) - \tilde{\varphi}_\varepsilon^0(x - \varepsilon z, a)] \right) - 1 \right] \nu_{t/\varepsilon}(a + t/\varepsilon) \omega(z) dz da \end{aligned}$$

due to the fact that  $\nu_{t/\varepsilon}$  is not a probability measure over  $[0, 1 + t/\varepsilon]$ . Since  $\nu_{t/\varepsilon}$  does approach a probability measure for any  $t > 0$  as  $\varepsilon \rightarrow 0$ , this is not a major problem.

## 4.2 Space-dependent jump rate

Our study, as briefly mentioned in the Introduction, has a biological motivation. The random motion we model takes place in cellular media in which heterogeneities are often prevalent. Hence the relevance of considering a space-dependant jump rate  $\beta(x, a)$ . There are different pertinent ways of defining the jump rate, depending on what we intend to model. Here, we will only consider the simple case of a slow space variation of the jump rate, in the sense that follows. We define

$$\beta(x, a) = \frac{\mu(x)}{1+a}, \quad (69)$$

where  $0 < \mu < 1$  is Lipschitz continuous, and consider the following problem:

$$\begin{cases} \partial_t n_\varepsilon(t, x, a) + \frac{1}{\varepsilon} \partial_a n_\varepsilon(t, x, a) + \frac{1}{\varepsilon} \beta(x, a) n_\varepsilon(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n_\varepsilon(t, x, 0) = \int_0^{1+t/\varepsilon} \int_{\mathbb{R}} \beta(x - \varepsilon z, a) \omega(z) n_\varepsilon(t, x - \varepsilon z, a) dz da \\ n_\varepsilon(0, x, a) = n_\varepsilon^0(x, a) = n^0(x/\varepsilon, a). \end{cases} \quad (70)$$

*Remark.* It follows that  $n(t, x, a) = n_\varepsilon(\varepsilon t, \varepsilon x, a)$  satisfies the problem below, with a jump rate that varies slowly in space:

$$\begin{cases} \partial_t n(t, x, a) + \partial_a n(t, x, a) + \beta(\varepsilon x, a) n(t, x, a) = 0, & t \geq 0, \quad a > 0, \quad x \in \mathbb{R} \\ n(t, x, 0) = \int_0^{1+t} \int_{\mathbb{R}} \beta(\varepsilon x', a) \omega(x - x') n(t, x', a) dx' da \\ n(0, x, a) = n^0(x, a). \end{cases} \quad (71)$$

Since  $\mu$  is Lipschitz continuous,

$$\beta(x - \varepsilon z, a) = \frac{\mu(x) + \mathcal{O}(\varepsilon z)}{1+a}. \quad (72)$$

The formulation of (70) along characteristic lines allows us to recover, for  $\psi_\varepsilon$  and  $\phi_\varepsilon^0$  defined as in (8),

$$\begin{aligned} 1 &= \int_0^{t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(x - \varepsilon z, a) \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \psi_\varepsilon(t - \varepsilon a, x - \varepsilon z)] \right) dz da \\ &+ \int_{t/\varepsilon}^{1+t/\varepsilon} \int_{\mathbb{R}} \omega(z) \Phi(x - \varepsilon z, a) \exp \left( \frac{1}{\varepsilon} [\psi_\varepsilon(t, x) - \phi_\varepsilon^0(x - \varepsilon z, a - t/\varepsilon)] + \int_0^{a-t/\varepsilon} \beta(x, s) ds \right) dz da, \end{aligned} \quad (73)$$

where

$$\Phi(x, a) = \beta(x, a) \exp \left( - \int_0^a \beta(x, s) ds \right). \quad (74)$$

Thanks to (72) and since  $\omega$  is a Gaussian, it follows that (73) admits a formal limiting Hamilton-Jacobi equation, similar to the space-independent case (10). Here however, the Hamiltonian depends on space:

$$1 = \int_0^\infty \Phi(x, a) \exp(a \partial_t \psi_0(t, x)) da \int_{\mathbb{R}} \omega(z) \exp(z \partial_x \psi_0(t, x)) dz. \quad (75)$$

Yet again, that is a Hamilton-Jacobi equation, since it is equivalent to:

$$\partial_t \psi_0(t, x) + H(x, \partial_x \psi_0)(t, x) = 0, \quad (76)$$

with  $H$  defined as follows, where  $(\hat{\Phi}(x, \cdot))^{-1}$  is the inverse function of the Laplace transform of  $\Phi(x, \cdot)$ :

$$H(x, p) = - \left( \hat{\Phi}(x, \cdot) \right)^{-1} \left( \frac{1}{\int_{\mathbb{R}} \omega(z) \exp(zp) dz} \right). \quad (77)$$

Passing to the limit rigorously is left for further work.

## Acknowledgements

This work was initiated within the framework of the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 639638).

The second author was supported by the ANR project KIBORD. (ANR-13-BS01-0004).

We wish to thank Thomas Lepoutre and Hugues Berry for many fruitful discussions.

## References

- [1] V. A. Volpert A. A. Nepomnyashchy. An exactly solvable model of subdiffusion–reaction front propagation, 2013. doi:10.1088/1751-8113/46/6/065101.
- [2] Guy Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*. 1994.
- [3] Hugues Berry, Thomas Lepoutre, and Álvaro Mateos González. Quantitative convergence towards a self similar profile in an age-structured renewal equation for subdiffusion. *Acta Applicandae Mathematicae*, 2016.
- [4] Émeric Bouin, Vincent Calvez, Emmanuel Grenier, and Grégoire Nadin. Large deviations for velocity-jump processes and non-local hamilton-jacobi equations. arXiv:1607.03676v1, 2016.
- [5] I. Bronstein, Y. Israel, E. Kepten, S. Mai, Y. Shav-Tal, E. Barkai, and Y. Garini. Transient Anomalous Diffusion of Telomeres in the Nucleus of Mammalian Cells. *Physical Review Letters*, 103(018102):1–4, 2009.
- [6] Carmine Di Rienzo, Vincenzo Piazza, Enrico Gratton, Fabio Beltram, and Francesco Cardarelli. Probing short-range protein brownian motion in the cytoplasm of living cells. *Nat Commun*, 5:5891, 2014.
- [7] L.C. Evans and P.E. Souganidis. A pde approach to geometric optics for certain semilinear parabolic equations. *Indiana University Mathematics Journal*, (Vol. 38, No. 1), 1989.
- [8] Sergei Fedotov and Steven Falconer. Subdiffusive master equation with space-dependent anomalous exponent and structural instability. *Physical Review E*, 85(031132):1–6, 2012.
- [9] Sergei Fedotov and Steven Falconer. Nonlinear degradation-enhanced transport of morphogens performing subdiffusion. *Phys. Rev. E*, 89:012107, Jan 2014.
- [10] M. I. Freidlin. Geometric optics approach to reaction-diffusion equations. *SIAM Journal on Applied Mathematics*, (Vol. 46, No. 2), 1986.
- [11] D. Froemberg, H. Schmidt-Martens, I. M. Sokolov, and F. Sagués. Front propagation in  $A + B \rightarrow 2A$  reaction under subdiffusion. *Phys. Rev. E*, 78:011128, Jul 2008.
- [12] D. Froemberg, H. H. Schmidt-Martens, I. M. Sokolov, and F. Sagués. Asymptotic front behavior in an  $a + b \rightarrow 2a$  reaction under subdiffusion. *Phys. Rev. E*, 83:031101, Mar 2011.
- [13] Ido Golding and Edward C. Cox. Physical Nature of Bacterial Cytoplasm. *Physical Review Letters*, 96(098102):1–4, 2006.
- [14] B. I. Henry, T A M. Langlelands, and S. L. Wearne. Anomalous diffusion with linear reaction dynamics: from continuous time random walks to fractional reaction-diffusion equations. *Phys Rev E Stat Nonlin Soft Matter Phys*, 74(3 Pt 1):031116, Sep 2006.
- [15] Felix Höfling and Thomas Franosch. Anomalous transport in the crowded world of biological cells. *Reports on Progress in Physics*, 76(4):046602, 2013.
- [16] Izeddin I, Récamier V, Bosanac L, Cisse II, Boudarene L, Dugast-Darzacq C, Proux F, Bénichou O, Voituriez R, Bensaude O, Dahan M, Darzacq X. Single-molecule tracking in live cells reveals distinct target-search strategies of transcription factors in the nucleus. *eLife*, Jun 2014.
- [17] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, 339(1):1 – 77, 2000.

- [18] Andreas Hellander Per Lötstedt Émilie Blanc, Stefan Engblom. Mesoscopic modeling of stochastic reaction-diffusion kinetics in the subdiffusive regime. arXiv:1503.07009v1, 2015.
- [19] Sylvie Méléard and Sepideh Mirrahimi. Singular limits for reaction-diffusion equations with fractional laplacian and local or nonlocal nonlinearity. *Communications in Partial Differential Equations*, 40(5):957–993, 2015.
- [20] Vicenç Méndez, Sergei Fedotov, and Werner Horsthemke. *Reaction-Transport Systems*. 2010.
- [21] E.W. Montroll and G.H. Weiss. Random walks on lattices. ii. *J. Math. Phys.*, 6:167–181, 1965.
- [22] A. A. Nepomnyashchy. Mathematical modelling of subdiffusion-reaction systems. *Math. Model. Nat. Phenom.*, 11(1):26–36, 2013.
- [23] Bradley R. Parry, Ivan V. Surovtsev, Matthew T. Cabeen, Corey S. O’Hern, Eric R. Dufresne, and Christine Jacobs-Wagner. The bacterial cytoplasm has glass-like properties and is fluidized by metabolic activity. *Cell*, 156(1-2):183–194, Jan 2014.
- [24] Benoît Perthame. *Équations de transport non linéaires et systèmes hyperboliques*. Cours DEA Analyse Numérique Paris 6.
- [25] Benoît Perthame. *Transport Equations in Biology*. Frontiers in Mathematics. Birkäuser, 2007.
- [26] Jan Kochanowski Katarzyna D. Lewandowska Tomasz Klinkosz Tadeusz Kosztołowicz, Mateusz Piwnik. The solution to subdiffusion-reaction equation for the system with one mobile and one static reactant. *Acta Physica Polonica B*, (5), 2013.
- [27] Paul Vigneaux. *Méthodes Level Set pour des problèmes d’interface en microfluidique*. PhD thesis, Université Bordeaux I, 2007.
- [28] Nepomnyashchy AA. Volpert VA, Nec Y. Fronts in anomalous diffusion–reaction systems. *Phil Trans R Soc A*, 2013. <http://dx.doi.org/10.1098/rsta.2012.0179>.
- [29] Santos Bravo Yuste, Katja Lindenberg, and Juan Jesus Ruiz-Lorenzo. *Anomalous Transport*, chapter Subdiffusion-Limited Reactions, pages 367–395. Wiley-VCH Verlag GmbH & Co. KGaA, 2008.