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ON THE KREIN-RUTMAN THEOREM AND BEYOND

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Contents

1. Introduction

1.1. **Framework and the main result.** In this work, we revisit the Krein-Rutman theory for semigroups of positive operators in a Banach lattice framework and we provide some very general, efficient and handy results with constructive estimates about
- the existence of a solution to the first eigentriplet problem;
- the geometry of the principal eigenvalue problem;
- the asymptotic stability of the first eigenvector with possible constructive rate of convergence.

This abstract theory is motivated and illustrated by several examples of differential, integro-differential and integral operators. In particular, we revisit the first eigenvalue problem and the asymptotic stability of the first eigenvector for
- some parabolic equations in a bounded domain and in the whole space;
- some transport equations in a bounded or unbounded domain, including some growth-fragmentation models and some kinetic models;
- the kinetic Fokker-Planck equation in bounded domain;
- some mutation-selection models.

The results we establish on these examples are more general and more accurate that what we can find in the literature. Our approach is in the same time able to tackle some critical cases, but also it is very natural and makes possible to bring out the main important properties for each example and to get rid of many technical issues.

The present work is motivated by new problems and ideas presented in the lectures on the Krein-Rutman theorem by P.-L. Lions at Collège de France [252] and by the recent contributions by Bansaye et al [35] and by Cañizo and Mischler [81] developing Harris techniques. Bringing and developing these ideas and techniques together with the more classical *spectral analysis* approach developed or synthesized in previous contributions by Krein and Rutman [238], by Arendt et al [15], by Mischler and Scher [278], by Bátkai et al [41] and many others, we are then able to significantly generalize and improve the Krein-Rutman theory for positive semigroups.

The abstract results are developed in the framework of a quite general Banach lattice X, that is a Banach space $(X, \|\cdot\|)$ endowed with a compatible order relation $\geq$ and thus with associated positive cone $X_+ := \{f \in X; \, f \geq 0\}$, which satisfies either $X = Y'$ or $X' = Y$ for another dual Banach lattice Y. The precise (and standard) framework will be presented in Section 2.1, and some additional properties will be added when needed (these ones always hold in usual Banach lattices used in PDE and stochastic processes theory). On the other hand, all the applications we will presented are made in the following examples of usual Banach lattices :

• $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm, or $X := C_{0,m}(E)$ its weighted variant;

• $X := L^p(E) = L^p(E, \mathscr{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ-algebra $\mathscr{E}$, a positive σ-finite measure μ and an exponent $p \in [1, \infty]$, or $X := L^p_m(E)$ its weighted variant;

• $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$, or $X := M^1_m(E)$ its weighted variant.

In all the above examples, E denotes a σ-compact metric space, and we write $E = \cup E_R$, with $E_R \subset E_{R+1}$, E_R compact.

We next consider a positive one-parameter semigroup of operators $S = S_\mathcal{L}$ on X (we will indifferently writes $S_t = S(t) = S_\mathcal{L}(t)$ for $t \geq 0$), and we denote by $\mathcal{L}$ its generator, by $D(\mathcal{L}) \subset X$ the domain of $\mathcal{L}$, by $\rho(\mathcal{L}) \subset \mathbb{C}$ the resolvent set of $\mathcal{L}$ and by $\Sigma(\mathcal{L}) = \mathbb{C}\backslash\rho(\mathcal{L})$ the spectrum of $\mathcal{L}$. We also denote by S^* and $\mathcal{L}^*$ the corresponding semigroup and generator on the dual space Y, and we refer to Section 2.1 for more notations.

As announced, we may split the issue into several pieces concerning the stationary and the evolution associated problems.

• **Existence.** We are first interested in the existence part of the first or principal eigentriplet problem, namely we wish to bring out very general conditions under which

(S1) there exists a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$ to the eigentriplet problem

$$\mathcal{L} f_1 = \lambda_1 f_1, \quad f_1 \geq 0, \quad f_1 \neq 0, \tag{1.1}$$

$$\mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \quad \phi_1 \geq 0, \quad \phi_1 \neq 0, \tag{1.2}$$

and furthermore λ_1 coincides with the spectral bound, namely

$$\lambda_1 = s(\mathcal{L}) := \sup\{\Re e\lambda; \lambda \in \Sigma(\mathcal{L})\} = \inf\{\kappa \in \mathbb{R};\ \Delta_\kappa \subset \rho(\mathcal{L})\}, \tag{1.3}$$

where Δ_α is the open half plan $\Delta_\alpha := \{z \in \mathbb{C};\ \Re ez > \alpha\}$.

We emphasize on the fact that this problem is named as the principal eigenvalue problem because

$$\lambda_1 \in \Sigma(\mathcal{L}) \subset \{z \in \mathbb{C},\ \Re e(z) \leq \lambda_1\}.$$

• **Geometry.** A second issue is about an accurate analysis of the principal eigentriplet solution and of the geometry of the (principal part of the) spectrum.

On the one hand, concerning the eigentriplet solution, we investigate conditions such that

(S2) f_1 is strictly positive (we refer to Section 4.1 for a definition) and f_1 is the unique (up to normalization) positive eigenvector for $\mathcal{L}$, ϕ_1 is strictly positive and ϕ_1 is the unique (up to normalization) positive eigenvector for $\mathcal{L}^*$, and finally λ_1 is geometrically and algebraically simple for both $\mathcal{L}$ and $\mathcal{L}^*$. We then may make the (usual) normalization choice

$$\big(\|f_1\| = 1,\ \langle f_1, \phi_1\rangle = 1\big) \ \text{ or } \ \big(\|\phi_1\| = 1,\ \langle f_1, \phi_1\rangle = 1\big). \tag{1.4}$$

We are next interested by describing the boundary point spectrum

$$\Sigma^+_P(\mathcal{L}) := \Sigma_P(\mathcal{L}) \cap \Sigma_+(\mathcal{L}),$$

where we define the boundary spectrum $\Sigma_+(\mathcal{L}) := s(\mathcal{L}) + i\mathbb{R}$ and $\Sigma_P(\mathcal{L})$ as the point spectrum (or set of eigenvalues). More precisely, we exhibit some conditions such that

(S3$_1$) $\Sigma^+_P(\mathcal{L}) - \lambda_1$ is a (discrete) additive subgroup of $i\mathbb{R}$;

(S3$_2$) $\Sigma^+_P(\mathcal{L})$ is trivial, namely

$$\Sigma^+_P(\mathcal{L}) = \{\lambda_1\}; \tag{1.5}$$

or

(S3$_3$) $\Sigma^+_P(\mathcal{L})$ is trivial and $\Sigma(\mathcal{L})$ enjoys a spectral gap property (on its principal part), namely

$$\exists\, \kappa < \lambda_1; \quad \Sigma(\mathcal{L}) \cap \Delta_\kappa = \{\lambda_1\}. \tag{1.6}$$

In the last situation (1.6), a band separates the spectral value λ_1 to the remainder of the spectrum, while there is no spectral gap when (1.5) holds but (1.6) does not.

The importance of such a eigentriplet comes from the fact that we may associate the Malthusian function

$$F_1(t) := e^{\lambda_1 t} f_1,$$

which is a particular solution to the evolution equation (with maximal growth) and a natural candidate to capture the main asymptotic feature of generic semigroup flow.

• **Asymptotic stability.** In order to formulate our third main issue, namely the asymptotic stability of F_1, we introduce the rescaled operators $\widetilde{\mathcal{L}} = \mathcal{L} - \lambda_1$ and $\widetilde{\mathcal{L}}^* = \mathcal{L}^* - \lambda_1$, so that

$$\widetilde{\mathcal{L}} f_1 = 0, \quad \widetilde{\mathcal{L}}^* \phi_1 = 0,$$

or in other words, f_1 is a stationary state of the semigroup $\widetilde{S} = S_{\widetilde{\mathcal{L}}}$ and ϕ_1 is a stationary state of the semigroup $\widetilde{S}^* = S_{\widetilde{\mathcal{L}}^*}$, and thus a conservation law for $\widetilde{S}$:

$$\widetilde{S}(t) f_1 = f_1, \quad \widetilde{S}^*(t)\phi_1 = \phi_1, \quad \langle S(t) f, \phi_1 \rangle = \langle f, \phi_1 \rangle,$$

for any $t \geq 0$ and any $f \in X$. Because of the property of the eigentriplet and of the normalization assumption (1.4), we may reduce the issue to considering the case $f \in X$ satisfies $\langle \phi_1, f \rangle = 0$ when **(S3$_2$)** or **(S3$_3$)** holds and more generally $f \in Y_0^\perp$ when **(S3$_1$)** holds, where Y_0 stands for the eigenspace associated to the eigenvalues belonging to $\Sigma_P^+(\mathcal{L})$. Depending of the hypotheses we made on $\mathcal{L}$ and $S_{\mathcal{L}}$, we are able to establish some

(E1) mean ergodic property, namely

$$\frac{1}{T} \int_0^T \widetilde{S}_t f dt \to 0 \quad \text{as} \quad T \to \infty;$$

(E2) ergodic property, namely

$$\widetilde{S}_t f \to 0 \quad \text{as} \quad t \to \infty;$$

(E3) quantitative asymptotic stability, which may be geometric (or exponential) in the spectral gap (1.6) case, namely

$$\textbf{(E3}_1\textbf{)} \ \ \|\widetilde{S}(t) f\| \leq C\, e^{-\varepsilon t} \|f\|, \quad \forall\, t \geq 0, \ \forall\, f \in X, \ \langle f, \phi_1 \rangle = 0,$$

for possible constructive constants $\varepsilon > 0$ and $C \geq 1$, or under the weaker condition (1.5) only subgeometric, namely

$$\textbf{(E3}_2\textbf{)} \ \ \|\widetilde{S}(t) f\|_1 \leq \Theta(t) \|f\|_2, \quad \forall\, t \geq 0, \ \forall\, f \in X, \ \langle f, \phi_1 \rangle = 0,$$

where $\| \cdot \|_2 = \| \cdot \|_X$, $\| \cdot \|_1$ is a weaker norm and $\Theta : \mathbb{R}_+ \to \mathbb{R}_+$ is a constructive decay function satisfying $\Theta(t) \searrow 0$ when $t \nearrow \infty$.

We aim now to allude some general hypotheses on the semigroup $S_{\mathcal{L}}$ or its generator $\mathcal{L}$ such that the above three main issues may be tackled. Additionally to the yet mentioned fact that $S_{\mathcal{L}}$ is positive (which is almost equivalent to the fact that its resolvent is a positive operator, that $\mathcal{L}$ enjoys a weak maximum principle or that $\mathcal{L}$ enjoys Kato's inequality) our hypotheses are mainly of two kinds :
- strict positivity conditions;
- regularity conditions;

and these ones may be formulated at the stationary level directly on the generator $\mathcal{L}$ or its resolvent $\mathcal{R}_{\mathcal{L}}$ or they may be formulated at the evolution level on the semigroup of operators $S_{\mathcal{L}}$. Of course, in order to establish constructive results these hypotheses will have to be formulated in a quantitative way.

The strict positivity we will introduce and use are of different kinds:
- strong maximum principle on the generator, or equivalently irreducibility of the semigroup;
- reverse Kato's inequality for the generator or aperiodicity condition of the semigroup;
- Doblin-Harris condition on the semigroup, which may be formulated as

$$S_T f \geq g_0 \langle \psi_0, f \rangle, \quad \forall\, f \in X_+, \tag{1.7}$$

for some for some $T > 0$ and convenient $g_0 \in X_+\backslash\{0\}$, $\psi_0 \in Y_+\backslash\{0\}$.
Less systematically but in a crucial way, we will make use of somehow related
- barrier functions and positive subeigenfunctions, which for the last one typically writes

$$\exists\, \kappa_0 \in \mathbb{R},\ \exists\, \phi_0 \in Y_+\backslash\{0\}, \quad \mathcal{L}^*\phi_0 \ge \kappa_0\phi_0. \tag{1.8}$$

On the other hand, some regularity is needed on the dominant part of the semigroup. In order to briefly explain the issue, we assume that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{B}$ is the generator of a positive semigroup $S_\mathcal{B}$. In such a context, we may write the resolvent factorization identity

$$\mathcal{R}_\mathcal{L} = \mathcal{R}_\mathcal{B} + \mathcal{R}_\mathcal{B}\mathcal{A}\mathcal{R}_\mathcal{L}$$

on the resolvent $\mathcal{R}_\mathcal{L}$ of $\mathcal{L}$ and $\mathcal{R}_\mathcal{B}$ of $\mathcal{B}$, and its iterated version

$$\mathcal{R}_\mathcal{L} = \mathcal{V} + \mathcal{W}\mathcal{R}_\mathcal{L}, \quad \mathcal{V} := \mathcal{R}_\mathcal{B} + \dots + \mathcal{R}_\mathcal{B}(\mathcal{A}\mathcal{R}_\mathcal{B})^{N-1}, \quad \mathcal{W} := (\mathcal{R}_\mathcal{B}\mathcal{A})^N. \tag{1.9}$$

At the level of the generator, our regularity assumption then typically writes

$$\sup_{z\in\Delta_\kappa} \|\mathcal{V}(z)\|_{\mathscr{B}(X)} < \infty, \quad \sup_{z\in\Delta_\kappa} \|\mathcal{W}(z)\|_{\mathscr{B}(X,X_1)} < \infty, \tag{1.10}$$

for some $\kappa \in \mathbb{R}$ and $X_1 \subset X$, which is nothing but the classical Voigt's power compact condition when $X_1 \subset X$ with compact embedding. Similarly, at the level of the semigroup, we may write the associated Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + (S_\mathcal{B}\mathcal{A}) * S_\mathcal{L},$$

(we refer to Section 3.1 for a precise definition) and its iterated version

$$S_\mathcal{L} = V + W * S_\mathcal{L}, \quad V := \sum_{\ell=0}^{N-1} S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)}, \quad W := (S_\mathcal{B}\mathcal{A})^{(*N)}, \tag{1.11}$$

with $N \ge 1$. At the level of the semigroup, our regularity assumption then typically writes

$$\sup_{t\ge 0} \|V(t)e^{-\kappa t}\|_{\mathscr{B}(X)} < \infty, \quad \sup_{t\ge 0} \|W(t)e^{-\kappa t}\|_{\mathscr{B}(X,X_1)} < \infty, \tag{1.12}$$

for some $\kappa \in \mathbb{R}$ and $X_1 \subset X$ in the dissipative framework and a variant of these estimates in a weak dissipative framework. The crucial information is $\kappa < \kappa_0$ (dissipative framework) or $\kappa = \kappa_0$ (more involved weak dissipative framework).
We are now in position to state in a very informal way our main result at the level of the abstract Banach lattice framework.

Theorem 1.1 (rough version)**.** *Let us consider a Banach lattice X picked up in the examples listed above and a positive semigroup $S_\mathcal{L}$ on X which enjoys the above splitting structure* (1.9), (1.10), (1.11), (1.12).

(1) Conclusion **(S1)** *holds under the localization of the principal spectrum assumption $\kappa < \kappa_0$ and a weak compactness assumption on the regular part $\mathcal{W}$ or W in the splitting.*

(2) Under an additional strong maximum principle the conclusion **(S2)** *holds. When $X \subset L^1_{\rm loc}$, we additionally conclude that* **(S3$_1$)** *and* **(E1)** *hold.*

In order to make one step further, we have the three next possibilities

(3) Under an additional inverse Kato's condition or an aperiodicity property, the conclusion **(S3$_2$)** *holds, as well as* **(E2)** *when $X \subset L^1_{\rm loc}$.*

(4) Alternatively, under an additional strong compactness assumption on the regular part W of the semigroup, the quantitative exponential asymptotic stability **(E3$_1$)** *holds without constructive constants, and thus also the spectral gap conclusion* **(S3$_3$)** *holds (in a not constructive way).*

(5) Alternatively, under the additional Doblin-Harris condition (1.7) *and an appropriate regularity estimate on the regular part of the splitting W, the quantitative asymptotic stability* **(E3)** *holds for both the geometric and subgeometric framework with now constructive constants.*

More general and precise statements will be presented in Sections 2, 3, 4, 5 and 6, where in particular some variants in a weak dissipative framework ($\kappa = \kappa_0$) will be presented. It is worth emphasizing that the assumptions in (4) and (5) may be optimal in the sense that reciprocal implications are likely to be true. We do not follow that line of investigation but rather refer to [278, 35] where such kind of results are established.

1.2. **Discussion about Theorem 1.1.** We discuss several works related to the main Theorem 1.1 as well as the hypotheses and the techniques used during the proof.

1.2.1. **The Krein-Rutman work and related approaches.** For a strictly positive matrix in a finite dimensional space, Perron [311] and Frobenius [167] establish at the beginning of the 20th Century that the eigenvalue with largest real part is unique, real and simple. In their pioneer work, Krein and Rutman establish in [238] for the very first time possible infinite dimensional functional space versions of the Perron-Frobenius theorem.

Theorem 1.2 (Krein-Rutman). *Consider a Banach lattice with positive cone X_+ and strictly positive cone $X_{++} := \mathrm{int} X_+ \neq \emptyset$. Consider a linear and compact operator $\mathcal{R} : X \to X$ such that $\mathcal{R} : X_+ \to X_+$ and $\mathcal{R} : X_+ \backslash \{0\} \to X_{++}$. Then there exists a unique eigentriplet (μ_1, f_1, ϕ_1) such that $\mu_1 > 0$, $f_1 \in X_{++}$, $f_1 = \mu_1 \mathcal{R} f_1$, $\phi_1 \in X'_{++}$, $\phi_1 = \mu_1 \mathcal{R}^* \phi_1$.*

The non-emptiness of X_{++} and the strict positivity assumption $\mathcal{R} : X_+ \backslash \{0\} \to X_{++}$ can be relaxed, to the price of loosing the uniqueness and strict positivity properties of the eigenvectors. For a bounded operator $\mathcal{R}$ on X, we denote by $r(\mathcal{R})$ the spectral radius

$$r(\mathcal{R}) := \sup\{|\lambda|;\ \lambda \in \Sigma(\mathcal{R})\} \leq \|\mathcal{R}\|.$$

Theorem 1.3 (Krein-Rutman). *Consider a Banach lattice with positive cone X_+ and a linear and compact operator $\mathcal{R} : X \to X$ such that $\mathcal{R} : X_+ \to X_+$ and $r(\mathcal{R}) > 0$. Then there exists an eigentriplet (μ_1, f_1, ϕ_1) with $\mu_1 = r(\mathcal{R})$, $f_1 \in X_+ \setminus \{0\}$, $f_1 = \mu_1 \mathcal{R} f_1$, $\phi_1 \in X'_+ \setminus \{0\}$, $\phi_1 = \mu_1 \mathcal{R}^* \phi_1$.*

In Theorems 1.2 and 1.3, the operator $\mathcal{R}$ corresponds to a resolvent operator $\mathcal{R} := (\kappa - \mathcal{L})^{-1}$ for $\kappa > 0$ large enough, so that when it applies, we deduce in particular that the first eigenvalue problem (1.1)-(1.2) has a solution with $\lambda_1 = \kappa - \mu_1$. The two conditions $\mathrm{int} X_+ \neq \emptyset$ and $\mathcal{R} : X_+ \backslash \{0\} \to X_{++}$ are very strong. The first one essentially imposes to work in the space of continuous function and the second one to work in a bounded domain. The result is however suitable and directly applicable (and somehow restricted) to an elliptic operator with smooth coefficients set in a bounded domain with suitable boundary conditions or to a Fredholm integral operator with positive kernel also set in a bounded domain. In the elliptic context, the property $\mathcal{R} : X_+ \backslash \{0\} \to X_{++}$ is nothing but the strong maximum principle while the compactness property of $\mathcal{R}$ comes from the elliptic regularity. We refer to Section 2.3 for further discussions. The weaker condition $r(\mathcal{R}) > 0$ is less restrictive and is in particular always satisfied for irreducible operators, by virtue of de Pagter's theorem [128]. In the same framework, Theorems 1.2 and 1.3 have been next slightly extended by Bonsall [65], Schaefer [336], Karlin [228] or Nussbaum [300] for instance. We also refer to the book by Dautray and Lions [125] for a clear and comprehensible presentation and several possible versions.
In his paper [58], G. Birkhoff derived the Perron-Frobenius theorem by proving a contraction principle in Hilbert's projective metric for positive matrices. His result actually applies to any "uniformly positive bounded" linear operators of a Banach lattice, such as integral operators with positive kernels, and also provides geometric stability estimates. A closely related result was proved by E. Hopf [216], and this Birkhoff-Hopf contraction theorem was subsequently generalized and sharpened, and its proof simplified, by several authors, see in particular [42, 73, 78, 155, 156, 233, 301, 305]. This approach of the Krein-Rutman theorem requires some "uniform positivity and boundedness" of the operator, which is quite restrictive, but it nevertheless allows to recover, through an approximation procedure, the standard result of Theorem 1.2, see [73, Thm. 6.18]. The contraction in Hilbert's projective metric has the advantage to be applicable in partially order linear vector spaces without any topological structure [156], and to nonlinear maps [301].

1.2.2. **Spectral analysis approach.** In his paper [315], R.S. Phillips formalized the notion of *positive semigroup* acting on a Banach lattice paving the way to a new field of research. In the precursory work [351] by Vidav and next in a series of papers by Greiner and co-authors [187, 189, 15], Webb [359, 360] and Bürger [75] (see also [15, C-III, Cor. 2.12, Thm. 3.12], [152, Thm. VI.1.12, Cor. VI.1.13] or more recently Theorem 14.17 in the very pedagogical book [41]) significant generalizations of the Krein-Rutman theory were established leading to, roughly speaking, the following result.

Theorem 1.4. *Consider a positive semigroup $S_\mathcal{L}$ on a (suitable) Banach lattice X which is irreducible and such that $s(\mathcal{L}) > -\infty$ is a pole, then*

• *$s(\mathcal{L})$ is a first-order pole with one-dimensional and strictly positive residue, so that in particular there exists a solution (λ_1, f_1, ϕ_1) to the eigentriplet problem;*
• *There exists $\alpha \in \mathbb{R}$ such that $\Sigma_+(\mathcal{L}) = \{s(\mathcal{L}) + i\alpha\mathbb{Z}\}$ consists of first-order poles with one-dimensional residue.*
• *A practical way for verifying that $s(\mathcal{L}) > -\infty$ is a pole consists in assuming that $\mathcal{L}$ enjoys the splitting structure $\mathcal{L} = \mathcal{A} + \mathcal{B}$, as described above, with $s(\mathcal{B}) < s(\mathcal{L})$ and $\mathcal{A}$ is $\mathcal{B}$ power compact, that is to say $\mathcal{W}$ is compact, on $\Delta_{s(\mathcal{B})}$.*
Assuming furthermore that $\widetilde{S}_\mathcal{L}$ is quasi-compact then
• *$\widetilde{S}_\mathcal{L}$ is exponential asymptotically stable in* $\mathrm{Span}\{\phi_1\}^\perp$ *(without constructive constants).*

The most important improvements here are the fact that the condition $\mathrm{int}X_+ \neq \emptyset$ and the strong compactness of the resolvent operator $\mathcal{R}_\mathcal{L}$ are removed, and also that the exponential asymptotically stability is established. The hypotheses seem stronger to those stated in Theorem 1.1-(1), where only weak compactness is required what is not the case here. It is however worth emphasizing that in an AL-space and an AM-space (what includes the examples $C_0(E)$ and $L^1(E)$) a power weak compactness implies a power strong compactness (see [75, Rk. 2.1] and [337, Cor. 1 of Thm. II.9.9]). The hypotheses and conclusions are similar to those stated in Theorem 1.1-(4). The proof is based on the one hand on the Banach lattices theory as formalized for instance by Schaefer [337] (see also [11, 12, 13, 15] for significant developments) using notions as ideals and quasi-interior points. On the other hand, it takes advantage on the perturbation techniques initiated by Phillips in [314] and developed further by Jörgens [225], Vidav [331, 352] and Voigt [355] leading to the notions of power compact resolvent and quasi-compact semigroup, essential spectrum and Calkin algebra.
The above theorem in particular applies to a positive and irreducible semigroup which is eventually norm continuous and its generator has compact resolvent (see for instance Corollary VI.1.13 in [152] and for the definition of an eventually norm continuous). In that case indeed, one can show that $s(\mathcal{L}) > -\infty$, $\Sigma_+(\mathcal{L})$ is bounded and consists of poles, so that $\Sigma_+(\mathcal{L}) = \{s(\mathcal{L})\}$ and the essential growth bound $\omega_{\mathrm{ess}}(S)$ associated to the essential spectrum (see for instance [41, Sec 14.1] for a definition) satisfies $\omega_{\mathrm{ess}}(S) < \omega(S) = s(\mathcal{L})$. The theorem was motivated and successfully applied to Boltzmann like transport operator [351], cell division operator [134], age structured equation [360] and selection-mutation dynamics [75]. We also refer to [152, Ch. VI] and [41] for other numerous applications. Although very general and quite efficient, we formulate several criticisms about the above result.
- The exponential convergence result is definitively not constructive and that approach is not able to say anything about the weak dissipative case (a framework we will introduce latter, see in particular Section 3.3).
- We may observe that Theorem 1.4 is not so popular in the probability and the PDE communities and still many works in these domains refer to the original Krein and Rutman theorem even when some additional (approximation) arguments are needed rather than applying directly Theorem 1.4. By the way, we did not find in the literature where Theorem 1.4 is stated in such an handy way (the closer formulation is probably [152, Thm. VI.1.12] which is given without proof).
- The proof of Theorem 1.4 that we may find in the above quoted references is written in a very specific and abstract language which make it quite obscure.

In [278, 273], one of the authors proposes the following variant.

Theorem 1.5. *Consider a positive semigroup $S_\mathcal{L}$ which satisfies* (1.8) *with $\kappa_0 \in \mathbb{R}$, it is irreducible and its generator enjoys the splitting structure* (1.9)-(1.10) *for some $\kappa < \kappa_0$ and $X_1 \subset X$ with compact imbedding. Assuming furthermore that*

$$\exists\, \alpha > 0, \quad \sup_{z \in \Delta_\kappa} \langle z \rangle^\alpha \|\mathcal{W}(z)\|_{\mathscr{B}(X)} < \infty, \tag{1.13}$$

the quantitative exponential asymptotic stability $(\mathbf{E3}_1)$ *holds (without constructive constants).*

The proof of Theorem 1.5 is based on a partial (but principal) spectral mapping and Weyl's theorem (in the spirit of Voigt [355]) coupled with a simple analysis of the first eigenelement problem based on the irreducibility of the semigroup, but which is really simpler than the deep result on irreducible semigroup stated in Theorem 1.4. On the other hand, that approach is unable to tackle the situation when $\Sigma_P^+(\mathcal{L})$ is not a singleton. One of the main features in Theorem 1.5

and the other results established in [278, 273] is the clear identification of the simple localisation of the principal spectrum condition with (1.8).

1.2.3. **Dynamical and probabilistic approach.**
It is well known from the mean ergodicity theory of Von Neumann and Birkhoff introduced in the 1930s in [356, 60] that for a bounded semigroup a possible stationary state (and thus a first eigenvector associated to the first eigenvalue $\lambda_1 = 0$) can be obtain through a dynamical approach by establishing that the Cesàro mean of the semigroup appropriately converges. A classical reference is [239], see also [152, Sec. V.4] for a short presentation.

The existence of invariant measures for Markov chains/processes can be derived through a contraction approach by using coupling arguments reminiscent from the ideas of Doeblin [140] and Harris [202]. This yields a simplified Krein-Rutman theorem in the Banach lattice of finite measures for Markov operators, providing the existence of f_1 whilst $\lambda_1 = 0$ and $\phi_1 = 1$ are known by definition. Doeblin's condition is a handy criterion which ensures contraction in total variation norm, and hence existence, uniqueness, and geometric stability of the invariant measure, see for instance [168, 81] for this very classical and easy result. It turns out that this contraction is related to the contraction in Hilbert's metric, see [174]. The drawback of Doeblin's condition is that it is quite demanding and typically requires the state space to be bounded. Harris's idea allows an extension to the unbounded setting by localizing Doeblin's condition in a "small set" which is visited infinitely often. The return to small sets is often obtained by using a Lyapunov function. When the Lyapunov function is strong enough for ensuring exponential return, contraction in weighted total variation norm can be established and geometric stability of the invariant measure is inferred [265, 266, 267, 268, 198, 81], leading to the following result (which is made constructive in the two last references).

Theorem 1.6. *Consider a positive semigroup S on the Banach space $X = M^1_m(E)$ for some weight function $m : E \to [1, \infty)$. Suppose that S is conservative, in the sense that*
(1) $S^*_t \mathbf{1} = \mathbf{1}$ for all $t \ge 0$,
and assume that, for some subset $K \subset E$ on which m is bounded and some time $T > 0$,
(2) $S^*_T m \le \alpha m + \theta \mathbf{1}_K$, for some $\alpha \in (0, 1)$ and $\theta > 0$;
(3) $S_T f \ge \langle f, \mathbf{1}_K \rangle g_0$, for all $f \in X_+$ and some $g_0 \in X_+$ such that $\langle g_0, \mathbf{1}_K \rangle > 0$.
Then there exists a unique probability measure $f_1 \in M^1_m$ such that $(\lambda_1 = 0, f_1, \phi_1 = \mathbf{1})$ is solution to the first eigentriplet problem, and the quantitative exponential stability $(\mathbf{E3}_1)$ *holds with constructive constants. Moreover, some reciprocal implication holds true.*

When only a weak version of the above *Lyapunov condition (2)* is available, an extension of the theory to a *weakly dissipative* framework is possible and has been developed in [347, 145, 144, 197, 81] leading to existence, uniqueness, but only sub-geometric stability of the invariant measure. We also mention that ergodicity properties of Feynman-Kac semigroups were investigated in [130, 131] and [235, 236].

Using a condition proposed in [131, Condition $\mathscr{Z}$], the Doeblin-Harris method was extended to non-conservative semigroups in [35, 103, 105, 104, 109]. In [35] necessary and sufficient conditions for the geometric stability of (λ_1, f_1, ϕ_1) in weighted total variation norm are obtained. To our knowledge, no extension to the above mentioned weakly dissipative setting is available.
The following result is an immediate consequence of [35, Thm. 2.1].

Theorem 1.7. *Consider the same situation as in Theorem 1.6 but relax the conservativeness assumption (1) by the assumption that there exists a function $\phi_0 : E \to (0, \infty)$, bounded from above and below by positive constants on K, such that $\phi_0 \le m$ on E, and satisfying*
(1a) $S^*_T \phi_0 \ge \beta \phi_0$, for some $\beta > 0$;
(1b) $\mathbf{1}_K S^*_t \phi_0 \le C \langle g_0, \mathbf{1}_K S^*_t \phi_0 \rangle$, for all $t \ge 0$ and some $C > 0$;
and replace the condition $\alpha \in (0, 1)$ by $\alpha \in (0, \beta)$ in the assumption (2).
Then, there exists a unique solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem and the quantitative exponential stability $(\mathbf{E3}_1)$ *holds with constructive constants. Moreover, some reciprocal implication holds true.*

Positivity conditions required for the Doeblin-Harris approach are less restrictive than for Birkhoff contraction. Conversely, unlike contraction in Hilbert's metric, Doeblin-Harris method strongly uses the linearity of the operators, and may thus not be easily extendable to nonlinear operator. However, since it is based on contraction arguments, it can be extended to time-inhomogeneous semigroups [34]. Finally, the existence of a first eigenmeasure in a non-conservative setting were established in [111, 112] through a Lyapunov function property, a suitable renormalization and a fixed point argument.

The key point in this approach is that it provides a constructive rate of convergence while its drawback is that it is somehow restricted to a M^1_m (or L^1_m) framework and that some of the conditions (typically (1b) in Theorem 1.7) are not fully intuitive and may be hard to verify in the applications.

1.2.4. **PDE approaches.**

At least as far as the existence issue is concerned, one of the most common way in PDE papers in order to tackle the existence part of the first eigentriplet problem consists in approximating (by regularization of the coefficients, add of a small viscosity, discretization) the eigentriplet problem, then use the most classical Perron-Frobenius Theorem [311, 167] or Krein-Rutman Theorem [238, 125] and next to derive appropriate estimates and pass to the limit through a *"stability argument"*.

Recently, in order to circumvent the above approximation step, a new abstract and general version of the existence part of the Krein-Rutman theory has been developed by Lions in [252] which, as for the early works [254, 253], is also adapted to nonlinear operators and it includes the following statement (in the linear operators framework).

Theorem 1.8. *Consider a Banach lattice with positive cone X_+ and a linear and bounded operator $\mathcal{R} : X \to X$ such that*
(i) $\mathcal{R} : X_+ \to X_+$;
(ii) $\exists\, g_2 \in X_+\backslash\{0\}$, $\exists\, C_2 > 0$ such that $\mathcal{R} g_2 \le C_2 g_2$, and set $K_2 := \{g \in X_+;\ \exists C,\ g \le C g_2\}$;
(iii) $\mu_1 := \sup J < +\infty$, where

$$J := \{\mu \ge 0;\ \exists h \in K_2, h \ge \mu \mathcal{R} h + g_2\};$$

(iv) any sequence (g^n) of almost first eigenvectors is relatively (possibly weakly) compact, where we say that (g^n) is a sequence of almost first eigenvectors if $g^n = \mu^n \mathcal{R} g^n + \varepsilon^n$, (g^n) is bounded, $\mu^n \nearrow \mu_1$ and $\varepsilon^n \to 0$.
Then there exists $f_1 \in K_2$ such that $f_1 = \mu_1 \mathcal{R} f_1$ and $\|f_1\| = 1$.

The statement and proof of Theorem 1.8 somehow generalize the existence part of the Krein-Rutman theorem presented in Theorem 1.4 because the required splitting structure and associated power compactness are replaced by the very natural stability principle *(iv)*. Applications to elliptic operator with strong or critical confinement property in the whole space $\mathbb{R}^d$ setting are also presented in [252].

Let us also mention the huge literature on the characterization of the first eigenvalue by a min-max formula. As explained with more details below, this approach has first been introduced in the Courant-Fischer min-max theorem [160, 114, 115] providing a variational characterization of eigenvalues in an abstract Hilbert setting for self-adjoint elliptic operators. Inspired next by pointwise minmax formula established for simple self-adjoint operators [149, 323, 212] using a technique which goes back to Picard [316], it has been next generalized to non self-adjoint elliptic operators in [324, 47] among others. More recently, the same approach has been generalized to non elliptic operators, see for instance [118] and the references therein.

On the other hand, and beyond the eigentriplet problem, the convergence towards the first eigenfunction may be proved using the *general relative entropy (GRE) method* which has been applied to a large class of evolution PDE in [269] which principle is as follows. Assume that $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times X'$ is a solution to the first eigenvalue problem, that $\lambda_1 = 0$ (a case to which one can always reduces from the general case by a mere change of operator and unknown), that $X, X' \subset L^1_{\rm loc}(\mathcal{O})$ and then define the generalized relative entropy

$$\mathcal{J}(f) := \int_{\mathcal{O}} j(f/f_1)\, f_1\, \phi_1\, dz$$

for any given convex function $j : \mathbb{R} \to \mathbb{R}_+$. For any solution $f(t) \in X$ to the (appropriate) evolution PDE, one may establish (at least formal) the identity

$$\mathcal{J}(f(t)) + \int_0^t \mathcal{D}_{\mathcal{J}}(f(s))\, ds = \mathcal{J}(f(0)), \qquad \forall\, t \ge 0, \tag{1.14}$$

where $\mathcal{D}_{\mathcal{J}} \ge 0$ is the associated generalized dissipation of relative entropy, so that $\mathcal{J}$ is a Lyapunov functional (it is decreasing along the flow associated to the evolution PDE). Under suitable positivity hypothesis, one has $\mathcal{D}_{\mathcal{J}}(f) = 0$ if and only if $f \in \mathrm{Vect}(f_1)$, and then one may deduce from (1.14) and some lower semicontinuity assumption on the operator $\mathcal{D}_{\mathcal{J}}$ that $f(t) \to cf_1$ as $t \to \infty$ (without rate and with $c \in \mathbb{R}$). The GRE method is of course connected to j-divergence in information theory and statistics [121, 122, 74, 262] and to j-entropy in probability and PDE theory [101, 176], where however here it is crucial to identify the associated operator $\mathcal{D}_{\mathcal{J}}$ and that this last one enjoys suitable properties.

1.2.5. **Hypotheses and proof.**

We now briefly discuss the strategy of the proof of Theorem 1.1 and how it is connected to the above material. Additional comments will be made in the corresponding Sections 2 to 6. As already said, the first eigenvalue problem is mainly split into three steps: existence, geometry and asymptotic stability. From a general point of view, our approach is more general than the initial Krein-Rutman theorem as well as less abstract than the usual semigroup school approach. We believe it is more intuitive and handy for the possible applications since it is presented as a series of estimates to be checked and the necessary assumptions are made clearer at each step.

• Concerning the existence of a solution to the first eigentriplet problem, our result improves the previous known results because (1) only weak compactness property is needed (while Theorems 1.4 & 1.5 require strong compactness assumptions), (2) it is more flexible than Theorems 1.4, 1.5, 1.7 & 1.8 (the two first ones being restricted to the generator of a strongly continuous semigroup, the third one being restricted to a M^1_m framework and involving the tricky condition (1b) and the last one being somehow restricted to a weighted L^∞ framework), (3) it applies to weakly dissipative cases (so that no spectral gap is needed). We present two different proofs: one based on a stationary problem approach and another one based on a dynamical problem approach (with which we are able to tackle the weakly dissipative case).

Our stationary problem approach mixes in a first step the (clearly formulated) approximation argument of [41, proof of Thm. 12.15] together with the stability argument of [252], where it is worth emphasizing that the condition $\kappa < \kappa_0$ in Theorem 1.1 is nothing but a practical (and possibly constructive) condition ensuring that assumption $s(\mathcal{B}) < s(\mathcal{L})$ holds in Theorem 1.4. On a second step, we exhibit several practical situations where the required stability condition is fulfilled recovering as a particular case the existence part in Theorems 1.4 & 1.7. We would like to point out here that the splitting hypothesizes (1.11)-(1.12) on the semigroup is a generalization of the Lyapunov condition (2) in Theorem 1.6 on the semigroup which in turn generalizes the classical Lyapunov condition on the generator, namely for instance

$$\mathcal{L}^*\psi_2 \le \kappa\psi_1 + K\psi_0$$

with $\psi_i \in X'$, $\psi_1, \psi_2 \ge \psi_0$ together with $\psi_2 \le \psi_1$ (super Lyapunov condition), $\psi_2 = \psi_1$ (standard Lyapunov condition), $\psi_2 \ge \psi_1$ (weak Lyapunov condition). We refer to [81] and to Sections 2 and 3 for further discussions on that question.

On the other hand, our dynamical approach mixes the splitting method yet alluded above together with some argument picked up from Von Neumann & Birkoff mean ergodic theory in the spirit of but in a more elaborate way than in [81, Sec. 6].

• The proof about the geometry of the principal eigenvalue problem in Theorem 1.1 is a refinement of many arguments already developed in the literature. More precisely, the uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) and the strict positivity of the eigenvectors is established by taking up again in a more general setting some arguments developed in [313, 278, 231]. The subgroup structure of the boundary point spectrum $\Sigma^+_P(\mathcal{L})$ is next established under suitable (but not very restrictive) geometrical properties on the Banach lattice X, these ones being always true for the usual examples we have in mind and thath we have already listed above. The proof mainly mimics the usual proof (as for instance presented in [41, Sec. 14.3]) but it is less abstract and more general.

Especially, the proof does not refer to the notions of ideals, quasi-interior points or Calkin algebra nor uses the Kakutani lattice isomorphism theorem but rather uses the simpler notion of strict positivity (defined by duality) and some convient structural properties of the signum operator. In order to go one step further and to prove the triviality property $\Sigma_P^+(\mathcal{L}) = \{\lambda_1\}$, we propose one quite original approach (which we believe to be new at this level of generality) based on an *inverse Kato's inequality condition* of $\mathcal{L}$ (by refining some arguments picked up from [278, 231]) and some more standard ones based on an *aperiodicity condition* on the semigroup $S_\mathcal{L}$, on a localisation of the point spectrum condition or on a *quasi-compacteness condition* on the semigroup $S_\mathcal{L}$.

• Finally, the proof on the asymptotic stability of the first eigenvector picks up and mixes some spectral analysis, dynamical system, entropy method and Doblin-Haris coupling arguments. On a first step, we mainly rewrite some very classical dynamical system results mixed together with some arguments coming from the General Relative Entropy method in order to get our mean ergodicity and ergodicity results which are really general and very little demanding about the trajectories. We also rewrite the most classical result about the exponential asymptotic stability (without constructive constants) of the first eigenfunction proposing a very simple (and self-contained) proof which does not make any references to abstract notions as Calkin algebra, essential spectrum or essential growth bound. Last, we adapt the Doblin-Harris approach as qualitatively formulated in [198, 81, 35] in order to get the quantitative asymptotic stability of the first eigenfunction with constructive constants.

1.3. **Some examples of applications.**

The abstract Krein-Rutman theory developed in these notes and alluded above have been cooked up in order to answer to the first eigenvalue problem for PDEs. We show its efficiency by applying it to several examples of evolution PDEs. These examples must be thus considered both as a motivation and an illustration of simultaneously developed abstract theory.

1.3.1. *Parabolic equations.* In Part 7, we are interested by parabolic equations in divergence form

$$\partial_t f = \partial_i(a_{ij}\partial_j f) + \partial_i(\beta_i f) + b_j\partial_j f + cf \quad \text{in} \quad (0,\infty)\times\Omega,$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \Omega$, with general conditions on the coefficients a_{ij}, β_i, b_j, c and in both the case of a bounded domain $\Omega \subset \mathbb{R}^d$ (and we then complement the equation with a Dirichlet boundary condition) and the case $\Omega = \mathbb{R}^d$. The importance of parabolic equations for Physics, Chemistry, Biology and Economy modeling is well known and we do not discuss it here. We consider the four following casses.

• For a bounded domain $\Omega \subset \mathbb{R}^d$, we consider a general elliptic operator in divergence form

$$\mathcal{L}f := \partial_i(a_{ij}\partial_j f) + b_i\partial_i f + \partial_i(\beta_i f) + cf, \quad f \in H^1_0(\Omega),$$

under the very general assumption about the regularity of the coefficients $a_{ij} \in L^\infty(\Omega)$, $a_{ij} \ge \nu\delta_{ij}$, for some $\nu > 0$, $b_i, \beta_j \in L^r(\Omega)$, $c \in L^{r/2}(\Omega)$, $r > d$.

• In the case when $\Omega = \mathbb{R}^d$, we focus first our analysis by considering

$$\mathcal{L}f := \Delta f + b\cdot\nabla f + cf, \quad f \in H^1(\mathbb{R}^d),$$

with drift $b \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, potential $c \in L^2_{\text{loc}}(\mathbb{R}^d)$ and a confinement condition that (roughly speaking) we impose through the properties $c \to -\infty$ as $|x| \to \infty$ and b is dominated by c at the infinity. A typical case is given by $c \sim -|x|^\gamma$ and $b \sim x|x|^{\beta-1}$ as $|x| \to \infty$, with $\gamma > \max(0, \beta - 1)$.

• Still in the case when $\Omega = \mathbb{R}^d$, we next consider the similar problem

$$\mathcal{L}f := \Delta f + b\cdot\nabla f + rcf, \quad f \in H^1(\mathbb{R}^d),$$

with now $c \in C_0(\mathbb{R}^d)$, $b \in C_0(\mathbb{R}^d)$ and $r \in \mathbb{R}_+$ a parameter. That hypotheses correspond to a critical confinement case and we further assume that $r > 0$ is large enough.

• In the case when $\Omega = \mathbb{R}^d$ again, we finally consider the elliptic operator

$$\mathcal{L}f := \Delta f + b\cdot\nabla f + cf,$$

with the drift confinement

$$b = \nabla U, \quad U(x) = \frac{1}{\beta}\langle x\rangle^\gamma, \quad \gamma > 0,$$

and with c dominated by b at the infinity. We further assume $c \geq \text{div}b$ when $\gamma \in (0,1]$. It is worth emphasizing that this corresponds to a perturbation of the classical Fokker-Planck operator associated to the potential U.

For each of these operators we are able to complete the existence, geometric and stability program as stated in Theorem 1.1, with constructive estimates on the first eigentriplet solution and more or less explicit rate of convergence to the first eigenfunction. Few suitable additional assumptions on the coefficients and on the regularity of Ω as well as the precise results will be discussed in the corresponding sections.

The first eigenvalue problem in the three first situations has been studied in [252, 8th and 9th courses] which inspired our study and to which we refer for motivations and possible extensions. Since mainly the existence issue is considered in [252], our results supplement the previous analysis by tackling the geometry of the principal spectrum and the exponential asymptotic stability of the first eigenfunction. On the other hand, the fourth situation in the conservative case ($c = \text{div}b$) is very classical and we refer to [27, 28, 190, 274, 231] and the references therein. We believe that the extension to a non conservative case as considered here is new.

Of course, when the operator $\mathcal{L}$ is the Laplace operator or more generally is a self-adjoint elliptic operator, there exists a huge literature about the analysis of its spectrum and in particular about its first eigenvalue problem because among other things this is related to the ground state problem in quantum mechanic. We do not have the precise historical reference where similar results to the ones developed here are established for the first time. We may for instance refer to the contributions by Poincaré [319] and by Courant and Hilbert [114, 115]. We also refer to the textbook [179, Thm 8.38] for the quite general and modern proof which mixes minimisation technique, strong maximum principle and Hilbert structure arguments. It is worth mentioning that in earlier works, the Krein-Rutman theorem has been proved using elementary ODE method when considering the Sturm-Liouville operator (in dimension $d = 1$), see for instance [63]. Still for a self-adjoint elliptic operator, the Courant-Fischer min-max theorem [160, 114] gives a variational characterization of eigenvalues through Rayleigh quotient [329] and the Weyl theorem [362, 363, 288, 312] provides some information about the distribution of the eigenvalues. More specifically, some constructive lower bound on the best constant in Poincaré inequality and thus on the first eigenvalue may be obtain through the Faber-Krahn [157, 237] isoperimetric inequality as presented in [92], see also Polya-Svzego [321, 320] and Payne-Weinberger [307, 308]. Other results on that direction but based on the Lyapunov condition are obtained in [27, 28] and we also refer to [30] and the references therein.

On the other hand, in the case of an elliptic operator which is not self-adjoint the first result on the principal eigenvalue problem seems to be Protter, Weinberger [324, Rk. 2] who consider the case of smooth domain and coefficients (without precise statement about the regularity) and use minmax formula and the Krein-Rutman Theorem 1.2, see also [310]. Next, Chicco [106, 107] establishes the existence, uniqueness and some monotony properties of the first eigenvalue-eigenfunction in the weak solutions framework of Stampacchia [339, 340] with mild regularity assumptions on the coefficients and which corresponds to the framework we will consider here (when we will consider the case of a bounded domain). These work has been followed by several papers by Donsker and Varadhan [143, 142] and next by the famous work of Berestycki, Nirenberg, Varadhan [47] opening a new field of research. These works are mainly based on strong maximum principle technique, see [325]. We also mention the recent works by Champagnat and Villemonais [104, 105] where similar results to ours for smooth enough coefficients are established using a variant of the probabilistic Doblin-Harris argument as already mentioned in Section 1.2.3. We also emphasize that in the conservative case, the long time behavior problem has been widely studied and some constructive estimates has been obtained in [29, 44, 215, 343, 344] by the mean of log-Sobolev inequality, in [27, 28, 332, 231] by the mean of Poincaré inequality and in [190, 231] by the mean of semigroup arguments.

1.3.2. *Transport equation.* In Part 8, we are interested in the general transport equation

$$\partial_t f + a \cdot \nabla_y f = \mathscr{K}[f] - Kf \quad \text{in} \quad (0,\infty) \times \mathcal{O}, \tag{1.15}$$

on the function $f = f(t, y)$, $t \geq 0$, $y \in \mathcal{O}$, with $\mathcal{O} \subset \mathbb{R}^D$, $D \geq 1$, a smooth open connected set. We assume that $a : \mathcal{O} \to \mathbb{R}^D$, $K : \mathcal{O} \to \mathbb{R}_+$, and that the collision operator $\mathcal{K}$ is linear and defined by

$$(\mathscr{K} g)(y) := \int_{\mathcal{O}} k(y, y_*)\, g(y_*)\, dy_*,$$

for some kernel $k : \mathcal{O} \times \mathcal{O} \to \mathbb{R}_+$. When $\mathcal{O} \neq \mathbb{R}^D$, we complement the equation with a boundary condition on the incoming boundary Σ_- which writes

$$(\gamma_- f)(t, y) = \mathcal{R}_{\mathcal{O}}[f(t, \cdot)] + \mathcal{R}_\Sigma[\gamma_+ f(t, .)](y) \quad \text{on} \quad (0, \infty) \times \Sigma_-,$$

where $\gamma_\pm f$ are the trace functions on the incoming and out going set $\Sigma_\pm$ and

$$(\mathcal{R}_{\mathcal{O}} g)(y) := \int_{\mathcal{O}} r_{\mathcal{O}}(y, y_*)\, g(y_*)\, dy_*, \quad (\mathcal{R}_\Sigma h)(y) := \int_{\Sigma_+} r_\Sigma(y, y_*)\, h(y_*)\, d\sigma_{y_*},$$

for some kernels $r_{\mathcal{O}} : \Sigma_- \times \mathcal{O} \to \mathbb{R}_+$, $r_\Sigma : \Sigma_- \times \Sigma_+ \to \mathbb{R}_+$. All the (quite usual) notations will be explained at the begin of Part 8. It is worth emphasizing here that this framework in particular covers the cases of the renewal equation, the growth-fragmentation equation and the kinetic linear Boltzmann equation on which we will come back below. This framework is motivated by and generalizes the transport theory developed in [37, 43, 139, 70, 119].

In a first step, we consider a very general vector field a by assuming that it satisfies the usual Sobolev regularity condition of DiPerna-Lions transport theory [139]. We also make general assumptions on $\mathcal{R}_{\mathcal{O}}$ and $\mathcal{R}_\Sigma$, but a very strong and somehow restrictive positivity condition on $\mathscr{K}$. Such an equation can be motivated by the abstract transport theory developed [43] as well as non-local reaction-diffusion models [109, 118, 249] and selection-mutation models in changing environment [162, 207]. Under these general conditions and additional ones we will detail later, we are able to solve the existence and geometrical part of the first eigenvalue problem and to prove an ergodicity result (without rate of convergence) generalizing some similar results obtained in [109, 118, 249].

Because of the strong positivity condition made on $\mathscr{K}$, the above mentioned result does not apply to the growth-fragmentation equation and the kinetic linear Boltzmann equation. We thus consider separately these important particular cases in the two next parts. Other singular jump kernels lacking strong positivity can appear in other models, for instance in neurosciences [150], and must also be treated through a specific study.

Another related model is the age structured (or renewal) equation

$$\begin{aligned} &\partial_t f + \partial_y f = -K f \quad \text{in} \quad (0, \infty) \times (0, \infty), \\ &f(t, 0) = (\mathcal{R} f(t, \cdot))(y) = \int_0^\infty r(y_*) f(t, y_*) dy_*. \end{aligned}$$

It corresponds to the case $D = 1$, $\mathcal{O} = (0, \infty)$, $a = 1$, $\mathcal{R}_\Sigma = 0$, $\Sigma_- = \{0\}$ and $\mathscr{K} = 0$ in the transport equation (1.15). The age structured equation is very popular because it is useful for describing dynamic of populations [338, 19, 123, 359, 263] and simple neuronal dynamic [295, 306]. The long time behaviour can be analyzed though Laplace transform technique [158, 159, 221], relative entropy method [275, 269, 196], spectral analysis tool [358, 187, 278, 280, 277] and Doblin approach [34, 82, 168]. Because $\mathscr{K} = 0$, our previous result on the first eigenvalue problem does not apply. We just briefly observe that the method can be applied on the dual equation, thus guaranteeing the existence of (λ_1, ϕ_1), and then that the validity of Doblin's condition ensures the existence and uniqueness of the triplet (λ_1, f_1, ϕ_1), its positivity, and the exponential ergodicity.

1.3.3. *Growth-fragmentation equation.* In Section 9, we consider the growth-fragmentation equation

$$\partial_t f = \mathcal{L} f = \mathcal{G} f + \mathcal{F} f$$

posed on $\mathbb{R}_+$, with the growth operator $\mathcal{G} f = -\partial_x (a f)$ and the fragmentation operator

$$(\mathcal{F} f)(x) = \int_x^\infty k(y, x) f(y)\, dy - K(x) f(x), \quad K(x) := \int_0^x k(y, x) \frac{y}{x} dy.$$

Since the work of Diekmann, Heijmans and Thieme [134], many authors studied this equation by using various methods. We can mention, among many others, [269, 313, 246, 148, 84, 31, 50] for studies based on suitable weak distance, entropy and functional inequalities, [263, 317, 83, 33, 51, 52, 278, 287] in the framework of positive semigroups, [56, 57] for a probabilistic approach via the Feynman-Kac formula, [35, 80, 169, 354] for Harris's method, and also [163] for a recent new approach based on the reformulation of the equation as an abstract renewal problem. Our aim here is not to treat the most general cases of coefficients, but rather to illustrate the variety of the possible behaviors of the equation together with the efficiency and flexibility of the method developed in the first sections. We thus focus on a specific case of fragmentation operator, namely the equal mitosis kernel

$$k(x,y) = 2K(x)\delta_{x/2}(dy) = 4K(x)\delta_{2y}(dx),$$

so that the equation writes

$$\partial_t f(t,x) = -\partial_x\big(a(x)f(t,x)\big) - K(x)f(t,x) + 4K(2x)f(t,2x).$$

In particular, we are interested in the case when the growth rate a is such that $a(2x) = 2a(x)$ for all sizes x, for which the boundary spectrum is not trivial and the solutions then exhibit persistent oscillations in time. When this condition is not satisfied, we recover the more usual exponential convergence to the first eigenfunction.

We also aim at studying the variant of this equation where a variability v is introduced as a growth speed parameter which is inherent to any individual, in the spirit of [275, 333] where such a variable is added in the renewal equation. More precisely we consider the growth-fragmentation equation with variability $v \in [1,2]$ and the equal mitosis division kernel which reads

$$\partial_t f(t,x,v) = -v\partial_x\big(a(x)f(t,x,v)\big) - K(x)f(t,x,v) + 4\int_1^2 K(2x)\wp(v,v_*)f(t,2x,v_*)dv_*.$$

This model was introduced in [146], and then also considered in [304]. We prove that, unlike the case without variability, it exhibits exponential relaxation to the first eigenfunction even when $a(2x) = 2a(x)$ for all x.

1.3.4. *Kinetic linear Boltzmann equation.* In Part 10, we are interested in another important subclass of transport equations, namely in the kinetic linear Boltzmann equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi(x)\cdot\nabla_v f = \mathscr{K}[f] - Kf \quad \text{in} \quad (0,\infty)\times\mathcal{O}, \tag{1.16}$$

on the function $f = f(t,x,v)$, $t \ge 0$, $(x,v) \in \mathcal{O} = \Omega\times\mathcal{V}$, $\Omega \subset \mathbb{R}^d$, $\mathcal{V} \subset \mathbb{R}^d$, $d \ge 1$. We assume that $K : \mathcal{O} \to \mathbb{R}_+$, that $\mathscr{K}$ is a linear integral operator defined by

$$\mathscr{K}[g] := \int_{\mathbb{R}^d} rk(x,v,v_*)\, g(v_*)\, dv_*,$$

for some real number $r > 0$ and some kernel $k : \Omega\times\mathcal{V}\times\mathcal{V} \to \mathbb{R}_+$, and that Φ is a space confining potential $\Phi : \Omega \to \mathbb{R}$. We restrict our analysis to the case $\mathcal{V} := \mathbb{R}^d$ and Ω is either the torus $\Omega := \mathbb{T}^d$ (and we assume $\Phi = 0$) or it is the whole space $\Omega := \mathbb{R}^d$ (and we assume that Φ is a power function). This equation is vey famous because it provide a model for neutron transport theory in nuclear reactors [93, 45] and for cells migration in a chemotactic gradient [7]. We refer to [40, 183, 39, 38, 282] for a mathematical analysis of the neutron transport equation and its diffusive approximation and to [214, 102] for the same concerning kinetic models for chemotaxis. Because the linear integral operator $\mathscr{K}$ is local in the position variable, this problem does not fall in the class of transport equation covered by the Krein-Rutman theorem established in Part 8 and a specific analysis is necessary. Under suitable positivity and regularity conditions on the kernel, we are able to complete the existence, geometric and stability program as stated in Theorem 1.1, with constructive estimates in the torus case, generalizing and improving previous works [59, 351, 352, 186, 257, 281, 283, 284, 285] where spectral analysis arguments are used and [103] based on a probability approach. It is worth emphasizing that these works are concerning the same equation in a bounded domain with no-flow boundary condition. Most of the literature is about the conservative case (when $\lambda_1 = 0$ and $\phi_1 = 1$) which has been tackled by the mean of spectral analysis method [53, 285, 286], of entropy method [132, 54], of geometric control method [200, 135], by hypocoercivity method [208, 141, 154] or by Harris coupling approach [79].

1.3.5. *Kinetic Fokker-Planck equation.* In Part 11, we consider a kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_y f = \Delta_v f + b \cdot \nabla_v f + cf \quad \text{in} \quad (0,\infty) \times \mathcal{O}, \tag{1.17}$$

on the function $f = f(t,x,v)$, $t \geq 0$, $(x,v) \in \mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is a bounded domain, $b : \mathcal{O} \to \mathbb{R}^d$ is a given vector field and $c : \mathcal{O} \to \mathbb{R}$ is a given function. In contrast with the previous part, collisions are typically modeled by a Fokker-Planck operator $\Delta_v f + \mathrm{div}_v(vf)$ which takes into account a thermal bath of (Gaussian) white-noise, see Kolmogorov [234], instead of the integral collisional operator $\mathscr{K}[f] - Kf$ in the linear Boltzmann equation (1.16). The above equation is complemented with the Maxwell boundary condition

$$\gamma_- f = \alpha(x)\mathcal{D}_x \gamma_+ f + \beta(x)\Gamma_x \gamma_+ f,$$

where $\gamma_\pm f$ stand for the outgoing and incoming trace functions, α and β are accommodation coefficients, $\mathcal{D}_x$ is a boundary diffusive operator and Γ_x is the specular reflection operator. All these classical objects will be precisely defined in Part 11. We refer to [217, 298, 129, 66, 91, 67, 272, 353, 365] for a mathematical analysis of the kinetic Fokker-Planck equation or related problems. Under suitable boundedness and regularity conditions on the coefficients we are able to complete the existence, geometric and stability (without constructive estimates) program as stated in Theorem 1.1, generalizing the previous works [248, 193] (partially based on [330, 213, 247]) where similar results are established for the same kind of equation in a bounded domain with no-flow boundary condition. From a technical point of view, our proof is based on trace results as those developed in [257], boundary estimates picked up from [18, 257, 55] and regularity estimates recently obtained in [210, 181, 192]. We also emphasize that in the conservative case, many works have been done related to hypocoercivity and constructive rate of convergence to the steady state in [133, 195, 151, 209, 205, 353] or more recently in [141, 274, 87, 68, 5].

1.3.6. *Mutation-selection equation.* Last, in Section 12, we consider the mutation-selection evolution equation

$$\partial_t f = \mathcal{L}f = J * f - W(x)f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d,$$

This nonlocal-diffusion equation appears for instance in the modeling of genetic variability in evolutionary biology. In this context, $f = f(t,x)$ represents the density of a population, at time $t \geq 0$, of phenotypical trait x on the multi-dimensional phenotypic trait space $\mathbb{R}^d$. The rate of change in f per generation is given by the convolution term with kernel J which models the mutations, and the fitness function $-W$ which stands for the difference between birth and death. This model has been widely used in the literature; we refer, for example, to the works of Kimura [232], Lande [245], Fleming [161] and Bürger [76] as examples of biological applications.

On the mathematical analysis point of view, the Krein-Rutman problem was investigated by Bürger in [75, 77] and more recently by Coville and co-authors [116, 249], as well as by Alfaro and co-authors in [6] where a quantified spectral gap is obtained for symmetric kernels J. A main difference of this equation compared to more classical "local" diffusion models, where the convolution is replaced by a Laplacian, is that the first eigenvector f_1 might be a measure with atoms [75, 77, 117]. Some conditions are then needed relating W and J for guaranteeing that the first eigenvector is an eigenfunction [6, 75, 249].

All the above mentioned results deal with kernels J which are densities, namely absolutely continuous with respect to the Lebesgue measure. In our study, we allow the convolution kernel to have a singular part. In Section 12.1 we extend the results of the literature to the case of a small enough singular part. In Section 12.2 we consider a specific kernel which is purely singular, supported by the canonical axes of $\mathbb{R}^d$, and we extend the recent result of Velleret [350] to more general confining functions W.

1.4. **Organization of the paper.** The paper is organized in two main parts: the sections 2 to 6 are dedicated to the development of the abstract results about the Krein-Rutman problem, and the last sections 7 to 12 aim at illustrating the applicability of these results to various linear positivity preserving PDEs.

More precisely, with start with the existence part of the Krein-Rutman theorem, namely the conclusion **(S1)**. This question is addressed through a stationary approach in Section 2 and through a dynamical approach in Section 3. Section 4 is devoted to the stronger conclusion of uniqueness of the first eigentriplet in the sense of **(S2)**, as well as to the mean ergodic property **(E1)**. In

Section 5, we are interested in the geometry of the boundary point spectrum, deriving conditions that guarantee **(S3$_1$)**, **(S3$_2$)** or **(S3$_3$)**, as well as in the ergodic properties **(E2)** and **(E3$_1$)**. Finally, in Section 6, we tackle the problem of quantifying the conclusions **(S3$_3$)** and **(E3)** by using constructive contraction arguments of the Doeblin-Harris type.
The purpose of the last six sections is to apply the theory developed in the first sections to the examples of PDEs presented in Section 1.3: some parabolic equations (Section 7), transport equations with integral terms (Section 8) and in particular growth-fragmentation equations (Section 9) and kinetic equations (Section 10), kinetic Fokker-Planck equations (Section 11), and purely integral mutation-selection equations (Section 12).

2. Existence through a stationary problem approach

In this part we provide a general existence result for the first eigentriplet problem by considering a family of approximating stationary problems and using a stability argument. We start by presenting the basic material about the Banach lattice framework and conclude with a comparison with several previous works.

2.1. The Banach lattice framework.

We start recalling the Banach lattice framework by stating (most of the time without proof) some well-known facts that one can find in reference monographs as [69, Chapitre II: Espaces de Riesz] or [337, 15, 32, 41].

Banach lattice. A real Banach lattice is a real Banach space $(X, \|\cdot\|)$ endowed with a partial order denoted by $\geq$ (or $\leq$) such that the following holds:
(1) The set $X_+ := \{f \in X;\ f \geq 0\}$ is a nonempty convex cone (compatibility of the order with the vector space structure).
(2) For any $f \in X$, there exist some unique positive part $f_+ \in X_+$ and negative part $f_- \in X_+$ such that $f = f_+ - f_-$ which are minimal: $f = g - h$, $g, h \geq 0$ imply $g \geq f_+$ and $h \geq f_-$ (generation and properness of the positive cone). We set $|f| := f_+ + f_- \in X_+$ the absolute value of $f \in X$.
(3) For any $f, g \in X$, $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ (compatibility of norm and order structures).

Under these assumptions, one can show that
- The convex cone X_+ is closed, pointed $X_+ \cap (-X_+) = \{0\}$ and generating $X = X_+ - X_+$.
- The lattice operations $f \mapsto f_+$, $f \mapsto f_-$ and $f \mapsto |f|$ are continuous (1-Lipschitz).
- The order intervals $\{h \in X;\ g \leq h \leq f\}$ are closed and bounded for any given $f, g \in X$, $f \geq g$.

It is worth emphasizing that one commonly defines the supremum and infimum operations by

$$f \vee g := g + (f-g)_+ \geq f, g, \quad f \wedge g := g - (g-f)_+ \leq f, g,$$

for any $f, g \in X$, and these operations can be used as an alternative way for defining a Banach lattice (the lattice structure refers indeed to these supremum and infimum operations). We may note the following elementary formulas

$$f_+ \wedge f_- = 0, \quad \||f|\| = \|f\|, \quad \forall f \in X. \tag{2.1}$$

We write $f \perp g$ when $|f| \wedge |g| = 0$ or equivalently when $|f| + |g| = |f| \vee |g|$. In that case, we have

$$|f| + |g| = |f+g|.$$

Dual Banach lattice. On the dual space X', we may naturally associate a dual order $\geq$ (or $\leq$) by writing for $\varphi \in X'$

$$\varphi \geq 0 \ \text{ (or } \varphi \in X'_+) \quad \text{iff} \quad \forall f \in X_+ \ \langle \varphi, f\rangle \geq 0.$$

For $\varphi \in X'$, there exist some unique $\varphi_\pm \in X'_+$ such that $\varphi = \varphi_+ - \varphi_-$ which also satisfy (and are defined by)

$$\forall f \in X_+, \quad \langle \varphi_\pm, f\rangle = \sup_{0 \leq g \leq f} \langle \pm\varphi, g\rangle.$$

One can show that the above conditions (1), (2) and (3) of a Banach lattice are fulfilled, and thus $X' = (X', \|\cdot\|_{X'}, \geq)$ is a Banach lattice. We observe that for any $f \in X_+$ there exists $f^* \in X'_+$ such that

$$\langle f^*, f\rangle = \|f\|^2 = \|f^*\|^2_{X'}, \tag{2.2}$$

as a classical corollary of the Hahn-Banach dominated extension theorem. Moreover, for any $f \in X$,

$$f \ge 0 \quad \text{iff} \quad \langle \varphi, f \rangle \ge 0, \ \forall \varphi \in X'_+, \tag{2.3}$$

as an immediate application of the Hahn-Banach separation theorem. In other words, the restriction to X of the dual order in X'' associated to the order defined (by duality) on X' is nothing but the initial order, in particular the positive cone X'_+ is weakly $*$ closed.

The functional framework : The duality bracket. We consider two Banach lattices X, Y such that $X = Y'$ with Y separable or such that $Y = X'$. We emphasize on the facts that

$$\text{for } f \in X: \quad f \in X_+ \text{ iff } \langle f, \varphi \rangle \ge 0, \ \forall \varphi \in Y_+, \tag{2.4}$$

$$\text{for } \varphi \in Y: \quad \varphi \in Y_+ \text{ iff } \langle f, \varphi \rangle \ge 0, \ \forall f \in X_+, \tag{2.5}$$

which are immediate consequences of (2.3) and of the definition of the dual order.

Examples. For the space $C_0(E)$, the order is defined by $f \ge 0$ iff $f(x) \ge 0$ for any $x \in E$. For a space $L^p(E, \mathscr{E}, \mu)$, $1 \le p \le \infty$, the order is defined by $f \ge 0$ iff $f(x) \ge 0$ for μ-a.e. $x \in E$. For the space $M^1(E)$, the order is defined by $f \ge 0$ iff in the Hahn decomposition $f = f_+ - f_-$, there holds $f_- = 0$, or equivalently, by duality: $f \ge 0$ iff $\langle f, \varphi \rangle \ge 0$ for any $\varphi \in C_0(E)$, $\varphi \ge 0$.

Because confinement will play a major role in our analysis, we will use some weighted version of the above space associated to a weight (continuous or Borel measurable) function $m : E \to (0, \infty)$ that we introduce now. We recall that E always denotes a σ-compact metric space, and we write $E = \cup E_R$, with $E_R \subset E_{R+1}$, E_R compact. In that context, we write $x_n \to \infty$ if for any $R \ge 1$ there exists n_R such that $x_n \notin E_R$ for any $n \ge n_R$.

- We denote by $C_{m,0}(E)$ the space

$$C_{m,0}(E) := \{\varphi \in C(E); \ |\varphi(x)|/m(x) \to 0 \text{ as } x \to \infty\}$$

endowed with the norm $\|\varphi\|_{C_{m,0}} := \|\varphi/m\|_{C_0}$.

- We denote by $M^1_m(E) := (C_{m,0}(E))'$ the associated space of Radon measures.
- We denote by $L^p_m(E) = L^p_m(E, \mathscr{E}, \mu)$ the space

$$L^p_m(E) := \{f \in L^1_{\text{loc}}(E); \ \|f\|_{L^p_m} := \|fm\|_{L^p} < \infty\}.$$

It is worth emphasizing that $L^p_m(E, \mathscr{E}, \mu) = L^p(E, \mathscr{E}, m^p \mu)$ when $p \in [1, \infty)$.

Positive operator. We denote by $\mathscr{B}(X)$ the set of linear and bounded operators on X. We also denote by $\mathscr{K}(X)$ the subspace of compact operators. We say that a bounded operator $A \in \mathscr{B}(X)$ is positive, and we write $A \ge 0$, if

$$Af \in X_+, \quad \forall f \in X_+.$$

We will also sometimes abuse notations by writing $A \in \mathscr{B}(X_+)$ for meaning that $A \ge 0$. For a positive operator $A \in \mathscr{B}(X)$, we have

$$|Af| \le A|f|, \quad \forall f \in X, \quad \text{and} \quad \|A\| = \sup_{0 \le f \in B_X} \|Af\|, \tag{2.6}$$

where B_X is the unit closed ball. More generally, we have

$$(Af) \vee (Ag) \le A(f \vee g), \quad \forall f, g \in X. \tag{2.7}$$

For X and Y in duality, and $A \in \mathscr{B}(X)$ and $A^* \in \mathscr{B}(Y)$ in duality, in the sense that

$$\langle Af, \phi \rangle = \langle f, A^* \phi \rangle, \quad \forall f \in X, \ \phi \in Y,$$

there holds

$$A \ge 0 \quad \text{iff} \quad A^* \ge 0. \tag{2.8}$$

Let us present the elementary and classical but instructive proof of the direct implication, the reciprocal way being similar. We assume thus $A \ge 0$. We take $\varphi \in Y_+$ and we define $\psi := A^* \varphi$. We then take $f \in X_+$ and we define $g := Af$, so that $g \ge 0$ by assumption. We compute

$$\langle \psi, f \rangle = \langle A^* \varphi, f \rangle = \langle \varphi, Af \rangle = \langle \varphi, g \rangle \ge 0.$$

Since $f \in X_+$ is arbitrary, we get $\psi \in Y_+$, and thus $A^* \ge 0$.

Semigroup, generator and spectrum. In this work, a semigroup $S = S(t) = (S_t)$ on X will always denote a semigroup of linear and bounded operators on a Banach lattice X which trajectories are
- either strongly continuous, namely, the mapping $t \mapsto S_t f$ is continuous for the norm of X for any fixed $f \in X$;
- either weakly $*$ continuous, namely $X = Y'$ for some separable Banach lattice Y such that $\forall f \in X$, $\forall \phi \in Y$, $t \mapsto \langle S_t f, \phi \rangle_{X,Y}$ is continuous and $\forall t \geq 0$, $\forall \phi \in Y$, $f \mapsto \langle S_t f, \phi \rangle_{X,Y}$ is continuous. That is in particular the case when there exists a strongly continuous semigroup P on Y such that $S_t = P_t^*$ for any $t \geq 0$.

For a semigroup S, we denote by $\mathcal{L}$ its generator and $D(\mathcal{L})$ the associated domain, and thus we sometimes write $S = S_\mathcal{L}$. We also denote the iterated domain defined recursively by $D(\mathcal{L}^k) := \{f \in D(\mathcal{L}^{k-1}),\, \mathcal{L}f \in D(\mathcal{L}^{k-1})\}$ for any $k \geq 2$ and $D(\mathcal{L}^\infty) := \bigcap_{k\geq 1} D(\mathcal{L}^k)$. We recall that $D(\mathcal{L})$ is dense in X and the graph of $\mathcal{L}$ is closed in $X \times X$. We define the growth bound

$$\omega = \omega(S) := \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\| \in \mathbb{R} \cup \{-\infty\}, \tag{2.9}$$

so that

$$\forall \omega' > \omega, \quad \exists M \geq 1, \quad \|S(t)\|_{\mathscr{B}(X)} \leq M\, e^{\omega' t}, \quad \forall t \geq 0, \tag{2.10}$$

and ω is the infimum of $\omega' \in \mathbb{R}$ such that (2.10) holds. We say that S is a semigroup of contractions when S satisfies (2.10) with $M = 1$ and $\omega' = 0$.

The resolvent set $\rho(\mathcal{L})$ is the set of $z \in \mathbb{C}$ such that if $z - \mathcal{L} : D(\mathcal{L}) \to X$ is bijective and its inverse belongs to $\mathscr{B}(X)$. We define the resolvent operator by

$$\mathcal{R}(z) = \mathcal{R}_\mathcal{L}(z) := (z - \mathcal{L})^{-1}, \quad \forall z \in \rho(\mathcal{L}), \tag{2.11}$$

and the spectrum by $\Sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$. Denoting the half complex plane of abscissa $\alpha \in \mathbb{R}$ by

$$\Delta_\alpha := \{z \in \mathbb{C};\ \Re e(z) > \alpha\}, \tag{2.12}$$

we have $\rho(\mathcal{L}) \supset \Delta_\omega$ and, for any $z \in \Delta_\omega$, there holds

$$\mathcal{R}(z) = \int_0^\infty S(t) e^{-zt}\, dt. \tag{2.13}$$

Positive semigroup. We say that a semigroup (S_t) on a Banach lattice X is positive if

$$S_t \geq 0, \quad \forall t \geq 0.$$

Lemma 2.1. *For a semigroup S on a Banach lattice X, there is equivalence between*
(a) S is positive;
(b) the associate resolvent operator $\mathcal{R}$ is positive: $\mathcal{R}(\kappa) \geq 0$ for all $\kappa > \omega$ (or for all sufficiently large κ).

It is immediate from Hille's identity (2.13) that (a) implies (b). The reciprocal implication comes from the relation $S(t) = \lim_{n\to\infty} [n/t\mathcal{R}(n/t)]^n$ at the foundation of the Hille-Yosida theory, see for instance [309, Thm. I.8.3].

2.2. **Existence part of the Krein-Rutman theorem.** From now on in this section, we consider a Banach lattice X and an operator $\mathcal{L}$ with dense domain and closed graph. Our goal is mainly to prove the existence part for the primal problem in the Krein-Rutman theorem, namely

$$\exists \lambda_1 \in \mathbb{R},\ \exists f_1 \in X_+\backslash\{0\}, \quad \mathcal{L} f_1 = \lambda_1 f_1. \tag{2.14}$$

We will also discuss the existence part for the dual problem at the end of the section.

We first assume

(H1) $\exists \kappa_1 \in \mathbb{R}$ such that $\lambda - \mathcal{L}$ is invertible and $(\lambda - \mathcal{L})^{-1} : X_+ \to X_+$ for any $\lambda \geq \kappa_1$.

Note that an operator $\mathcal{L}$ satisfying **(H1)** is sometimes called a *resolvent positive* operator after the paper of Arendt [14].

We then set

$$\mathcal{I} := \{\kappa \in \mathbb{R};\ \lambda - \mathcal{L} \text{ is invertible},\ (\lambda - \mathcal{L})^{-1} \geq 0 \ \text{ for any } \ \lambda \geq \kappa\}, \tag{2.15}$$

which is a non empty and non upper bounded interval due to **(H1)**. We finally set

$$\lambda_1 := \inf \mathcal{I} \in [-\infty, \kappa_1]. \tag{2.16}$$

For the sake of completeness, we recall now some general facts about $\mathcal{I}$ and λ_1 when $\mathcal{L}$ is the generator of a positive semigroup. We also refer to [152, Sec. 1.b, Chap. VI] or [41, Chapter 12] and the references therein for more details.

Lemma 2.2. *When $\mathcal{L}$ is the generator of a positive semigroup $S = S_\mathcal{L}$, then*
- **(i)** **(H1)** *automatically holds with any $\kappa_1 > \omega(S)$, so that $\lambda_1 \le \omega(S)$;*
- **(ii)** $\Sigma(\mathcal{L}) \cap \Delta_{\lambda_1} = \emptyset$ *and the representation formula* (2.13) *holds true for any $z \in \Delta_{\lambda_1}$;*
- **(iii)** *it may happen that $\lambda_1 = -\infty$.*

The important property **(ii)** is probably due to [189].

Proof of Lemma 2.2. The claim **(i)** is an immediate consequence of the representation formula (2.13) for any $\kappa_1 > \omega(S)$ and the positivity of $S(t)$ for any $t \ge 0$ (that is nothing but Lemma 2.1).
We prove **(ii)**. Take $\lambda > \lambda_1$. From the classical identity

$$S(t)e^{-\lambda t} - I = (\mathcal{L} - \lambda) \int_0^t S(s)e^{-\lambda s}\, ds, \quad \forall\, t \ge 0,$$

and the positivity property of S, we have

$$0 \le V(t) := \int_0^t S(s)e^{-\lambda s}\, ds = \mathcal{R}(\lambda) - \mathcal{R}(\lambda)S(t)e^{-\lambda t} \le \mathcal{R}(\lambda),$$

for any $t \ge 0$. From that estimate, we get $\|V(t)\| \le \|\mathcal{R}(\lambda)\|$. For any $z \in \Delta_\lambda$, an integration by part yields

$$\int_0^t e^{-zs} S(s)\, ds = e^{-(z-\lambda)t} V(t) + (z - \lambda) \int_0^t e^{-(z-\lambda)s} V(s)\, ds.$$

The estimate on V makes possible to pass to the limit $t \to \infty$ in the above identity, and we deduce

$$\mathcal{U}(z) := \int_0^\infty e^{-zs} S(s)\, ds = (z - \lambda) \int_0^\infty e^{-(z-\lambda)s} V(s)\, ds \in \mathscr{B}(X).$$

In that situation, one classically knows that $z \in \rho(\mathcal{L})$ and $(z - \mathcal{L})^{-1} = \mathcal{U}(z)$. We have thus established $\Sigma(\mathcal{L}) \cap \Delta_\lambda = \emptyset$ and we conclude the proof of **(ii)** by observing that (2.13) is then nothing but the above formula.
(iii) On $L^p(0,1)$, $1 \le p < \infty$, the translation semigroup defined for $a > 0$ by

$$S(t)f(x) := f(x + at)\mathbf{1}_{x+at\le 1}, \quad \forall\, t \ge 0,\ x \in (0,1),$$

is strongly continuous and positive. Since $S(t) \equiv 0$ for any $t \ge 1/a$, we have $\omega(S) = -\infty$, and thus $\lambda_1 = -\infty$ because of **(i)**. □

For further discussion, we give some probably classical results about the condition **(H1)** and some equivalent definitions of the set $\mathcal{I}$.

Lemma 2.3. *The operator $\mathcal{L}$ satisfies* **(H1)** *if and only if the operator $\mathcal{L}^*$ satisfies* **(H1)**. *Furthermore, under condition* **(H1)** *for $\mathcal{L}$ (or $\mathcal{L}^*$), we have*

$$\mathcal{I} = \mathcal{I}_i, \quad \forall\, i = 2, 3, 4, \tag{2.17}$$

with

$$\begin{aligned}
\mathcal{I}_2 &:= \{\kappa \in \mathbb{R};\ \lambda - \mathcal{L} \text{ is invertible for any } \lambda \ge \kappa\},\\
\mathcal{I}_3 &:= \{\kappa \in \mathbb{R};\ \lambda - \mathcal{L}^* \text{ is invertible, } (\lambda - \mathcal{L}^*)^{-1} \ge 0 \text{ for any } \lambda \ge \kappa\},\\
\mathcal{I}_4 &:= \{\kappa \in \mathbb{R};\ \lambda - \mathcal{L}^* \text{ is invertible for any } \lambda \ge \kappa\}.
\end{aligned}$$

Proof of Lemma 2.3. The equivalence of condition **(H1)** for the operators $\mathcal{L}$ and $\mathcal{L}^*$ is an immediate consequence of the identity $\rho(\mathcal{L}) = \rho(\mathcal{L}^*)$ (see for instance [229, Thm. III.6.22]) and the fact that $(\lambda - \mathcal{L})^{-1} : X_+ \to X_+$ iff $(\lambda - \mathcal{L}^*)^{-1} : Y_+ \to Y_+$ as recalled in (2.8). As a consequence, we have $\mathcal{I} = \mathcal{I}_3$ and $\mathcal{I}_2 = \mathcal{I}_4$.

We obviously have $\mathcal{I}_2 \subset \mathcal{I}$ and let us show the reverse inclusion. We denote $\mathcal{R} = \mathcal{R}_\mathcal{L}$. On the one hand, for any $z_0 \in \rho(\mathcal{L})$ and any $z \in \mathbb{C}$, $|z - z_0| < \|\mathcal{R}(z_0)\|^{-1}$, we know that

$$\mathcal{R}(z) = \mathcal{R}(z_0) \sum_{k=0}^{\infty} (z_0 - z)^k \mathcal{R}(z_0)^k, \tag{2.18}$$

which gives a proof of the fact that resolvent set $\rho(\mathcal{L})$ is open and that $\mathcal{R}$ is an holomorphic function on $\rho(\mathcal{L})$. Formula (2.18) also ensures that for $\lambda_0, \lambda \in \mathbb{R}$, the condition $\mathcal{R}(\lambda_0) \ge 0$ implies that $\mathcal{R}(\lambda) \ge 0$ provided that $\lambda_0 - \lambda > 0$ is small enough and thus $\mathcal{R}(\lambda) \ge 0$ for any λ in the non upper bounded connected component of the set $\rho(\mathcal{L}) \cap \mathbb{R}$ thanks to a continuation argument. In particular, $\mathcal{I}$ is an open set and $\mathcal{I} = \mathcal{I}_2$. □

We next assume

(H2) $\exists\, \kappa_0 \in \mathbb{R}$ such that $\inf \mathcal{I} \ge \kappa_0$.

We do not further consider in these notes the case when $\inf \mathcal{I} = -\infty$ and moreover we will particularly focus on the possibility to exhibit constructive lower bound κ_0.

We point out several conditions under which **(H2)** is satisfied.

Lemma 2.4. *Condition* **(H2)** *holds under one of the four following conditions*

(i) $\exists\, \kappa_0 \in \mathbb{R}$, $\exists\, \phi_0 \in Y_+\backslash\{0\}$ *such that* $\mathcal{L}^*\phi_0 \ge \kappa_0 \phi_0$, *which means*

$$\forall\, f \in D(\mathcal{L}) \cap X_+, \quad \langle \phi_0, (\kappa_0 - \mathcal{L}) f \rangle \le 0;$$

(ii) $\exists\, \kappa_0 \in \mathbb{R}$, $\exists\, f_0 \in X_+\backslash\{0\}$ *such that* $\mathcal{L} f_0 \ge \kappa_0 f_0$, *which means*

$$\forall\, \phi \in D(\mathcal{L}^*) \cap Y_+, \quad \langle (\kappa_0 - \mathcal{L}^*)\phi, f_0 \rangle \le 0;$$

(iii) $\mathcal{L}^*$ *is the generator of a positive semigroup* $S^* = (S^*_t)$ *and*

$$\exists\, \kappa_0 \in \mathbb{R},\ \exists\, \phi_0 \in Y_+\backslash\{0\},\ \exists\, T > 0 \ \ \textit{such that} \ \ S^*_T \phi_0 \ge e^{\kappa_0 T} \phi_0;$$

(iv) $\mathcal{L}$ *is the generator of a positive semigroup* $S = (S_t)$ *and*

$$\exists\, \kappa_0 \in \mathbb{R},\ \exists\, f_0 \in X_+\backslash\{0\},\ \exists\, T > 0 \ \ \textit{such that} \ \ S_T f_0 \ge e^{\kappa_0 T} f_0.$$

Proof of Lemma 2.4. In the three cases, we claim that $\kappa_0 \notin \mathcal{I}$ and thus $\inf \mathcal{I} \ge \kappa_0$. We argue by contradiction, assuming $\lambda_1 < \kappa_0$, so that $\kappa_0 \in \mathcal{I} = \mathcal{I}_i$ for any $i = 2, 3, 4$.

We assume **(i)**. For any $g \in X_+$, we define $f := (\kappa_0 - \mathcal{L})^{-1} g \in X_+$ and we compute

$$0 \le \langle \phi_0, g \rangle = \langle \phi_0, (\kappa_0 - \mathcal{L}) f \rangle \le 0.$$

That implies $\langle \phi_0, g \rangle = 0$ for any $g \ge 0$, so that $\phi_0 = 0$ and a contradiction.

We assume **(ii)**. For any $\psi \in Y_+$, we define $\phi := (\kappa_0 - \mathcal{L}^*)^{-1} \psi \in Y_+$ and we compute

$$0 \le \langle \psi, f_0 \rangle = \langle (\kappa_0 - \mathcal{L}^*)\phi, f_0 \rangle \le 0.$$

That implies $\langle \psi, f_0 \rangle = 0$ for any $\psi \ge 0$, so that $f_0 = 0$ and a contradiction.

We assume first that **(iii)** holds for any $T > 0$. For any $f \in D(\mathcal{L}) \cap X_+\backslash\{0\}$, we compute

$$\langle \phi_0, (\kappa_0 - \mathcal{L}) f \rangle = -\frac{d}{dt} \langle \phi_0, e^{-\kappa_0 t} S_t f \rangle \le 0,$$

which is precisely **(i)**. We assume now that **(iii)** holds. If $\kappa_0 \in \mathcal{I}$, for any $g \in X_+$, we may define $f = (\kappa_0 - \mathcal{L})^{-1} g \in X_+ \cap D(\mathcal{L})$ and from condition **(iii)**, we have

$$0 \le \langle e^{-n\kappa_0 T} S_{nT} f - f, \phi_0 \rangle = \Big\langle (\mathcal{L} - \kappa_0) \int_0^{nT} e^{-\kappa_0 t} S_t f \, dt, \phi_0 \Big\rangle,$$

for any $n \in \mathbb{N}$. From the very definition of f, we also have

$$(\mathcal{L} - \kappa_0) \int_0^{nT} e^{-\kappa_0 t} S_t f \, dt = \int_0^{nT} e^{-\kappa_0 t} S_t (\mathcal{L} - \kappa_0) f \, dt = -\int_0^{nT} e^{-\kappa_0 t} S_t g \, dt \le 0.$$

The two pieces of information together imply

$$\Big\langle \int_0^{nT} e^{-\kappa_0 t} S_t g \, dt, \phi_0 \Big\rangle = 0.$$

Passing to the limit $n \to \infty$ thanks to Lemma 2.2-**(ii)** and using (2.11)-(2.13), we obtain

$$0 = \Big\langle \int_0^\infty e^{-\kappa_0 t} S_t g \, dt, \phi_0 \Big\rangle = \langle f, \phi_0 \rangle = \langle g, (\kappa_0 - \mathcal{L}^*)^{-1} \phi_0 \rangle.$$

That implies $(\kappa_0 - \mathcal{L}^*)^{-1}\phi_0 = 0$ since g is arbitrary, what is not possible since $\phi_0 \neq 0$. The proof of **(H2)** under assumption **(iv)** is similar and thus skipped. □

Remark 2.5. *(1) In practice, we may build f_0 or ϕ_0 through an explicit computation or use a barrier fonction and strong maximum principle techniques. We refer to Lemma 4.12 for a possible general result in that direction.*
(2) When **(ii)** *holds with $f_0 \in X_+ \backslash \{0\} \cap D(\mathcal{L})$ and $\mathcal{L}$ is the generator of a positive semigroup S, then* **(iv)** *holds for any $T > 0$. In that case, we may indeed compute*

$$S_T e^{-\kappa_0 T} f_0 - f_0 = \int_0^T S_t e^{-\kappa_0 t} (\mathcal{L} - \kappa_0) f_0 ds \geq 0.$$

Lemma 2.6. *Under conditions* **(H1)** *and* **(H2)**, *there hold*

$$\lambda_1 \in [\kappa_0, \kappa_1] \tag{2.19}$$

and

$$\exists\, \lambda_n \searrow \lambda_1, \ \exists\, \hat f_n \in D(\mathcal{L}) \cap X_+, \ \varepsilon_n := \lambda_n \hat f_n - \mathcal{L} \hat f_n \geq 0, \ \|\hat f_n\| = 1, \ \|\varepsilon_n\| \to 0. \tag{2.20}$$

Proof of Lemma 2.6. We obviously have $\lambda_1 \leq \kappa_1$ from assumption **(H1)** and $\lambda_1 \geq \kappa_0$ by assumption **(H2)**, so that (2.19) is proved.

Consider now a sequence $(\lambda_n)_{n \geq 2}$ such that $\lambda_n \searrow \lambda_1$ as $n \to \infty$. We eventually have $\|\mathcal{R}(\lambda_n)\| \to \infty$ as $n \to \infty$, where we denote by $\mathcal{R} = \mathcal{R}_\mathcal{L}$ the resolvent of $\mathcal{L}$. On the contrary, we would have $\|\mathcal{R}(\lambda_{n'})\| \leq M$ for some subsequence $\lambda_{n'} \searrow \lambda_1$ and some constant $M > 0$. Because of (2.18) this implies that $(\lambda_{n'} - \varepsilon, \lambda_{n'}) \subset \mathcal{I}$ for any n' and some $\varepsilon > 0$, and this is in contradiction with the definition of λ_1. The blow up $\|\mathcal{R}(\lambda_n)\| \to \infty$ means that

$$\exists\, f_n \in D(\mathcal{L}), \exists\, g_n \in X, \quad \mathcal{R}(\lambda_n) g_n = f_n, \ \|f_n\| \to \infty, \ \|g_n\| \leq 1.$$

By splitting $g_n = g_n^+ - g_n^-$, we get

$$f_n = \mathcal{R}(\lambda_n) g_n^+ - \mathcal{R}(\lambda_n) g_n^-$$

with

$$\|g_n^\pm\| \leq 1 \quad \text{and} \quad (\|\mathcal{R}(\lambda_n) g_n^+\| \to \infty \text{ or } \|\mathcal{R}(\lambda_n) g_n^-\| \to \infty).$$

Changing notations, we have thus

$$\exists\, f_n \geq 0, \exists\, g_n \geq 0, \quad \mathcal{R}(\lambda_n) g_n = f_n, \ \|f_n\| \to \infty, \ \|g_n\| \leq 1.$$

We get (2.20) by defining $\hat f_n := f_n / \|f_n\|$ and $\varepsilon_n := g_n / \|f_n\|$. □

We learn a very similar proof in [252], from which our own proof is adapted. The same type of arguments can also be found in [41, proof of Theorem 12.15].

We finally assume that

(H3) for any sequence $(\hat f_n)$ of X such that (2.20) holds, there exist $f_1 \in X_+ \backslash \{0\}$ and a subsequence $(\hat f_{n'})$ such that $\hat f_{n'} \rightharpoonup f_1$ for the weak convergence or the weak $*$ convergence.

We discuss several situations in which assumption **(H3)** holds. We start with a very classical framework formalized for instance by Voigt [355], see also Karlin [228, Cor. 1] or Sasser [335] for earlier similar situations and results, which is however somehow restrictive since it is based on a strong compactness property assumed at the level of the associated semigroup of operators.

Lemma 2.7. *We assume that $\mathcal{L}$ generates a positive semigroup S, that* **(H2)** *holds for a constant $\kappa_0 \in \mathbb{R}$ and that there exists $T > 0$ such that the splitting*

$$S_T = V_T + K_T, \tag{2.21}$$

holds with $\|V_T\|_{\mathscr{B}(X)} \leq e^{\kappa T}$, $\kappa < \kappa_0$, and $K_T \in \mathscr{K}(X)$. Then condition **(H3)** *holds for the primal and the dual problems.*

Proof of Lemma 2.7. The condition **(H1)** holds because of Lemma 2.2-**(i)**. Let us then consider three sequences (λ_n), $(\hat f_n)$ and (ε_n) satisfying (2.20). Integrating along the rescaled flow, this yields

$$\begin{aligned} e^{-\lambda_n T} S_T \hat f_n - \hat f_n &= \int_0^T e^{-\lambda_n t} S_t (\mathcal{L} - \lambda_n) \hat f_n dt \\ &= -\int_0^T e^{-\lambda_n t} S_t \varepsilon_n \, dt =: \tilde\varepsilon_n, \end{aligned}$$

which also reads

$$V \hat f_n + K \hat f_n - e^{\lambda_n T} \hat f_n = e^{\lambda_n T} \tilde\varepsilon_n.$$

Since $e^{\lambda_n T} \ge e^{\kappa_0 T} > e^{\kappa T}$, the operator $e^{\lambda_n T} - V_T$ is invertible with inverse $Q(\lambda_n) := (e^{\lambda_n T} - V_T)^{-1}$ uniformly bounded and converging in $\mathscr{B}(X)$ to $Q(\lambda_1) = (e^{\lambda_1 T} - V_T)^{-1}$. We thus have

$$\hat f_n = w_n + v_n, \quad w_n := Q(\lambda_n) K_T \hat f_n, \quad v_n := -Q(\lambda_n) e^{\lambda_n T} \tilde\varepsilon_n,$$

with $\|v_n\|_X \to 0$ and (w_n) relatively compact in X. There exist thus a subsequence $(\hat f_{n_k})$ and $g \in X$ such that $K_T \hat f_{n_k} \to g$ and next

$$w_{n_k} - Q(\lambda_1) g = (Q(\lambda_{n_k}) - Q(\lambda_1)) K_T \hat f_{n_k} + Q(\lambda_1)(K_T \hat f_{n_k} - g) \to 0.$$

We deduce that $\hat f_{n_k} \to f_1$ strongly in X. Because of the positivity and normalized properties of $\hat f_n$, we get $f_1 \in X_+$, $\|f_1\|_X = 1$, and we conclude that **(H3)** holds for the primal problem Observing that the dual semigroup S^* satisfies $S^*_T = V^*_T + K^*_T$ with $\|V^*_T\|_{\mathscr{B}(Y)} \le e^{\kappa T}$ and $K^*_T \in \mathscr{K}(Y)$, the same proof implies that **(H3)** holds for the dual problem. □

In the six next lemmas, we will assume that **(H1)**-**(H2)** holds associated to some constants $\kappa_i \in \mathbb{R}$, $\kappa_0 < \kappa_1$, and we always make the following splitting structure hypothesis

(HS1) there exists a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ such that $\mathcal{B} - \alpha$ is invertible for any $\alpha \ge \kappa_0$ and

$$\mathcal{V}(\alpha) := \sum_{i=0}^{N-1} (\mathcal{R}_\mathcal{B}(\alpha)\mathcal{A})^i \mathcal{R}_\mathcal{B}(\alpha), \quad \mathcal{W}(\alpha) := (\mathcal{R}_\mathcal{B}(\alpha)\mathcal{A})^N, \tag{2.22}$$

are bounded in $\mathscr{B}(X)$ uniformly with respect to $\alpha \ge \kappa_0$ and for some $N \ge 1$, where we recall that $\mathcal{R}_\mathcal{B}(\alpha) := (\alpha - \mathcal{B})^{-1}$ is the resolvent of $\mathcal{B}$.

We first present a result also based on a strong compactness property which is assumed to hold however at the level of the resolvent operator. We will be able to use that result in most of the applications.

Lemma 2.8. *(1) We assume* **(H1)**-**(H2)**-**(HS1)** *and there exists $N \ge 1$ such that*

$$\mathcal{W}(\alpha) \text{ is strongly compact locally uniformly on } \alpha \ge \kappa_0, \tag{2.23}$$

in the sense that if $\alpha_n \to \alpha$, $\alpha_n \ge \kappa_0$, and (g_n) is a bounded sequence in X, then there exist $f \in X$ and a subsequence (g_{n_k}) such that $\mathcal{W}(\alpha_{n_k}) g_{n_k} \to f$ strongly in X. Then condition **(H3)** *holds.*
(2) We assume **(H1)**-**(H2)** *and* **(HS1)** *where $\mathcal{R}_\mathcal{B}(\alpha)$ is bounded uniformly in $\alpha \ge \kappa_0$, $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{W}(\alpha) \in \mathscr{K}(X)$ for any fixed $\alpha \ge \kappa_0$ and some $N \ge 1$. Then condition* **(H3)** *holds both for the primal and the dual problems.*

Remark 2.9. *(1) The property* (2.23) *holds if we assume $\mathcal{W}(\alpha) : X \to \mathcal{X}_1$ is bounded uniformly in $\alpha \ge \kappa_0$ and $\mathcal{X}_1 \subset X$ with strong compact embedding.*
(2) The property (2.23) *holds if we assume* **(H1)**-**(H2)**-**(HS1)** *together with the facts that $\mathcal{R}_\mathcal{B}(\alpha)$ and $\mathcal{R}_\mathcal{B}(\alpha)\mathcal{A}$ are bounded uniformly in $\alpha \ge \kappa_0$ and $\mathcal{W}(\alpha) \in \mathscr{K}(X)$ for any fixed $\alpha \ge \kappa_0$. Consider indeed $\alpha_n \to \alpha$, $\alpha_n \ge \kappa_0$, and (g_n) a bounded sequence in X. On the one hand, there exist $f \in X$ and a subsequence (g_{n_k}) such that $\mathcal{W}(\alpha) g_{n_k} \to f$ strongly in X, because $\mathcal{W}(\alpha) \in \mathscr{K}(X)$. On the other hand, using the resolvent identity $\mathcal{R}_\mathcal{B}(\lambda) - \mathcal{R}_\mathcal{B}(\mu) = (\mu - \lambda)\mathcal{R}_\mathcal{B}(\lambda)\mathcal{R}_\mathcal{B}(\mu)$, we have*

$$\mathcal{W}(\alpha) - \mathcal{W}(\alpha_n) = (\alpha_n - \alpha) \sum_{j=1}^N (\mathcal{R}_\mathcal{B}(\alpha)\mathcal{A})^{N-j} \mathcal{R}_\mathcal{B}(\alpha) (\mathcal{R}_\mathcal{B}(\alpha_n)\mathcal{A})^j \to 0,$$

so that $\mathcal{W}(\alpha_{n_k}) g_{n_k} \to f$ strongly in X, and (2.23) *holds true.*

Proof of Lemma 2.8. We first assume (1). Taking advantage of the splitting structure **(HS1)**, we write equation (2.20) as

$$(\lambda_n - \mathcal{B})\hat{f}_n = \mathcal{A}\hat{f}_n + \varepsilon_n, \tag{2.24}$$

or equivalently

$$\hat{f}_n = \mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A}\hat{f}_n + \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n.$$

Iterating that last identity and using the notations (2.22), we get

$$\hat{f}_n = w_n + v_n, \quad w_n := \mathcal{W}(\lambda_n)\hat{f}_n, \quad v_n := \mathcal{V}(\lambda_n)\varepsilon_n. \tag{2.25}$$

We observe that (w_n) is strongly relatively compact from (2.23) and $\|\hat{f}_n\|_X = 1$, so that there exist a subsequence (w_{n_k}) and $f_1 \in X$ such that $w_{n_k} \to f_1$ strongly in X. Since $v_n \to 0$ strongly in X, we deduce that $\hat{f}_{n_k} \to f_1$ strongly in X. We conclude that condition **(H3)** holds as in the proof of Lemma 2.7.

We next assume (2). As observed in Remark 2.9-(2), the property (2.23) holds and thus also the condition **(H3)** for the primal problem. We claim that the same locally uniform strong compactness property (2.23) holds for the dual problem at order $N+1$ and thus condition **(H3)** holds for the dual problem. We may indeed use Remark 2.9-(2) since then $\mathcal{R}_{\mathcal{B}^*}(\alpha)$ and $\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*}(\alpha)$ are bounded uniformly in $\alpha \ge \kappa_0$ and

$$(\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*}(\alpha))^{N+1} = \mathcal{A}^*\mathcal{W}(\alpha)^*\mathcal{R}_{\mathcal{B}^*}(\alpha) \in \mathscr{K}(Y), \quad \forall\, \alpha \ge \kappa_0,$$

as a product of two bounded operator with a compact operator. □

Remark 2.10. *Instead of* **(HS1)** *in Lemma 2.8, one can assume that there exists a splitting* $\mathcal{L} = \mathcal{A} + \mathcal{B}$ *and* $N \ge 1$ *such that* $\mathcal{B} - \alpha$ *is invertible for any* $\alpha \ge \kappa_0$ *and*

$$\mathcal{R}_{\mathcal{B}}(\alpha) := (\alpha - \mathcal{B})^{-1}, \quad \check{\mathcal{V}}(\alpha) := \sum_{i=0}^{N-1} (\mathcal{A}\mathcal{R}_{\mathcal{B}}(\alpha))^i, \quad \check{\mathcal{W}}(\alpha) := (\mathcal{A}\mathcal{R}_{\mathcal{B}}(\alpha))^N$$

are respectively bounded in $\mathscr{B}(X)$ *uniformly with respect to* $\alpha \ge \kappa_0$ *and strongly compact locally uniformly on* $\alpha \ge \kappa_0$. *Starting indeed again from* (2.24) *and defining* $h_n := (\lambda_n - \mathcal{B})\hat{f}_n$, *we may write*

$$h_n = \mathcal{A}\mathcal{R}_{\mathcal{B}}(\lambda_n)h_n + \varepsilon_n.$$

Observing that $\|h_n\|_X \ge \|\mathcal{R}_{\mathcal{B}}(\lambda_n)\|^{-1}_{\mathscr{B}(X)} \ge c > 0$ *by assumption, we deduce that* $\hat{h}_n := h_n/\|h_n\|_X$ *satisfies*

$$\hat{h}_n = \check{w}_n + \check{v}_n, \quad \check{w}_n := \check{\mathcal{W}}(\lambda_n)\hat{h}_n, \quad \check{v}_n := \check{\mathcal{V}}(\lambda_n)\hat{\varepsilon}_n,$$

with $\|\hat{h}_n\| = 1$ *and* $\hat{\varepsilon}_n := \varepsilon_n/\|h_n\|_X \to 0$. *Similarly as in the proof of Lemma 2.8, we conclude to the existence of subsequence* $(\hat{h}_{n_k})$ *and* $h_1 \in X_+ \setminus \{0\}$ *such that* $\hat{h}_{n_k} \to h_1$ *strongly in* X. *Defining* $f_1 := \mathcal{R}_{\mathcal{B}}(\lambda_1)h_1/\|\mathcal{R}_{\mathcal{B}}(\lambda_1)h_1\|$, *we have again* $\hat{f}_{n_k} \to f_1$ *strongly in* X *and next that condition* **(H3)** *holds.*

As we see now, strong compactness is not really necessary.

Lemma 2.11. *We assume* **(H1)**-**(H2)**-**(HS1)** *and there exists* $N \ge 1$ *such that*

$$\mathcal{W}(\alpha) : X \to \mathcal{X}_1 \subset X \quad \text{is positive and uniformly bounded in} \quad \alpha \ge \kappa_0$$

and, denoting $\mathcal{X}_0 := X$, *we assume that for any* $R_1 \ge R_0 > 0$ *the set*

$$\mathcal{C} = \mathcal{C}_{R_0,R_1} := \{g \in X_+;\ \|g\|_{\mathcal{X}_0} \ge R_0,\ \|g\|_{\mathcal{X}_1} \le R_1\} \tag{2.26}$$

is relatively sequentially compact for the weak topology $\sigma(X,Y)$ *and* $0 \notin \overline{\mathcal{C}}$, *where the closure is taken in the sense of the weak topology* $\sigma(X,Y)$. *Then condition* **(H3)** *holds.*

Remark 2.12. *When* $\mathcal{X}_1 \subset \mathcal{X}_0$ *with strongly compact embedding the above set* $\mathcal{C}$ *clearly satisfies the required conditions. In particular, Lemma 2.11 generalizes the result stated in Remark 2.9-(1).*

Proof of Lemma 2.11. We go back to the proof of Lemma 2.8 and we start with (2.25). We recall that $\|\hat f_n\|_{\mathcal{X}_0} = 1$ and $\|v_n\|_{\mathcal{X}_0} \to 0$ from (2.20) and that $w_n \ge 0$ because $\mathcal{W}(\lambda_n)$ is a positive operator. We also observe that

$$\|w_n\|_{\mathcal{X}_1} \le C_{\mathcal{W}} \|\hat f_n\|_{\mathcal{X}_0} = C_{\mathcal{W}}$$

and

$$\|w_n\|_{\mathcal{X}_0} \ge 1 - \|v_n\|_{\mathcal{X}_0} \ge 1/2$$

for any $n \ge n_*$, with $n_* \ge 1$ large enough, so that $w_n \in \mathcal{C} := \mathcal{C}_{1/2, C_{\mathcal{W}}}$ for any $n \ge n_*$. By the compactness properties of $\mathcal{C}$, there exist a subsequence (w_{n_k}) and $f_1 \in X_+\backslash\{0\}$ such that $w_{n_k} \rightharpoonup f_1$ weakly $\sigma(X,Y)$. Since $v_n \to 0$ strongly in X, we deduce that $\hat f_{n_k} \rightharpoonup f_1$ weakly $\sigma(X,Y)$ and that ends the proof of **(H3)**. □

We present a typical concrete application of the preceding result.

Lemma 2.13. *We assume $X = L^p(E, \mathscr{E}, \mu)$, $p \in [1,\infty)$,* **(H1)**-**(H2)**-**(HS1)** *with $\mathcal{A} \ge 0$, $\mathcal{R}_{\mathcal{B}}(\alpha) \ge 0$ for $\alpha \ge \kappa_0$, and there exists $N \ge 1$ such that*

$$\mathcal{W}(\alpha) : X \to \mathcal{X}_1 \quad \text{is uniformly bounded in} \quad \alpha \ge \kappa_0, \tag{2.27}$$

for a subspace $\mathcal{X}_1 \subset X$ such that $\{g^p;\ g \ge 0,\ \|g\|_{\mathcal{X}_1} \le R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$. Then condition **(H3)** *holds.*

Remark 2.14. *(1) A typical example in the above statement is $\mathcal{X}_1 := L^q \cap L^p_m$ for some exponent $q > p$ and some weight function $m : E \to [1,\infty)$ such that $m(x) \to \infty$ as $x \to \infty$.*
(2) The same result holds under the condition that if (u_n) is a nonnegative and bounded sequence in L^p then the nonnegative sequence $w_n := \mathcal{W}(\lambda_n) u_n$ is such that w_n^p is weakly compact in L^1.

Proof of Lemma 2.13. For $0 < R_0 < R_1$, we define $\mathcal{C}$ by (2.26) with $\mathcal{X}_0 := L^p$. From the weak compactness property made on $\mathcal{X}_1$, we observe that

$$\alpha(R) := \sup_{g \in \mathcal{C}} \|g \mathbf{1}_{E_R^c}\|_{L^p} \to 0, \text{ as } R \to \infty,$$

and

$$\beta(M) := \sup_{g \in \mathcal{C}} \|g \mathbf{1}_{g \ge M}\|_{L^p} \to 0, \text{ as } M \to \infty.$$

For $g \in \mathcal{C}$, we may then write

$$R_0 \le \|g\|_{L^p} \le \|g \wedge M \mathbf{1}_{E_R}\|_{L^p} + \|g \mathbf{1}_{E_R^c}\|_{L^p} + \|g \mathbf{1}_{g \ge M}\|_{L^p}$$

and thus

$$M^{1-1/p} \|g \mathbf{1}_{E_R}\|_{L^1}^{1/p} \ge \|g \wedge M \mathbf{1}_{E_R}\|_{L^p} \ge R_0 - \alpha(R) - \beta(M) \ge R_0/2,$$

for some $R, M > 0$ large enough. On the one hand, from the reflexivity of L^p or the Dunford-Pettis theorem, the set $\mathcal{C}$ is relatively sequentially compact for the weak topology $\sigma(L^p, L^{p'})$. On the other hand, because $\mathbf{1}_{E_R} \in L^{p'}$ the last estimate implies that any element $g^* \in \overline{\mathcal{C}}$, where the closure is taken in the sense of the weak topology $\sigma(L^p, L^{p'})$, satisfies

$$\langle g^*, \mathbf{1}_{E_R} \rangle = \|g^* \mathbf{1}_{E_R}\|_{L^1} \ge M^{1-p} (R_0/2)^p > 0,$$

and in particular $0 \notin \overline{\mathcal{C}}$. We deduce that **(H3)** holds as a consequence of Lemma 2.11. □

We present a second kind of result where some weak compactness is involved.

Lemma 2.15. *We assume* **(H1)**-**(H2)**-**(HS1)** *and there exists $N \ge 1$ such that*

$$\mathcal{W}(\alpha) : \mathcal{X}_0 \to X \subset \mathcal{X}_0 \quad \text{is uniformly bounded in} \quad \alpha \ge \kappa_0 \tag{2.28}$$

and, denoting $\mathcal{X}_1 := X$, the set $\mathcal{C}$ defined by (2.26) *satisfies the same properties as the ones stated in Lemma 2.11. Then condition* **(H3)** *holds.*

Remark 2.16. *If we replace the norm $\|\cdot\|_{\mathcal{X}_0}$ by a seminorm $\|f\|_{\mathcal{X}_0} := \langle |f|, \varphi_0 \rangle$, $\varphi_0 \in Y_+$, and we define $\mathcal{C}$ accordingly by* (2.26)*, and if we assume that $X = Y'$ with Y separable, then $\mathcal{C}$ satisfies the same compactness properties as required in the statement of Lemma 2.11. If we further assume that* (2.28) *holds where $\mathcal{X}_0$ is endowed with the above seminorm, we may repeat the proof below in order to obtain that* **(H3)** *holds in that situation (see also Lemma 2.19 and its proof for a slightly generalized situation).*

Proof of Lemma 2.15. We start here again with (2.25). We have

$$1 = \|\hat f_n\|_{\mathcal{X}_1} \le C_{\mathcal{W}} \|\hat f_n\|_{\mathcal{X}_0} + \|v_n\|_{\mathcal{X}_1},$$

and thus

$$\|\hat f_n\|_{\mathcal{X}_0} \ge C_{\mathcal{W}}^{-1}(1 - \|v_n\|_{\mathcal{X}_1}) \ge (2C_{\mathcal{W}})^{-1}$$

for any $n \ge n_*$, with $n_* \ge 1$ large enough, so that $\hat f_n \in \mathcal{C} := \mathcal{C}_{(2C_{\mathcal{W}})^{-1},1}$, for $n \ge n_*$. By the compactness properties of $\mathcal{C}$, there exist a subsequence $(\hat f_{n_k})$ and $f_1 \in X_+\backslash\{0\}$ such that $\hat f_{n_k} \rightharpoonup f_1$ weakly $\sigma(X,Y)$. ☐

We present a variant of Lemma 2.13 which is also a concrete consequence of Lemma 2.11 and Lemma 2.15.

Corollary 2.17. *We assume* **(H1)**-**(H2)**-**(HS1)** *in $X = L^{p_0}_{m_0}$, $1 \le p_0 < \infty$, together with the facts that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is positive and bounded in $\mathscr{B}(L^{p_0}_{m_0})$ and $\mathscr{B}(L^{p_1}_{m_1})$ uniformly in $\alpha \ge \kappa_0$, $0 \le \mathcal{A} \in \mathscr{B}(L^{p_0}_{m_0})$ and $(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N$ is bounded in $\mathscr{B}(L^{p_0}_{m_0}, L^{p_1}_{m_1})$ uniformly in $\alpha \ge \kappa_0$ for some $N \ge 1$, with $p_1 > p_0$ and m_1 such that $m_0/m_1 \in L^\vartheta$, $1/\vartheta := 1/p_0 - 1/p_1$. Then condition* **(H3)** *holds for both the primal and the dual problems.*

Proof of Corollary 2.17. On the one hand, we have

$$\begin{aligned}&\mathcal{R}_{\mathcal{B}}(\alpha) + \dots + (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^{N-1}\mathcal{R}_{\mathcal{B}}(\alpha) \text{ is bounded in } \mathscr{B}(X) \text{ uniformly in } \alpha \ge \kappa_0,\\ &\mathcal{W}(\alpha) := (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N \text{ is bounded in } \mathscr{B}(X, \mathcal{X}_1) \text{ uniformly in } \alpha \ge \kappa_0,\end{aligned}$$

with $\mathcal{X}_1 := L^{p_1}_{m_1} \subset X$ and thus $\{(gm_0)^{p_0};\ g \ge 0,\ \|g\|_{\mathcal{X}_1} \le R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$. Condition **(H3)** holds for the direct problem thanks to Lemma 2.13.
On the other hand, we set $Y := X' = L^{q_0}_{\nu_0}$, $q_0 := p_0'$, $\nu_0 := m_0^{-1}$, and we first observe that

$$\mathcal{R}_{\mathcal{B}^*}(\alpha) + \dots + (\mathcal{R}_{\mathcal{B}^*}(\alpha)\mathcal{A}^*)^{N-1}\mathcal{R}_{\mathcal{B}^*}(\alpha) \text{ is bounded in } \mathscr{B}(Y) \text{ uniformly in } \alpha \ge \kappa_0.$$

We next observe that

$$(\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*}(\alpha))^{N+1} = \mathcal{A}^*\mathcal{W}(\alpha)^*\mathcal{R}_{\mathcal{B}^*}(\alpha) \text{ is bounded in } \mathscr{B}(\mathcal{Y}_0, Y) \text{ uniformly in } \alpha \ge \kappa_0,$$

with $\mathcal{Y}_0 := L^{q_1}_{\nu_1}$, $q_1 := p_1'$, $\nu_1 := m_1^{-1}$. Because $\{(g\nu_1)^{q_1};\ g \ge 0,\ \|g\|_Y \le R_1\}$ is a weakly compact subset of $L^1(E)$ for any $R_1 > 0$, we have from the proof of Lemma 2.13 that the set $\mathcal{C}$ defined by (2.26) for the norms of $\mathcal{Y}_0$ and $\mathcal{Y}_1 := Y$ satisfies the weak compactness property required in the statement of Lemma 2.11. We may thus apply Lemma 2.15 and we deduce that condition **(H3)** holds for the dual problem. ☐

Another concrete consequence of Lemma 2.11 and Lemma 2.15 is the following.

Lemma 2.18. *We assume $X = M^1_{m_i}(E)$ for a continuous weight function m_i on E, $i = 0$ or $i = 1$,* **(H1)**-**(H2)**-**(HS1)** *and there exists $N \ge 1$ such that $(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N : M^1_{m_0}(E) \to M^1_{m_1}(E)$ uniformly in $\alpha \ge \kappa_0$ for another continuous weight function m_{1-i} on E such that $m_1(x)/m_0(x) \to \infty$ as $x \to \infty$. We additionally assume that $\mathcal{A} \ge 0$ and $\mathcal{R}_{\mathcal{B}}(\alpha) \ge 0$ for $\alpha \ge \kappa_0$ when $i = 0$. Then condition* **(H3)** *holds.*

Proof of Lemma 2.18. We define $\mathcal{X}_i := M^1_{m_i}(E)$ and we consider the set $\mathcal{C}$ defined by (2.26) which is clearly compact for the weak $*$ $\sigma(M^1_{m_1}, C_{m_1,0})$ topology. When $X = M^1_{m_0}$, the result follows from Lemma 2.11 while when $X = M^1_{m_1}$, the result is a consequence of Lemma 2.15. ☐

We may slightly improve the preceding results by considering a more abstract framework and a somehow more general boundedness condition.

Lemma 2.19. *We assume $X = Y'$, Y separable,* **(H1)**-**(H2)**-**(HS1)** *and there exist $N \ge 1$, $\gamma \in [0,1)$ and $\varphi \in Y_+ \setminus \{0\}$ such that for any $\alpha \ge \kappa_0$, there holds*

$$\|\mathcal{W}(\alpha)f\|_X \le \gamma\|f\|_X + \langle f, \varphi\rangle_{X,Y}, \tag{2.29}$$

for all $f \in X_+$, or there holds

$$\|\mathcal{W}(\alpha)f\|_X \le \gamma\|f\|_X + \langle \mathcal{W}(\alpha)f, \varphi\rangle_{X,Y}, \tag{2.30}$$

for all $f \in X_+$. Then condition **(H3)** *holds true, and the limit f_1 satisfies $\langle f_1, \varphi\rangle_{X,Y} \ge 1 - \gamma > 0$.*

The case $X = M^1_{m_1}(E)$ in Lemma 2.18 corresponds here to the first situation where (2.29) holds with $X := M^1_{m_1}(E)$, $\gamma := 0$, $Y := C_{m_0,0}(E)$ and $\varphi := m_0/m_1$.

Proof of Lemma 2.19. Starting with (2.25) and using (2.29), we have

$$\begin{aligned} \|\hat f_n\|_X &\le \|\mathcal{W}(\lambda_n)\hat f_n\|_X + \|\mathcal{V}(\lambda_n)\varepsilon_n\|_X \\ &\le \gamma\|\hat f_n\|_X + \langle \hat f_n, \varphi\rangle_{X,Y} + \|v_n\|_X, \end{aligned}$$

so that

$$\langle \hat f_n, \varphi\rangle_{X,Y} \ge 1 - \gamma - \|v_n\|_X.$$

By compactness, there are $f_1 \ge 0$ and a subsequence $(\hat f_{n'})$ such that $\hat f_{n'} \rightharpoonup f_1$ weak $*$ $\sigma(X,Y)$. Passing to the limit as $n' \to \infty$ in the above estimate, we find

$$\langle f_1, \varphi\rangle_{X,Y} = \lim_{n'\to\infty} \langle \hat f_{n'}, \varphi\rangle_{X,Y} \ge 1 - \gamma, \tag{2.31}$$

and in particular $f_1 \ne 0$.

Under the assumption (2.30), modifying slightly the previous argument, we have

$$\|\hat f_n\|_X \le \gamma\|\hat f_n\|_X + \langle w_n, \varphi\rangle_{X,Y} + \|v_n\|_X,$$

which, together with

$$\langle \hat f_n, \varphi\rangle_{X,Y} = \langle w_n, \varphi\rangle_{X,Y} + \langle v_n, \varphi\rangle_{X,Y},$$

implies

$$\langle \hat f_n, \varphi\rangle_{X,Y} \ge 1 - \gamma - \|v_n\|_X + \langle v_n, \varphi\rangle_{X,Y}.$$

By compactness again, there are $f_1 \ge 0$ and a subsequence $(\hat f_{n'})$ such that $\hat f_{n'} \rightharpoonup f_1$ weak $*$ $\sigma(X,Y)$, and passing to the limit $n' \to \infty$ in the above estimate, we conclude again to (2.31). □

Let us comment on Lemma 2.19 and in particular the condition (2.30).

In the case when $X = L^\infty(E, \mathscr{E}, \mu) = \big(L^1(E, \mathscr{E}, \mu)\big)'$, we can relate condition (2.30) to the assumption that there exist $f_0 \in X_+$ and $\varphi \in Y_+ \setminus \{0\}$ such that

$$\|S_{\mathcal{L}}(t) f_0\|_X \le \langle S_{\mathcal{L}}(t) f_0, \varphi\rangle, \quad \forall\, t \ge 0. \tag{2.32}$$

This last condition is reminiscent from conditions that appear in probabilistic inspired methods for the ergodicity of semigroups, see the condition *(1b)* in Theorem 1.7 but also Assumption (A2) in [103], both in the vein of [131, Condition $\mathscr{Z}$]. Assume indeed (2.32), let $\eta > \kappa_1 - \kappa_0 > 0$ and consider the trivial decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B} = \eta + (\mathcal{L} - \eta)$. Then set $\kappa_{\mathcal{B}} := \kappa_1 - \eta < \kappa_0$, so that for any $\alpha > \kappa_{\mathcal{B}}$, $\mathcal{B} - \alpha = \mathcal{L} - (\eta + \alpha)$ is invertible since $\eta + \alpha > \eta + \kappa_{\mathcal{B}} = \kappa_1$. We thus have for any $\alpha > \kappa_{\mathcal{B}}$

$$\mathcal{W}(\alpha) := \eta(\alpha - \mathcal{B})^{-1} = \eta \int_0^\infty e^{-(\eta+\alpha)t} S_{\mathcal{L}}(t)\, dt$$

and (2.32) then ensures that

$$\|\mathcal{W}(\alpha) f_0\|_X \le \langle \mathcal{W}(\alpha) f_0, \varphi\rangle.$$

We recover (2.30) with $\gamma = 0$ and the difference that f_0 is fixed here.

As a Corollary of Lemma 2.18 or Lemma 2.19 and anticipating on the material of part 3, we present now a situation very classical in stochastic processes theory.

Corollary 2.20. *We consider a positive semigroup $S = S_{\mathcal{L}}$ defined on a Radon space $X = M^1_\psi(E)$ for some positive weight functions ψ on E, in particular* **(H1)** *holds. We also assume that* **(H2)** *holds for some $\kappa_0 \in \mathbb{R}$. We finally assume the Lyapunov condition*

$$\mathcal{L}^*\psi \le \kappa_{\mathcal{B}}\psi + M\chi, \tag{2.33}$$

with $\kappa_{\mathcal{B}} < \kappa_0$, $M \ge 0$ and $\chi \in C_{\psi,0}(E)$, $0 \le \chi \le \psi$. Then condition **(H3)** *holds true.*

Let us emphasize that we may assume some regularity on ψ by considering $\psi \in D(\mathcal{L}^*)$ so that (2.33) makes sense in X or just understand (2.33) in the weak sense:

$$\langle \mathcal{L} f, \psi\rangle \le \kappa_{\mathcal{B}}\langle f, \psi\rangle + M\langle f, \chi\rangle, \quad \forall\, f \in D(\mathcal{L}) \cap X_+.$$

Proof of Corollary 2.20. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is the bounded multiplicator operator $\mathcal{A} := M\chi/\psi$. As a bounded perturbation of $\mathcal{L}$, the operator $\mathcal{B}$ is the generator of a semigroup $S_{\mathcal{B}}$. Defining $\widetilde{S}_t := S_{\mathcal{L}}(t)e^{-Mt} \ge 0$ and $\mathcal{A}^c := M(1-\chi/\psi) \ge 0$, we have the Duhamel formula

$$S_{\mathcal{B}} = \widetilde{S} + \widetilde{S}\mathcal{A}^c * S_{\mathcal{B}}$$

and iterating infinitely many times, we deduce the Dyson-Philips formula

$$S_{\mathcal{B}} = \sum_{k=0}^{\infty} (\widetilde{S}\mathcal{A}^c)^{(*k)} * \widetilde{S}.$$

That implies that $S_{\mathcal{B}} \ge 0$ as a combination of positive operators. Alternatively, from the very definition of $\mathcal{B}$, we have $\kappa - \mathcal{B} \le (M+\kappa) - \mathcal{L}$ for any $\kappa \in \mathbb{R}$. Choosing $\kappa > \max(\omega(S_{\mathcal{L}}), \omega(S_{\mathcal{B}}))$ and using the direct implication in Lemma 2.1, we have $\mathcal{R}_{\mathcal{B}}(\kappa) \ge \mathcal{R}_{\mathcal{L}}(M+\kappa) \ge 0$. Using the reciprocal implication in Lemma 2.1, we obtain again that $S_{\mathcal{B}} \ge 0$.
Now, for $0 \le f_0 \in D(\mathcal{B})$ and setting $f_t := S_{\mathcal{B}}(t) f_0$, we may compute

$$\frac{d}{dt}\langle f_t, \psi\rangle = \langle \mathcal{B} f_t, \psi\rangle \le \kappa_{\mathcal{B}} \langle f_t, \psi\rangle,$$

so that

$$\|S_{\mathcal{B}}(t) f_0\|_{M^1_\psi} \le e^{\kappa_{\mathcal{B}} t} \|f_0\|_{M^1_\psi}.$$

Using (2.13) we immediately and classically deduce

$$\|\mathcal{R}_{\mathcal{B}}(\alpha)\|_{\mathscr{B}(M^1_\psi)} \le \frac{1}{\alpha - \kappa_{\mathcal{B}}}, \quad \forall\, \alpha > \kappa_{\mathcal{B}},$$

so that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is bounded in $\mathscr{B}(M^1_\psi)$ and $\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ is bounded in $\mathscr{B}(M^1_\chi, M^1_\psi)$ uniformly for $\alpha \ge \kappa_0$. We apply Lemma 2.18 or Lemma 2.19 ((2.29) with $N = 1$, $\gamma = 0$ and $\varphi = \frac{M}{\alpha - \kappa_{\mathcal{B}}}\chi$) in order to conclude. ☐

In the proof of Corollary 2.20, we may alternatively use the trivial splitting $\mathcal{L} = \tilde{\mathcal{A}} + \tilde{\mathcal{B}} = \eta + (\mathcal{L} - \eta)$ for some $\eta > \kappa_1 - \kappa_0$, so that $\alpha - \tilde{\mathcal{B}}$ is invertible for any $\alpha \ge \kappa_0$, and reformulate the Lyapunov condition

$$(\alpha - \tilde{\mathcal{B}}^*)\psi \ge (\alpha + \eta - \kappa_{\mathcal{B}})\psi - M\chi,$$

for any $\alpha \ge \kappa_0$. Observing that $\widetilde{\mathcal{W}}(\alpha) := \tilde{\mathcal{A}}\mathcal{R}_{\tilde{\mathcal{B}}}(\alpha) = \eta(\alpha - \tilde{\mathcal{B}})^{-1}$, we deduce

$$\widetilde{\mathcal{W}}^*(\alpha)\psi \le \frac{\eta}{\eta + \alpha - \kappa_{\mathcal{B}}}\psi + \frac{M}{\eta + \alpha - \kappa_{\mathcal{B}}}\widetilde{\mathcal{W}}^*(\alpha)\chi.$$

We equivalently have

$$\|\widetilde{\mathcal{W}}(\alpha) f\|_{M^1_\psi} \le \gamma \|f\|_{M^1_\psi} + \langle \widetilde{\mathcal{W}}(\alpha) f, \varphi\rangle,$$

uniformly for any $\alpha \ge \kappa_0$, with $\gamma := \frac{\eta}{\eta + \kappa_0 - \kappa_{\mathcal{B}}} < 1$ and $\varphi := \frac{M}{r + \kappa_0 - \kappa_{\mathcal{B}}}\chi$, which is nothing but condition (2.30).

We finally come to the existence of a solution to the first eigenvalue problem and the first eigentriplet problem.

Theorem 2.21. *Under conditions* **(H1)-(H2)-(H3)***, the first eigenvalue problem* (2.14) *has a solution* (λ_1, f_1) *with* λ_1 *satisfying* (2.19)*. When furthermore* **(H3)** *holds for the dual problem, then the first eigentriplet problem* (1.1)-(1.2) *admits a solution* $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$.

Theorem 2.21 generalizes some known versions of the existence part of the Krein-Rutman Theorem where either $\mathcal{L}$ is assumed additionally to be the generator of a semigroup or to have strongly power compact resolvent or even where some additional conditions are made on the positive cone X_+. As mentioned in the introduction, some possible references for these previous results are Krein-Rutman [238], Greiner in [187, Cor 1.2] and in [15, C-IV Thm. 2.1] and Webb [360, Prop. 2.5], see also [75, Thm. 2], [278, Thm. 5.3], [252], [35, Thm. 2.1], the textbook [41, Thm. 12.15] and the references therein.

Proof of Theorem 2.21. We first assume **(H1)-(H2)-(H3)**. Because of Lemma 2.6, there exists a sequence $(\hat{f}_n)$ of X such that (2.20) holds, and in particular

$$\langle \lambda_n \hat{f}_n, \phi\rangle - \langle \hat{f}_n, \mathcal{L}^*\phi\rangle = \langle \varepsilon_n, \phi\rangle, \tag{2.34}$$

for any $\phi \in D(\mathcal{L}^*)$. Because of condition **(H3)**, we may pass to the limit $n' \to \infty$ in equation (2.34) and we deduce that (λ_1, f_1) satisfies (2.14) and (2.19).

We now additionally assume that **(H3)** holds for the dual problem. As recalled during the proof of Lemma 2.3 and by definition of λ_1, we have $(\lambda_1, +\infty) \subset \rho(\mathcal{L}) = \rho(\mathcal{L}^*)$ and $\lambda_1 \in \Sigma(\mathcal{L}) = \Sigma(\mathcal{L}^*)$. Taking $\lambda_n \searrow \lambda_1$, we argue as in the proof of Lemma 2.6 and we get

$$\exists\, \widehat{\phi}_n \geq 0,\ \lambda_n \widehat{\phi}_n - \mathcal{L}^* \widehat{\phi}_n \to 0,\ \|\widehat{\phi}_n\| = 1.$$

Thanks to **(H3)** for the dual problem, we deduce that there exist a subsequence $(\widehat{\phi}_{n_k})$ and $\phi_1 \in X'$, $\|\phi_1\| = 1$ such that $\widehat{\phi}_{n_k} \to \phi_1$. We thus conclude that ϕ_1 is a solution to the dual problem (1.2) (for the same eigenvalue λ_1). ☐

Let us conclude this section by some remarks.

Remark 2.22. *(1) - As seen above, the condition* **(H1)**-**(H2)** *for the primal and the dual problems are equivalent, and thus one only has to check* **(H1)**-**(H2)**-**(H3)** *for the primal problem and* **(H3)** *for the dual problem in order to solve the first eigentriplet problem. It is worth emphasizing that condition* **(H3)** *on the dual problem is not a consequence of the condition* **(H3)** *on the primal problem. However, as presented in Lemma 2.7, Lemma 2.8 and Corollary 2.17, there exist several natural situations where both conditions* **(H3)** *for the primal and the dual problems hold together.*
(2) - Alternatively, one may also assume **(H1)**-**(H2)**-**(H3)** *for the dual problem, and then use a more classical fixed point theorem for proving the existence of a steady state for the rescaled semigroup by using for instance the Markov–Kakutani fixed point theorem* [226] *as in* [231, Thm. 5.1], *the Tychonov fixed point theorem as in* [172] *or* [153, Thm. 1.2] *or a Birkhoff-Von Neumann type Theorem as in* [81, Thm. 6.1]. *For these last techniques, we also refer to Section 3, where such a dynamical approach is adapted to the present context. One may also use the usual Doblin-Harris theory, see for instance* [198, 81] *and the references therein, and Sections 8.6 and 12.2 for applications of this approach.*

2.3. **Discussion.**

We discuss now the existence results presented in the preceding section.

For further references, let us first recall that when X is a Hilbert space and $\mathcal{L}$ is self-adjoint, the first eigenvalue may be simply obtained thanks to the variational problem

$$\lambda_1 = \sup_{f \in X_+ \backslash \{0\}} \frac{\langle \mathcal{L} f, f\rangle}{\|f\|^2}. \tag{2.35}$$

We now explain how Theorem 2.21 is a generalization of the classical Krein-Rutman theorem stated in Theorem 1.2. We thus consider a Banach lattice X such that $X_{++} := \mathrm{int} X_+ \neq \emptyset$ and an operator $\mathcal{L}$ such that, for $\kappa_1 \in \mathbb{R}$ and any $\kappa > \kappa_1$, $\mathcal{R} := (\kappa - \mathcal{L})^{-1} : X \to X$ is compact and $\mathcal{R} : X_+ \backslash \{0\} \to X_{++}$, in particular **(H1)** holds true. As a first step, we recall the following very classical technical lemma of the KR theory.

Lemma 2.23. *Assume $X_{++} := \mathrm{int}\, X_+ \neq \emptyset$. For $g \in X_+$ and $f \in X_{++}$, there exists $C \geq 0$ such that $g \leq C f$.*

Proof of Lemma 2.23. We argue by contradiction. Otherwise, for any $n \geq 1$, we would have $f - g/n \in X_+^c \subset X_{++}^c$ and that last set is closed. Passing to the limit, we get $f \in X_{++}^c$, which is in contradiction with the assumption $f \in X_{++}$. ☐

For a given $g_0 \in X_+ \backslash \{0\}$, we set $f_0 := \mathcal{R} g_0 \in X_{++}$. From Lemma 2.23, there exists $C_0 \geq 0$ such that $(\kappa - \mathcal{L}) f_0 = g_0 \leq C_0 f_0$. That implies that Lemma 2.4-**(ii)** holds with $\kappa_0 := \kappa - C_0$, and thus **(H2)** also holds. One may then define $\mu_1 := \kappa - \lambda_1$, with

$$\lambda_1 := \inf\{\lambda \in \mathbb{R};\ (\lambda' - \mathcal{L})^{-1} \in \mathscr{B}(X),\ \forall\, \lambda' \in [\lambda, \kappa]\} \geq \kappa_0.$$

We recall that because of Lemma 2.6 (or its proof), there exist (λ_n), $(\hat{f}_n)$ and (ε_n) such that (2.20) holds, namely

$$\lambda_n \searrow \lambda_1,\ \hat{f}_n \geq 0,\ \varepsilon_n := \lambda_n \hat{f}_n - \mathcal{L} \hat{f}_n \geq 0,\ \|\hat{f}_n\| = 1,\ \|\varepsilon_n\| \to 0.$$

We may rewrite the equation as

$$\hat{f}_n = \mathcal{R}[\varepsilon_n + (\kappa - \lambda_n) \hat{f}_n],$$

so that $(\hat{f}_n)$ belongs to a compact set of X because of the compactness assumption made on $\mathcal{R}$, so that **(H3)** holds true.
Because of Theorem 2.21, we deduce that there exists $f_1 \in X_+$ such that $\|f_1\| = 1$ and $\mathcal{L}f_1 = \lambda_1$. That implies $f_1 = \mu_1 \mathcal{R} f_1$, and thus that the existence part of Theorem 1.2 is a consequence of Theorem 2.21 for an operator $\mathcal{R}$ which is the positive resolvent of an operator $\mathcal{L}$.

We would like to emphasize on the fact that our definition of the first eigenvalue by (2.15)-(2.16) bears some strong similarity with the definition of the first eigenvalue for elliptic operators in non divergence form as presented in [47]. Indeed, if $\lambda \in \mathcal{I}$, then

$$\exists\, f \in X_+ \backslash \{0\}, \quad \mathcal{L} f \le \lambda f.$$

Assuming now that X is a space of functions (defined on a set E) and that $f(x) > 0$ for any $x \in E$, we deduce that

$$\lambda \ge \sup_E \frac{\mathcal{L} f}{f},$$

and thus λ_1 is characterized by

$$\lambda_1 = \inf_{f>0} \sup_E \frac{\mathcal{L} f}{f},$$

which is nothing but [47, (1.13)] (with a change of sign because of a different sign convention). We thus see that our formulation is a generalization at a more abstract level and for resolvent positive operators of that classical min-max approach for elliptic operators. Some more or less classical references on that subject are [142, 143], [302], [318], [48] and [46]. In particular in [48], two generalized principal eigenvalues

$$\lambda_1' := \sup\{\kappa_0 \in \mathbb{R};\, \exists\, g_0 \in \mathcal{C}_0 \ \mathcal{L} g_0 \ge \kappa_0 g_0\}$$

and

$$\lambda_1'' := \inf\{\kappa_1 \in \mathbb{R};\, \exists\, g_1 \in \mathcal{C}_1 \ \mathcal{L} g_1 \le \kappa_1 g_1\}$$

are defined for appropriate cones $\mathcal{C}_i \subset X_+ \backslash \{0\}$ for problems with lack of compactness. The links between the three quantities λ_1, λ_1' and λ_1'' are discussed as well as the possible non existence of a related principal eigenfunction f_1. The non existence of associated principal eigenfunction should not be a surprise since it is the case when one considers $\mathcal{L} = \Delta$ in $X = L^2(\mathbb{R}^d)$ where $\mathcal{L} g_1 = \mathcal{L}^* g_1 = \lambda_1'' \psi$ with $0 < \psi = 1 \notin X = X'$ and $\lambda_1'' = 0$, but no associated principal eigenfunction exists in X. We also refer to [252] where some examples of such a situation are discussed.

For its own interest and further discussions, we finally state and prove a slightly less general variant of Theorem 1.8.

Theorem 2.24. *Consider a Banach lattice with positive cone X_+ and a linear and bounded operator $\mathcal{R} : X \to X$ such that*
(i) $\mathcal{R} : X_+ \to X_+$;
(ii) $\exists\, g_2 \in X_+ \backslash \{0\}$, $\exists\, C_2 > 0$ such that $\mathcal{R} g_2 \le C_2 g_2$.
We define

$$K_2 := \{g \in X_+;\, \exists a > 0,\, g \le a g_2\},$$

and next

$$A(g) := \inf\{a > 0;\, g \le a g_2\}, \quad \text{if} \ \ g \in K_2,$$

as well as

$$\mathcal{J} := \{\mu \ge 0;\ \exists h \in K_2,\ h \ge \mu \mathcal{R} h + g_2\}.$$

We further assume
(iii) $\mu_1 := \sup \mathcal{J} < +\infty$.
(iv) Any upper bounded and increasing sequences (g^n) of X is convergent in the weak sense $\sigma(X, Y)$. More precisely, if $g_n \le g_{n+1} \le \bar g \in X$ for any $n \ge 1$, there exists $g \in X$, $g \le \bar g$, such that $g_n \rightharpoonup g$.
(v) Any sequence (g^n) of normalized almost first eigenvectors is relatively compact. More precisely, for any sequence (g^n) of K_2 such that $A(g^n) = 1$, $g^n = \mu^n \mathcal{R} g^n + \varepsilon^n$, $\mu^n \nearrow \mu_1$ and $\varepsilon^n \to 0$, there exists $g \in K_2$ and a subsequence (g^{n_k}) such that $g^{n_k} \to g$ and $A(g) = 1$.
Then there exists $f_1 \in X_2$ such that $f_1 = \mu_1 \mathcal{R} f_1$ and $A(f_1) = 1$.

Proof of Theorem 2.24. We split the proof into three steps.

Step 1. We first establish that for any $\mu \in \mathcal{J}$, there exists $\tilde g = \tilde g_\mu \in K_2$ such that

$$\tilde g = \mu \mathcal{R} \tilde g + g_2. \tag{2.36}$$

We set $\tilde g_0 = 0$, $\tilde g_1 = g_2$, and we define $(\tilde g_n)$ recursively by $\tilde g_{n+1} = \mu \mathcal{R} \tilde g_n + g_2$, for any $n \ge 1$. We claim that

$$0 \le \tilde g_n \le \tilde g_{n+1} \le h, \quad \text{for any} \quad n \ge 0,$$

where h enters in the definition of $\mu \in \mathcal{J}$. That is obviously true at order $n = 0$. Assuming that last inequality is proved at order $n-1$ for $n \ge 1$, we compute

$$\tilde g_{n+1} = \mu \mathcal{R} \tilde g_n + g_2 \ge \mu \mathcal{R} \tilde g_{n-1} + g_2 = \tilde g_n$$

and

$$\tilde g_{n+1} = \mu \mathcal{R} \tilde g_n + g_2 \le \mu \mathcal{R} h + g_2 \le h,$$

which proves the same inequality at order n, and thus for any $n \ge 0$. Using the convergence property (iv) of upper bounded increasing sequences in X, we deduce that there exists $\tilde g \in X_2$ such that $\tilde g_n \to \tilde g$ and thus (2.36) holds.

Step 2. We obviously have $0 \in \mathcal{J}$ and $\mathcal{J}$ is an interval because if (μ, h) satisfies the condition $\mu \in \mathcal{J}$ then so do (μ', h) for any $\mu' \in [0, \mu]$. We finally claim that $\mathcal{J}$ is open. Take indeed $\mu \in \mathcal{J}$ and $\tilde g \in K_2$ such that (2.36) holds, what is possible due to Step 1. By definition, there would exist $A > 0$ such that $\tilde g \le A g_2$. Choosing $0 < \varepsilon \le 1/(2AC_2)$ and $M \ge 2$, we compute

$$\begin{aligned} (M\tilde g) - (\mu + \varepsilon)\mathcal{R}(M\tilde g) &= M g_2 - M\varepsilon \mathcal{R} \tilde g \\ &\ge M g_2 - M\varepsilon A \mathcal{R} g_2 \ge M(1 - \varepsilon A C_2) g_2 \ge g_2, \end{aligned}$$

so that $\mu + \varepsilon \in \mathcal{J}$.

Step 3. We first establish by contradiction that $A(\tilde g_\mu) \nearrow \infty$ when $\mu \nearrow \mu_1$. If it was not the case, there exists $A \in (0, \infty)$ and a sequence (μ^n) such that $A(\tilde g_{\mu^n}) \le A$ as $\mu^n \nearrow \mu_1$. Choosing $0 < \varepsilon \le 1/(2AC_2)$ and $M \ge 2$ as in Step 2, the same computation gives

$$(M\tilde g_{\mu^n}) - (\mu + \varepsilon)\mathcal{R}(M\tilde g_{\mu^n}) \ge g_2,$$

so that $\mu^n + \varepsilon \in \mathcal{J}$. That means that $\mu^n + \varepsilon \le \mu_1$, and a contradiction with the fact $\mu^n \nearrow \mu_1$. We next consider $\mu^n \nearrow \mu_1$ and we define

$$A^n := A(\tilde g_{\mu^n}), \quad \hat g^n := \frac{\tilde g_{\mu^n}}{A^n}, \quad \varepsilon_n := \frac{g_2}{A^n}, \quad \hat g^n = \mu^n T \hat g^n + \varepsilon_n.$$

We observe that $\varepsilon_n \to 0$ and $A(\hat g^n) = 1$. Because of the compactness assumption *(v)*, we deduce that there exists $f_1 \in K_2$ and a subsequence $(\hat g^{n_k})$ such that $\hat g^{n_k} \to f_1$ and $A(f_1) = 1$. We conclude by passing to the limit in the above almost first eigenvalue equations. □

We may compare Theorem 2.24 with the results presented in the previous section. When $\mathcal{L}$ satisfies condition **(H1)**, we may set $\mathcal{R} := \mathcal{R}_{\mathcal{L}}(\kappa_1)$ so that $\mathcal{R} \in \mathscr{B}(X)$ and $\mathcal{R}$ satisfies (i). In that case, Theorem 2.24 claims the existence of $f_1 \in K_2$ such that $\mathcal{L} f_1 = \lambda_1 f_1$, with $\lambda_1 := \kappa_1 - \mu_1$. The condition (ii) on $\mathcal{R}$ translates as $\mathcal{L} g_2 \le (\kappa_1 - 1/C_2) g_2$ which may be seen as an equivalent of condition **(H1)** (when working in the space $X_2 := K_2 - K_2$ with norm $\|g\|_2 := A(|g|)$ and $\mathcal{L}$ generates a semigroup S. The hypothesis (iii) is nothing but **(H2)** and the hypothesis (iv) is very natural: it holds in the space $L^p(E)$ and $M^1(E)$ without additional condition on $\mathcal{R}$ and it holds in a space of continuous functions when some additional uniform continuity assumption is made on the range of $\mathcal{R}$. Assumption (v) has to be compared with condition **(H3)**. It is worth emphasizing that when $X \subset L^p(E)$ and $g_2 > 0$ a.e., we simply have $A(g) = \|g/g_2\|_{L^\infty}$ for any $g \in X_+$. As a conclusion, although Theorem 2.21 and Theorem 2.24 bear some similarities none seems to be a consequence of the other. We believe that Theorem 2.21 is more flexible since it does not impose to work with the normalization associated to the seminorm $g \mapsto A(|g|)$ of L^∞-type. It is also worth emphasizing the very similarity between Step 3 in the proof of Theorem 2.24 and the proof of Lemma 2.6 and, on the other hand, that Theorem 1.2 is a particular case of Theorem 2.24 by essentially exploiting the fundamental Lemma 2.23 as shown in [252]. We finally point out that when $Y = X'$, the condition (iv) is equivalent to a property of Banach lattices known as *order continuous norm*, see for instance [264, Definition 2.4.1], as a consequence of [264, Thm. 2.4.2 (iii)] along with the fact that weakly convergent increasing sequences in Banach lattices are automatically norm convergent, see *e.g.* [264, Prop. 1.4.1].

3. Existence through a dynamical approach

In this part, we develop a dynamical approach for proving the existence part of the Krein-Rutman Theorem. We thus always consider a positive semigroup $S = S_\mathcal{L}$ on a Banach lattice X. We recover Theorem 2.21 under slightly reinforced assumptions. Above all, we are able to extend the existence part of the Krein-Rutman Theorem to a more general framework, namely to the case when $\mathcal{L}$ only enjoys a suitable weakly dissipative condition.

3.1. **About dissipativity.**

Let us start by recalling some classical definitions and results. We say that an operator $\mathcal{L}$ defined in a Banach space X is dissipative if there is some number $\kappa \in \mathbb{R}$ such that

$$\forall\, f \in D(\mathcal{L}),\ \exists\, f^* \in J_f, \quad \Re e\langle f^*, \mathcal{L}f\rangle \le \kappa \|f\|^2,$$

where we define the associated dual set $J_f \subset X'$ of f by

$$J_f := \{\varphi \in X';\ \langle \varphi, f\rangle = \|f\| = \|\varphi\|_{X'}\}. \tag{3.1}$$

In that situation and in order to be more precise, we should say that $\mathcal{L} - \kappa$ is dissipative. It is worth emphasizing that $J_f \neq \emptyset$ thanks to the corollary (2.2) of the Hahn-Banach dominated extension theorem. We say that an operator $\mathcal{L}$ is hypodissipative in a Banach space X if there exist an equivalent norm $|\!|\!| \cdot |\!|\!|$ in X and a number $\kappa \in \mathbb{R}$ such that

$$\forall\, f \in D(\mathcal{L}),\ \exists\, f^* \in J_{f,|\!|\!|\cdot|\!|\!|}, \quad \Re e\langle f^*, \mathcal{L}f\rangle \le \kappa |\!|\!| f |\!|\!|^2, \tag{3.2}$$

where

$$J_{f,|\!|\!|\cdot|\!|\!|} := \{\varphi \in X';\ \langle \varphi, f\rangle = |\!|\!| f |\!|\!|^2 = |\!|\!| \varphi |\!|\!|^2_{X'}\}. \tag{3.3}$$

The only difference between the two definitions (3.1) and (3.3) comes from the norms in which the normalization is performed. When $\mathcal{L}$ is the generator of a semigroup $S_\mathcal{L}$, one can show that the growth bound $\omega = \omega(S_\mathcal{L})$ defined in (2.9) also satisfies

$$\omega = \inf\{\kappa \in \mathbb{R},\ (3.2) \text{ holds for some equivalent norm } |\!|\!| \cdot |\!|\!|\},$$

and $S_\mathcal{L}$ is a semigroup of contraction when $\mathcal{L}$ is dissipative with $\kappa = 0$. At least formally, denoting $f_t := S(t)f$, for $f \in D(\mathcal{L})$, we deduce from (3.2) that

$$\frac{1}{2}\frac{d}{dt} |\!|\!| f_t |\!|\!|^2 = \Re e\langle (f_t)^*, \mathcal{L}f_t\rangle \le \kappa |\!|\!| f_t |\!|\!|^2,$$

and together with the Grönwall lemma, we deduce

$$|\!|\!| S(t)f |\!|\!| \le e^{\kappa t} |\!|\!| f |\!|\!|, \quad \forall\, t \ge 0,$$

which is nothing but (2.10). That last estimate is actually equivalent to the hypodissipativity estimate (3.2). Quite similarly, when

$$\exists\, \psi \in Y_+\backslash\{0\},\ \exists \kappa \in \mathbb{R}, \quad \pm\mathcal{L}^*\psi \le \kappa\psi, \tag{3.4}$$

we may compute

$$\pm\frac{d}{dt}\langle f_t, \psi\rangle = \pm\langle \mathcal{L}f_t, \psi\rangle = \pm\langle f_t, \mathcal{L}^*\psi\rangle \le \kappa\langle f_t, \psi\rangle,$$

and together with the Grönwall lemma, we get

$$\pm\langle S_t f, \psi\rangle \le \pm e^{\pm\kappa t}\langle f, \psi\rangle, \quad \forall\, t \ge 0. \tag{3.5}$$

Two important more accurate versions of the previous ones are presented now. They will be of main importance in the sequel. On the one hand, we may assume that $\mathcal{L}$ satisfies a Lyapunov type condition, namely there exists $\psi_i \in Y_+$ and $\kappa \in \mathbb{R}$ such that

$$\mathcal{L}^*\psi_2 \le \kappa\psi_2 + \psi_0, \tag{3.6}$$

with $\psi_2 > 0$ and $\psi_0/\psi_2 \to 0$ at infinity. For $f_t = S_{\mathcal{L}}(t)f$, $f \in D(\mathcal{L}) \cap X_+$, a similar computation as above gives

$$\frac{d}{dt}\langle f_t, \psi_2\rangle = \langle f_t, \mathcal{L}^*\psi_2\rangle \le \kappa\langle f_t, \psi_2\rangle + \langle f_t, \psi_0\rangle.$$

Denoting $[f]_i := \langle |f|, \psi_i\rangle$ and using the Grönwall lemma, we classically deduce

$$[S(t)f]_2 \le e^{\kappa t}[f]_2 + \int_0^t e^{\kappa(t-s)}[S(s)f]_0\, ds, \quad \forall\, t \ge 0. \tag{3.7}$$

The Lyapunov condition (3.6) is particularly relevant and useful in a Radon measures space framework $X = M^1_{\psi_2}(E)$ for some weight function ψ_2 on E.

On the other hand, we may generalize the above Lyapunov condition by assuming the structure condition

(HS2) there exist a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\kappa_{\mathcal{B}} \in \mathbb{R}$ such that $\mathcal{A}$ is $\mathcal{B}$-bounded, that means

$$\exists\, C \ge 0,\ \forall\, f \in X, \quad \|\mathcal{A}f\| \le C(\|f\| + \|\mathcal{B}f\|),$$

the operator $\mathcal{B}$ generates a semigroup $S_{\mathcal{B}}$ and

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}(t)\|_{\mathscr{B}(X)} = \mathcal{O}(e^{\alpha t}), \qquad \forall\, t > 0, \tag{3.8}$$

for any $\ell \ge 0$ and $\alpha > \kappa_{\mathcal{B}}$.

Here and below, for two functions $U : \mathbb{R}_+ \to \mathscr{B}(\mathcal{X}_0, \mathcal{X}_1)$ and $V : \mathbb{R}_+ \to \mathscr{B}(\mathcal{X}_1, \mathcal{X}_2)$, we define the convolution function

$$(V * U)(t) := \int_0^t V(t-s)U(s)\, ds,$$

when the integral is well-defined. For $U : \mathbb{R}_+ \to \mathscr{B}(\mathcal{X})$, we also recursively define $U^{(*0)} = I$ and $U^{(*(\ell+1))} = U^{(*\ell)} * U$. Using this convolution notation, the Duhamel formula writes

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}},$$

and iterating this formula, for any $N \ge 1$, we get the following iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S_{\mathcal{L}}. \tag{3.9}$$

When $S_{\mathcal{L}}$ is well defined in another space $X_0 \supset X$ and the last iterated convolution term enjoys the regularity property $\|(S_{\mathcal{B}}\mathcal{A})^{(*N)}(t)\|_{\mathscr{B}(X_0,X)} = \mathcal{O}(e^{\alpha t})$ for all $t > 0$ and $\alpha > \kappa_{\mathcal{B}}$, we deduce from the above iterated Duhamel formula, the estimate

$$\|S(t)f\| \le C_0 e^{\alpha t}\|f\| + C_1 \int_0^t e^{\alpha(t-s)}\|S(s)f\|_0\, ds, \quad \forall\, t \ge 0,\ \alpha > \kappa_{\mathcal{B}},\ \forall\, f \in X, \tag{3.10}$$

for some constants $C_i \ge 1$ and where $\|\cdot\|_0$ stands for the norm in X_0. We may observe that the estimate (3.7) in the case of a Lyapunov condition is a particular case of (3.10) corresponding to the norms $\|\cdot\| = [\cdot]_2$ and $\|\cdot\|_0 = [\cdot]_0$. More specifically, in a Radon measures space framework, the splitting condition **(HS2)** is obtained by introducing the bounded operator $\mathcal{A}f := f\psi_0$ and the generator $\mathcal{B} := \mathcal{L} - \mathcal{A}$. Because of (3.6), we have $\mathcal{B}^*\psi_2 \le \kappa\psi_2$, and arguing as for establishing (3.7), we have $[S_{\mathcal{B}}(t)f]_2 \le e^{\kappa t}[f]_2$ for any $t \ge 0$ and $f \in X$. That last growth condition is equivalent to assuming that $\mathcal{B} - \kappa$ is dissipative for the norm $[\cdot]_2$, so that we have established that $\mathcal{L}$ enjoys the splitting condition **(HS2)**.

3.2. **Existence in the dissipative case.**

In this section, we give an existence result for a positive semigroup $S_{\mathcal{L}}$ on a Banach lattice X satisfying a kind of regularity/compactness assumption in the spirit of the structure condition **(HS2)** discussed above.

Theorem 3.1. *On a Banach lattice $X = Y'$, with Y separable Banach lattice, consider a positive semigroup $S = S_{\mathcal{L}}$ satisfying the growth bound (2.10), and set $\kappa_1 := \omega' + \log M$ for some $\omega' > \omega(S_{\mathcal{L}})$.*

We assume

(1) *$\exists\, \phi_0 \in Y_+\backslash\{0\}$, $\exists\, \kappa_0 \in \mathbb{R}$ such that $[S(t)f]_0 \ge e^{\kappa_0 t}[f]_0$ for any $t \ge 0$ and $f \in X_+$, where we denote $[f]_0 := \langle |f|, \phi_0\rangle$;*

(2) *there exist $\kappa, C_0, C_1 \in \mathbb{R}$ with $\kappa < \kappa_0$, $C_0 \ge 1$ and $C_1 \ge 0$, such that*

$$\|S(t)f\| \le C_0 e^{\kappa t}\|f\| + C_1 \int_0^t e^{\kappa(t-s)}[S(s)f]_0\, ds, \quad \forall\, t \ge 0,\ \forall\, f \in X. \tag{3.11}$$

Then there exist $\lambda_1 \in [\kappa_0, \kappa_1]$ and $f_1 \in X_+ \backslash \{0\}$ such that $\mathcal{L} f_1 = \lambda_1 f_1$.

Let us mention that this result shares similarities with [260, Cor. 2.7] and [111, Thm. 4.2], see also [259, 94] for earlier works in that direction.

Remark 3.2. *(1) Assumption* **(2)** *in the statement of Theorem 3.1 holds when there exist V, W such that*

$$S = V + W * S, \quad W \ge 0, \tag{3.12}$$

and there exist $\kappa, C_V, C_W \in \mathbb{R}$, $\kappa < \kappa_0$, $C_V \ge 1$, $C_W > 0$ such that

$$\|V(t)\|_{\mathscr{B}(X)} \le C_V e^{\kappa t}, \quad \|W(t)\|_{\mathscr{B}(\mathcal{X}_0, X)} \le C_W e^{\kappa t}. \tag{3.13}$$

(2) Under the structural condition **(HS2)** *together with some regularization effect on the semigroup of the type*

$$\|(S_\mathcal{B}\mathcal{A})^{(*N)}(t)\|_{\mathscr{B}(\mathcal{X}_0, X)} = \mathcal{O}(e^{\kappa t}), \qquad \forall\, t > 0,\ \kappa \in (\kappa_\mathcal{B}, \kappa_0),$$

we recover the above condition (3.12)-(3.13) *with*

$$V := S_\mathcal{B} + \cdots + (S_\mathcal{B}\mathcal{A})^{*(N-1)} * S_\mathcal{B}, \quad W := (S_\mathcal{B}\mathcal{A})^{(*N)}, \tag{3.14}$$

because of the iterated Duhamel formula (3.9)*. In that case, the representation formula* (2.13) *holds true for any $z > \lambda_1$ from Lemma 2.2-***(ii)** *and we easily compute*

$$\mathcal{R}_L(z) = \mathcal{V}(z) + \mathcal{W}(z)\mathcal{R}_L(z), \quad \forall\, z > \lambda_1,$$

with

$$\mathcal{V}(z) := \int_0^\infty e^{-\lambda t} V(t) dt, \quad \mathcal{W}(z) := \int_0^\infty e^{-\lambda t} W(t) dt, \quad \forall\, z > \kappa.$$

We observe that $\mathcal{W}$ satisfies (2.28) *in Lemma 2.15 if W satisfies* (3.13) *and the set $\mathcal{C}$ defined by* (2.26) *satisfies the same compactness properties as required in the statement of Lemma 2.11. We may thus apply Lemma 2.15 (see also Remark 2.16) and deduce that* **(H3)** *holds for the primal problem. We finally obtain the same conclusion as in Theorem 3.1 thanks to Theorem 2.21.*

(3) Under the same structural condition **(HS2)** *as above, but assuming now that*

$$\|W(t)\|_{\mathscr{B}(X, \mathcal{X}_1)} = \mathcal{O}(e^{\kappa t}), \qquad \forall\, t > 0,\ \kappa \in (\kappa_\mathcal{B}, \kappa_0),$$

with $W := (S_\mathcal{B}\mathcal{A})^{(*N)}$ and $\mathcal{X}_1 \subset X$ with strongly compact embedding, we observe that S does not necessary satisfies the assumptions of Theorem 3.1, but it rather satisfies the assumptions of Lemma 2.7 with $K_T := (W * S)(T)$ and $T > 0$ large enough. In that situation, we also obtain the same conclusion as in Theorem 3.1 thanks to Lemma 2.7 and Theorem 2.21.

Proof of Theorem 3.1. We split the proof into two steps.

Step 1. We define the set

$$\mathcal{C} := \{f \in X_+,\ [f]_0 = 1,\ \|f\| \le R\},$$

for a convenient constant $R > 0$ to be fixed later. For any fixed $t > 0$, we next define the nonlinear weakly $\sigma(X, Y)$ continuous mapping

$$\Phi_t : \mathcal{C} \to X, \quad f \mapsto \frac{S_t f}{[S_t f]_0}.$$

Thanks to assumption **(1)**, we may observe that it is well defined because

$$[S_t f]_0 \ge e^{\kappa_0 t}[f]_0 = e^{\kappa_0 t} > 0. \tag{3.15}$$

For any $f \in \mathcal{C}$, we thus immediately have $\Phi_t f \ge 0$ and $[\Phi_t f]_0 = 1$. On the other hand, from assumption **(1)** again and the semigroup property, we have

$$[S(t)f]_0 \ge e^{\kappa_0(t-s)}[S(s)f]_0. \tag{3.16}$$

For $f \in \mathcal{C}$ and $t \geq 0$, we next compute

$$\begin{aligned} \|\Phi_t f\| &\leq C_0 e^{-\alpha t}\|f\| + C_1 \int_0^t e^{-\alpha(t-s)}\, ds \\ &\leq C_0 e^{-\alpha t} R + \frac{C_1}{\alpha}, \end{aligned}$$

where we have set $\alpha := \kappa_0 - \kappa_{\mathcal{B}} > 0$. Fixing T_0 such that $C_0 e^{-\alpha T_0} \leq 1/2$ and next $R \geq 2C_1/\alpha$, we have thus $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$. Thanks to the Tykonov fixed point Theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0} f_{T_0} = f_{T_0}$. In other words, we have established the existence of $f_{T_0} \in X$ such that

$$f_{T_0} \geq 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}, \tag{3.17}$$

with $\lambda_1 := (1/T_0)\log[S_{T_0} f_{T_0}]_0 \in [\kappa_0, \kappa_1]$.

Step 2. Rewriting equation (3.17) as

$$0 = e^{-\lambda_1 T_0} S_{T_0} f_{T_0} - f_{T_0} = (\mathcal{L} - \lambda_1) \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt$$

and defining

$$f_1 := \int_0^{T_0} e^{-\lambda_1 t} S_t f_{T_0} dt,$$

we get that $f_1 \in X_+ \backslash \{0\}$ satisfies $\mathcal{L} f_1 = \lambda_1 f_1$. □

We present now a second proof based on a large times dynamical argument which is classical in the mean ergodicity theory of Von Neumann and Birkhoff introduced in [356, 60] and which will be adaped in the weak dissipativity case in Section 3.5 below.

Alternative Step 2. We define $\widetilde{S}_t := S_t e^{-\lambda_1 t}$, so that f_{T_0} becomes a periodic state for $\widetilde{S}_t$ from (3.17), namely

$$\widetilde{S}_t f_{T_0} = \widetilde{S}_{t-kT_0} f_{T_0}, \quad k := [t/T_0], \quad \forall\, t > 0.$$

Using (3.15) and the above relation, we have

$$\begin{aligned} [\widetilde{S}_t f_{T_0}]_0 &= [\widetilde{S}_{t-kT_0} f_{T_0}]_0 \\ &\geq e^{(\kappa_0-\lambda_1)(t-kT_0)}[f_{T_0}]_0 \geq e^{(\kappa_0-\lambda_1)T_0} =: r_* > 0, \end{aligned}$$

for any $t \geq 0$. On the other hand, thanks to the growth bound (2.10), we have

$$\begin{aligned} \|\widetilde{S}_t f_{T_0}\| &= \|\widetilde{S}_{t-kT_0} f_{T_0}\| \\ &\leq M e^{(\kappa-\lambda_1)(t-kT_0)}\|f_{T_0}\| \leq M e^{(\kappa-\lambda_1)T_0} R =: R^* < \infty, \end{aligned}$$

for any $t \geq 0$. We finally define

$$u_T := \frac{1}{T}\int_0^T \widetilde{S}_t f_{T_0}\, dt.$$

From the previous estimates, both sequences $(\widetilde{S}_t f_{T_0})$ and (u_T) are bounded in

$$\mathbb{K} := \{f \in X;\ f \geq 0,\ [f]_0 \geq r_*,\ \|f\| \leq R^*\}.$$

By compactness, there exists a subsequence (u_{T_k}) and $f_1 \in \mathbb{K}$ such that $u_{T_k} \rightharpoonup f_1$ in a weak sense as $k \to \infty$. For any fixed $t > 0$, we observe that

$$\begin{aligned} \widetilde{S}_t f_1 - f_1 &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_0^{T_k} \widetilde{S}_t\widetilde{S}_s f_{T_0} ds - \frac{1}{T_k}\int_0^{T_k} \widetilde{S}_s f_{T_0} ds\Big\} \\ &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_{T_k}^{T_k+t} \widetilde{S}_s f_{T_0} ds - \frac{1}{T_k}\int_0^t \widetilde{S}_s f_{T_0} ds\Big\} = 0, \end{aligned}$$

where we have used that $(\widetilde{S}_s f_{T_0})$ is uniformly bounded in the last line. As a consequence, f_1 is a stationary state for the rescaled semigroup $\widetilde{S}_t$, and thus an eigenfunction associated to the eigenvalue λ_1 for the operator $\mathcal{L}$. □

3.3. About weak dissipativity.

In this section, we recall some definitions and results about the weak dissipativity. We say that the generator $\mathcal{B}$ of a semigroup $S_\mathcal{B}$ is weakly dissipative in a Banach space X_i if there exist a second Banach space $X_{i-1} \supset X_i$ and some numbers $\kappa \in \mathbb{R}$ and $\sigma > 0$ such that

$$\forall\, f \in D(\mathcal{B}_{|X_i}),\ \exists\, f^* \in J_{f,X_i}, \quad \langle f^*, \mathcal{B} f\rangle \le \kappa \|f\|^2_{X_i} - \sigma \|f\|^2_{X_{i-1}},$$

where we define the associated dual set $J_{f,X_i} \subset X_i'$ of f (for the norm $\|\cdot\|_{X_i}$) by

$$J_{f,X_i} := \{\varphi \in X_i';\ \langle \varphi, f\rangle = \|f\|^2_{X_i} = \|\varphi\|^2_{X_i'}\}. \tag{3.18}$$

By translation, we may assume that $\kappa = 0$, an hypothesis we will always make in the sequel of this section. We will furthermore assume the splitting structure $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A}$ a $\mathcal{B}$-bounded operator and $\mathcal{B}$ weakly dissipative generator.

More precisely, we assume that there exists one more Banach lattice $X_0 \supset X_1 \supset X_2 := X$, with norm denoted by $\|\cdot\|_k := \|\cdot\|_{X_k}$, such that $\mathcal{B}$ generates a semigroup and is weakly dissipative in each X_k: for any $k = 1, 2$

$$\forall\, f \in D(\mathcal{B}_{|X_k}),\ \exists\, f^* \in J_{f,X_k}, \quad \langle f^*, \mathcal{B} f\rangle_{X_k', X_k} \le -\sigma \|f\|^2_{k-1}. \tag{3.19}$$

This classically implies (or we can take the next inequality as a definition of the weak dissipativity condition) that

$$\frac{d}{dt}\|S_\mathcal{B}(t) f\|_k + \sigma \|S_\mathcal{B}(t) f\|_{k-1} \le 0, \quad \forall\, t \ge 0,\ \forall\, f \in X_k,\ \forall\, k = 1, 2. \tag{3.20}$$

We assume that X_k is dense into X_{k-1} for $k = 1, 2$ and that X_1 is an interpolated space between X_0 and X_2 in the sense that there exists a continuous and strictly decreasing function $\eta : (0, 1] \to [0, \infty)$, $\eta(\varepsilon) \to \infty$ when $\varepsilon \to 0$, $\eta(1) = 0$, such that

$$\|f\|_1 \le \varepsilon \|f\|_2 + \eta(\varepsilon) \|f\|_0, \quad \forall\, \varepsilon \in (0, 1],\ \forall\, f \in X_2. \tag{3.21}$$

From (3.20) with $k = 2$, we deduce

$$\|S_\mathcal{B}(t) f\|_2 \le \|f\|_2, \quad \forall\, t \ge 0,\ \forall\, f \in X_2. \tag{3.22}$$

Next, for $k = 1$, gathering the weak dissipativity condition (3.20), the interpolation condition (3.21) and the non expansion property (3.22) in the space X_2, we get

$$\begin{aligned} \frac{d}{dt}\|S_\mathcal{B}(t) f\|_1 + \frac{\sigma}{\eta(\varepsilon)} \|S_\mathcal{B}(t) f\|_1 &\le \frac{\sigma\varepsilon}{\eta(\varepsilon)} \|S_\mathcal{B}(t) f\|_2 \\ &\le \frac{\sigma\varepsilon}{\eta(\varepsilon)} \|f\|_2, \end{aligned}$$

for any $t \ge 0$, $\varepsilon \in (0, 1)$ and $f \in X_2$. We deduce

$$\frac{d}{dt}\Big(\|S_\mathcal{B}(t) f\|_1 e^{\frac{\sigma}{\eta(\varepsilon)} t}\Big) \le \frac{\sigma\varepsilon}{\eta(\varepsilon)} e^{\frac{\sigma}{\eta(\varepsilon)} t} \|f\|_2,$$

and thanks to the Grönwall lemma, we obtain

$$\|S_\mathcal{B}(t) f\|_1 \le \Theta(t) \|f\|_2, \tag{3.23}$$

for any $t \ge 0$ and $f \in X_2$, with

$$\Theta(t) := \inf_{\varepsilon \in (0,1)} \big(e^{-\frac{\sigma}{\eta(\varepsilon)} t} + \varepsilon\big) \to 0 \ \text{ as } \ t \to +\infty. \tag{3.24}$$

On the other hand, using the representation formula

$$\mathcal{R}_\mathcal{B}(z) f = \int_0^\infty e^{-zt} S_\mathcal{B}(t) f \, dt, \quad \forall\, z \in \Delta_0,\ \forall\, f \in X_2,$$

together with (3.20), we get

$$\sigma \|\mathcal{R}_\mathcal{B}(z) f\|_1 \le \int_0^\infty \sigma \|S_\mathcal{B}(t) f\|_1 \, dt \le \|f\|_2,$$

for any $z \in \overline{\Delta}_0$ and $f \in X_2$. We next assume that

$$\Theta(t)^{-1} \|\mathcal{A} S_\mathcal{B}(t) f\|_1 + \int_0^\infty \|\mathcal{A} S_\mathcal{B}(t) f\|_1 dt \lesssim \|f\|_1, \tag{3.25}$$

that there exist $\alpha > 1$, $N \ge 1$, $C \ge 1$ such that

$$\sup_{x+iy\in\Delta_0} \|\mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(x+iy)\dots\mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(x+iy)f\|_2 \le \frac{C}{\langle y\rangle^\alpha}\|f\|_2, \tag{3.26}$$

for any $\varepsilon \in \{0,1\}^N$, $\varepsilon_1 + \dots + \varepsilon_N \le 1$, and that

$$\sup_{z\in\Delta_0} \|(\mathcal{R}_{\mathcal{B}}(z)\mathcal{A})^N f\|_{\mathcal{X}_1} \le \|f\|_1, \tag{3.27}$$

with $\mathcal{X}_1$ compactly imbedded in X_1. The necessity to add (ε_i) in (3.26) is probably purely technical and not restrictive for applications. In examples, we can take $N = 2N'$, when

$$\sup_{x+iy\in\Delta_0} \|(\mathcal{A}\mathcal{R}_{\mathcal{B}})^{N'}(x+iy)f\|_3 \le \frac{C}{\langle y\rangle^\alpha}\|f\|_2, \tag{3.28}$$

for some convenient space X_3 such that $\mathcal{A} : X_1 \to X_3$ and $\sup_{z\in\Delta_0} \|\mathcal{R}_{\mathcal{B}}(z)\|_{\mathscr{B}(X_3,X_2)} < \infty$. At the level of the semigroup, (3.28) is typically a consequence of

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*N'')}(t)\|_{\mathscr{B}(X_2,X_3^\zeta)} \in L^1(\mathbb{R}_+),$$

with $\zeta > 0$, where $X_3^\zeta := \{f \in X_3,\ \mathcal{L}^\zeta f \in X_3\}$ stands for the (possibly fractional) domain for the operator defined in X_3. However, (3.26) is a bit more general than that last estimate. We refer to [278, 273, 280, 277] for precise definition, examples and discussion. For further references, we observe that (3.23) and (3.25) together imply

$$\begin{aligned}
\frac{1}{T}\int_0^T \|(S_{\mathcal{B}} * \mathcal{A}S_{\mathcal{B}})(t)f\|_1\,dt &\le \frac{1}{T}\int_0^T\int_0^t \|S_{\mathcal{B}}(t-s)\mathcal{A}S_{\mathcal{B}}(s)f\|_1\,dsdt \\
&\le \frac{1}{T}\int_0^T\int_0^T \|S_{\mathcal{B}}(u)\|_{\mathscr{B}(X_2,X_1)}\|\mathcal{A}S_{\mathcal{B}}(s)f\|_1\,duds \\
&\lesssim \frac{1}{T}\int_0^T \Theta(u)\,du\,\|f\|_2.
\end{aligned}$$

Arguing in a similar way for any $\ell \ge 1$, we establish

$$\frac{1}{T}\int_0^T \|(S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)})(t)f\|_1\,dt \lesssim \frac{1}{T}\int_0^T \Theta\,du\,\|f\|_2 \to 0 \quad\text{as}\quad T\to\infty. \tag{3.29}$$

For synthesizing and for further references, let us now bring out some possible general framework for semigroup enjoying weak dissipativity. We introduce the following structure condition on a semigroup $S_{\mathcal{L}}$ and its generator $\mathcal{L}$ by assuming

(HS3) there exist a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, some Banach lattices $X_2 \subset X_1$, an integer $N \ge 1$ and some decaying functions $\Theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Theta_1(t) \to 0$ as $t \to \infty$, $\Theta_2 \in L^1(\mathbb{R}_+)$ such that $\mathcal{A}$ is positive, $\mathcal{B}$ generates a positive semigroup $S_{\mathcal{B}}$ and the following estimates hold

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}}\|_{\mathscr{B}(X_2,X_1)} = \mathcal{O}(\Theta_1), \quad \forall\,\ell \in \{0,\dots,N-1\}, \tag{3.30}$$

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*N)}\|_{\mathscr{B}(X_1,X_2)} = \mathcal{O}(\Theta_2). \tag{3.31}$$

We now particularize our discussion to a Radon measures framework. We assume that there exist some weight functions ψ_i on E, $\psi_0 \lesssim \psi_1 \le \psi_2$, with $\psi_2(x)/\psi_1(x) \to \infty$ as $x \to \infty$ so that $M^1_{\psi_2} \subset\subset M^1_{\psi_1}$ (compact imbedding for the weak convergence), a function $\chi \in C_c(E)$, $0 \le \chi \le 1$, and a constant $M \ge 0$ such that

(i) $\mathcal{L}^*\psi_1 \le -\psi_0 + M\chi$;
(ii) $\mathcal{L}^*\psi_2 \le M\chi$;
(iii) $\psi_1 \le \varepsilon\psi_2 + \eta(\varepsilon)\psi_0$ for any $\varepsilon > 0$,

for a function $\eta : (0,1] \to (0,\infty)$ such that $\eta(1) = 0$, $\eta(\varepsilon) \to \infty$ when $\varepsilon \to 0$, and

$$t \mapsto \Theta(t) := \inf_{\varepsilon\in(0,1)} \left(e^{-\frac{t}{\eta(\varepsilon)}} + \varepsilon\right) \in L^1(0,\infty). \tag{3.32}$$

It is worth emphasizing that from the very definition, we have automatically that Θ is positive and decreasing, $\Theta(0) = 1$ and $\Theta(t) \to 0$ as $t \to \infty$. Arguing similarly as we did during the proof of Corollary 2.20 and the end of Section 3.1, we introduce the splitting

$$\mathcal{A} := M\chi, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

and we establish that $S_\mathcal{B}$ is a positive semigroup on $X = M^1_{\psi_2}(E)$. More precisely, for $0 \le f_0 \in D(\mathcal{B})$ in the domain of $S_\mathcal{B}$ and denoting $f_t := S_\mathcal{B}(t) f_0$, we may compute

$$\frac{d}{dt}\int f_t\,\psi_2 \le \int f_t\,\mathcal{B}^*\psi_2 \le 0$$

and similarly

$$\frac{d}{dt}\int f_t\,\psi_1 \le \int f_t\,\mathcal{B}^*\psi_1 \le -\int f_t\,\psi_0.$$

Integrating both differential inequalities, we deduce $S_\mathcal{B} \in L^\infty_t(\mathscr{B}(M^1_{\psi_i}))$, $i = 1, 2$ and

$$\int_0^\infty \|S_\mathcal{B}(t) f_0\|_{M^1_{\psi_0}}\,dt \le \|f_0\|_{M^1_{\psi_1}}, \quad \forall\, f_0 \in M^1_{\psi_1}.$$

We may make a slight (but important) improvement of the previous estimate by proceeding similarly as we did for proving (3.23). Using the same notations as in the above computation, we indeed have

$$\frac{d}{dt}\int f_t\,\psi_1 + \frac{1}{\eta(\varepsilon)}\int f_t\,\psi_1 \le \frac{\varepsilon}{\eta(\varepsilon)}\int f_t\,\psi_2 \le \frac{\varepsilon}{\eta(\varepsilon)}\int f_0\,\psi_2,$$

where we have used (i) and (iii) in the first inequality and the previous $L^\infty_t(\mathscr{B}(M^1_{\psi_2}))$ bound in the second inequality. Integrating in time, we deduce

$$\|S_\mathcal{B}(t) f\|_{M^1_{\psi_1}} \le \Theta(t)\|f\|_{M^1_{\psi_2}}, \quad \forall\, t > 0.$$

Taking $X_i := M^1_{\psi_i}$ and $N = 1$, we see that $\mathcal{L}$ then satisfies **(HS3)** with $\Theta_i = \Theta$.

3.4. **First existence result in the weakly dissipative case.** We first come back to the proof of Theorem 2.21 and explain what goes wrong when we try to adapt it to a weak dissipativity context. More precisely, we assume that $S_\mathcal{L}$ is a positive semigroup (so that **(H1)** holds) satisfying $\mathcal{L}^*\phi_0 \ge 0$ for some $\phi_0 \in X'\backslash\{0\}$ (so that **(H2)** holds) and the splitting structure **(HS3)** for some bounded operator $\mathcal{A}$ and some weakly dissipative operator $\mathcal{B}$, in the sense that (3.19) holds. In such a situation, we may define

$$\lambda_1 := \inf\{\lambda \in \mathbb{R};\ \mathcal{R}_\mathcal{L}(\kappa) \in \mathscr{B}(X),\ \forall\, \kappa \ge \lambda\} \ge 0,$$

and there exist sequences (λ_n) of $\mathbb{R}$ and $(\hat f_n)$ of X_+ such that

$$\lambda_n \searrow \lambda_1 \ge 0, \quad \|\hat f_n\| = 1, \quad \varepsilon_n := \lambda_n \hat f_n - \mathcal{L}\hat f_n \to 0 \text{ in } X,$$

thanks to Lemma 2.6. In the simplest situation, we may further assume that $\mathcal{R}_\mathcal{B}(\kappa) : X_1 \to X_0$ is uniformly bounded in $\kappa \ge \lambda_1$ and $\mathcal{A} : X_0 \to X_1$ with $X = X_1 \subset X_0$. The issue is that even in that case, we may write

$$\hat f_n = \mathcal{R}_\mathcal{B}(\lambda_n)\mathcal{A}\hat f_n + \mathcal{R}_\mathcal{B}(\lambda_n)\varepsilon_n,$$

but it is not clear how to conclude that $(\hat f_n)$ belongs to a compact set in X because it is not clear that $\mathcal{R}_\mathcal{B}(\lambda_n)\varepsilon_n \to 0$ in X.

The next result aim precisely to establish that last convergence under suitable quite strong (although natural and true in some examples) assumptions on the operator $\mathcal{L}$. The proof is adapted from [231, Sec. 6.3] and mixes some dynamical argument together with the stationary approach developed in Section 2.2.

Theorem 3.3. *Consider a positive semigroup $S_\mathcal{L}$ in a Banach lattice $X = X_2 \subset X_1 \subset X_0$ such that its generator $\mathcal{L}$ satisfies*
(1) *there exists $\phi_0 \in D(\mathcal{L}^*)$, $\phi_0 \ge 0$, $\phi_0 \ne 0$, such that $\mathcal{L}^*\phi_0 \ge 0$;*
(2) *$\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ satisfying* (3.23), (3.25), (3.26) *and* (3.27).
Then, there exist $\lambda_1 \ge 0$ and $f_1 \in X_1$ such that

$$\|f_1\|_{X_1} = 1, \quad f_1 \ge 0, \quad \mathcal{L} f_1 = \lambda_1 f_1. \tag{3.33}$$

Proof of Theorem 3.3. We split the proof into four steps.

Step 1. We know from Lemma 2.2 and Lemma 2.4-**(i)** that **(H1)** and **(H2)** hold. We may then define $\lambda_1 \ge 0$ with the help of (2.16). If $\lambda_1 > 0$, we see that $\mathcal{V}(\alpha)$ defined in (2.22) is bounded in $\mathcal{B}(X)$ uniformly on $\alpha \ge \kappa_0 := \lambda_1/2$ because of (3.23) and (3.25), and that $\mathcal{W}(\alpha)$ also defined in

(2.22) satisfies (2.23) because of (3.25) and Remark 2.9-**(1)**. Using Lemma 2.8, we get that **(H3)** holds, and we conclude thanks to Theorem 2.21 in that case.

In the sequel, we always assume $\lambda_1 = 0$.

Step 2. Let us fix $f_0 \in D(\mathcal{L})$ such that $f_0 \ge 0$ and $C_0 := \langle f_0, \phi_0 \rangle > 0$, which exists by definition of ϕ_0. Denoting $f(t) := S_{\mathcal{L}}(t) f_0$, we have

$$\frac{d}{dt}\langle f(t), \phi_0\rangle = \langle \mathcal{L} f(t), \phi_0\rangle = \langle f(t), \mathcal{L}^* \phi_0\rangle \ge 0,$$

which in turns implies

$$\langle f(t), \phi_0\rangle \ge C_0, \quad \forall\, t \ge 0.$$

Step 3. We claim that $\|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathscr{B}(X_2,X_1)} = +\infty$. That in particular implies $\|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathscr{B}(X)} = +\infty$ and thus $0 \in \Sigma(\mathcal{L})$. We assume by contradiction that $\mathcal{K}_{2,1} := \|\mathcal{R}_{\mathcal{L}}(0)\|_{\mathscr{B}(X_2,X_1)} < +\infty$. First, because $S_{\mathcal{L}}$ is positive, we have

$$|\mathcal{R}_{\mathcal{L}}(z) f| \le \int_0^\infty e^{-t\Re e z} S_{\mathcal{L}}(t)|f|\, dt = |\mathcal{R}_{\mathcal{L}}(\Re e z)|f|,$$

from which we deduce

$$\|\mathcal{R}_{\mathcal{L}}(z)\|_{\mathscr{B}(X_2,X_1)} \le \|\mathcal{R}_{\mathcal{L}}(\Re e z)\|_{\mathscr{B}(X_2,X_1)}, \quad \forall\, z \in \Delta_0.$$

As a consequence, we have

$$\sup_{y\in\mathbb{R}} \|\mathcal{R}_{\mathcal{L}}(iy)\|_{\mathscr{B}(X_2,X_1)} \le \mathcal{K}_{2,1}. \tag{3.34}$$

We write the representation formulas (taken from [278, (2.21)])

$$S_{\mathcal{L}}(t) f = \mathcal{T}_0(t) + \lim_{M\to\infty} \mathcal{T}_{1,M}(t)$$

with

$$\mathcal{T}_0(t) := \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}(t) f$$

and

$$\mathcal{T}_{1,M}(t) := \frac{i}{2\pi} \int_{a-\mathrm{i}M}^{a+\mathrm{i}M} \mathrm{e}^{zt}\, \mathcal{R}_{\mathcal{L}}(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N f\, dz,$$

for any $f \in D(\mathcal{L})$, $t \ge 0$ and $a > 0$. On the one hand, from (3.29), we have the Cesàro mean convergence

$$\frac{1}{T}\int_0^T \mathcal{T}_0(t)\, dt \to 0 \ \text{ in } X_1, \ \text{ as } \ T \to \infty. \tag{3.35}$$

On the other hand, we estimate the contribution of $\mathcal{T}_{1,M}$. Integrating by part, we have

$$\mathcal{T}_{1,M}(t) = \frac{1}{t}\frac{i}{2\pi} \int_{a-\mathrm{i}M}^{a+\mathrm{i}M} \mathrm{e}^{zt}\, \frac{d}{dz}\big[\mathcal{R}_{\mathcal{L}}(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N\big] f\, dz,$$

with

$$\frac{d}{dz}\big[\mathcal{R}_{\mathcal{L}}(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N\big] = \sum_{\varepsilon\in\mathbb{N}^{N+1},\, |\varepsilon|=1} \mathcal{R}_{\mathcal{L}}(z)^{1+\varepsilon_0} \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(z) \dots \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(z).$$

Together with condition (3.26) and estimate (3.34), we get

$$\begin{aligned}
&\Big\| \frac{d}{dz}\big[\mathcal{R}_{\mathcal{L}}(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^N\big] f \Big\|_1 \\
&\quad \le (\mathcal{K}_{2,1} + \mathcal{K}_{2,1}^2) N \sup_{\varepsilon\in\mathbb{N}^N,\, |\varepsilon|\le 1} \|\mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_1}(z) \dots \mathcal{A}\mathcal{R}_{\mathcal{B}}^{1+\varepsilon_N}(z) f\|_2 \\
&\quad \le \frac{C_1}{\langle y\rangle^\alpha} \|f\|_2,
\end{aligned}$$

uniformly for any $z = x + iy \in \Delta_0$, for some constant $C_1 > 0$. We deduce

$$\|\lim_{M\to\infty} \mathcal{T}_{1,M}(t)\|_1 \le \frac{1}{t}\frac{1}{2\pi} \int_{\mathbb{R}} \frac{C_1}{\langle y\rangle^\alpha}\, dy \|f\|_2 \to 0, \tag{3.36}$$

as $t \to \infty$. Gathering (3.35) and (3.36), we conclude in particular that

$$\frac{1}{T}\int_0^T S_{\mathcal{L}}(t) f_0 \, dt \to 0 \quad \text{in } X_1, \quad \text{as} \quad T \to \infty,$$

which is in contradiction with the estimate of Step 2.

Step 4. Conclusion. Taking advantage of the convenient blow up of $\mathcal{R}_{\mathcal{L}}(\lambda)$ as $\lambda \searrow 0$ established in the previous Step 3, we may now argue similarly as in the proof of Theorem 2.21. From Step 2, there exists a sequence (λ_n) such that $\lambda_n \to 0$ and

$$\|\mathcal{R}_{\mathcal{L}}(\lambda_n)\|_{\mathscr{B}(X_2,X_1)} \to \infty.$$

That means that there exist (f_n) and (g_n) such that

$$\|f_n\|_{X_1} \to \infty, \quad \|g_n\|_{X_2} = 1, \quad f_n = \mathcal{R}_{\mathcal{L}}(\lambda_n)\, g_n,$$

or equivalently that there exist $(\hat f_n)$ and (ε_n) (by defining $\hat f_n := f_{n\pm}/\|f_{n\pm}\|_{X_1}$, $\varepsilon_n := g_{n\pm}/\|f_{n\pm}\|_{X_1}$) satisfying

$$\|\hat f_n\|_{X_1} = 1, \quad \hat f_n \ge 0, \quad \|\varepsilon_n\|_{X_2} \to 0, \quad \varepsilon_n = (\lambda_n - \mathcal{L})\, \hat f_n. \tag{3.37}$$

As in the proof of Lemma 2.8, we deduce that (2.25) holds, that is

$$\hat f_n = \sum_{\ell=0}^{N-1} (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^\ell \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n + (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^N \hat f_n. \tag{3.38}$$

Using the uniform boundedness

$$(\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^\ell \mathcal{R}_{\mathcal{B}}(\lambda_n) \in \mathcal{B}(X_2, X_1), \quad (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^N \in \mathcal{B}(X_1, \mathcal{X}_1), \ \mathcal{X}_1 \subset\subset X_1,$$

we deduce that $(\hat f_n)$ belongs to a compact set of X_1, or in other words, that there exist a subsequence of $(\hat f_n)$ (not relabeled) and $f_1 \in X_1$ such that $\hat f_n \to f_1$ in X_1. We may pass to the limit in (3.37), and we get (3.33). □

3.5. **Second existence result in the weakly dissipative case.** Using a pure dynamical approach adapted from the second proof of Theorem 3.1 and from [81, Thm. 6.1], we establish a second existence result which is less demanding in terms of conditions on the semigroup $S_{\mathcal{L}}$.

Theorem 3.4. *Consider a positive semigroup $S = S_{\mathcal{L}}$ on a Banach lattice $X = Y'$ for a separable Banach lattice Y. We assume*

(i) there exists $\phi_0 \in Y_+\backslash\{0\}$ such that $[S_t f]_0 \ge [f]_0$ for any $f \in X_+$ and $f \mapsto [f]_0 := \langle |f|, \phi_0\rangle$ is a norm on X. We then denotes $\mathcal{X}$ the vector space X endowed with this norm $[\cdot]_0$;

(ii) there exist $v \in L^\infty(\mathbb{R}_+; \mathscr{B}(X))$ and $0 \le w \in L^1(\mathbb{R}_+; \mathscr{B}(\mathcal{X}, X))$ such that

$$S = v + w * S, \tag{3.39}$$

and we set

$$M := \sup_{t\ge0} \|v(t)\|_{\mathscr{B}(X)} < \infty, \quad \Theta(t) := \|w(t)\|_{\mathscr{B}(\mathcal{X},X)} \in L^1(\mathbb{R}_+). \tag{3.40}$$

Then there exists a pair $(\lambda_1, f_1) \in \mathbb{R}_+ \times X_+\backslash\{0\}$ such that $\mathcal{L} f_1 = \lambda_1 f_1$.

Remark 3.5. *(1) When $S_{\mathcal{L}}$ satisfies* **(HS3)** *then* (3.39) *holds with*

$$v := \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}, \quad w := (S_{\mathcal{B}}\mathcal{A})^{(*N)}. \tag{3.41}$$

(2) By definition of the norm $[\cdot]_0$ of $\mathcal{X}$, we see that $\mathcal{X}$ is a weighted L^1 space or a weighted Radon measures space. In many applications, when both $\mathcal{X}$ and X are Radon measures spaces, one can choose $N = 1$. On the other hand, when X is for instance a (possibly weighted) L^p space with $p > 1$, one must take $N \ge 2$ in most of the applications. In condition (ii), the first bound is not really demanding and almost automatic in view of the estimates exhibited in Section 3.3. The second bound is a kind of regularity estimate reminiscent of the enlarging and shrinkage technique developed in [294, 190, 274].

Proof of Theorem 3.4. We split the proof into three steps.

Step 1. We define

$$R := \max(2\|\Theta\|_{L^1}, \|g_0\|),$$

for some $g_0 \in X_+$ such that $[g_0]_0 = 1$, and next the nonempty convex and compact (in the weak $*\,\sigma(X,Y)$ sense) set

$$\mathcal{C} := \{f \in X_+;\, [f]_0 = 1,\ \|f\| \le R\},$$

as well as the increasing function

$$\lambda(t) := \inf_{f\in\mathcal{C}} [S(t)f]_0, \quad \forall\, t \ge 0.$$

We have the alternative

- **(1)** $\sup \lambda > 2M$,
- **(2)** $\sup \lambda \le 2M$.

Step 2. We assume that the first term **(1)** of the alternative holds true, or in other words, there exists $T_0 > 0$ such that

$$\forall\, f \in \mathcal{C}, \quad [S_{T_0} f]_0 \ge 2M. \tag{3.42}$$

We define as before

$$\Phi_{T_0} f := \frac{S_{T_0} f}{[S_{T_0} f]_0}, \quad \forall\, f \in \mathcal{C}.$$

By construction, for any $f \in \mathcal{C}$, we have $\Phi_{T_0} f \ge 0$ and $[\Phi_{T_0} f]_0 = 1$. On the other hand, using the splitting structure (3.39) and the estimates (3.40), we have

$$\|S(t)f\| \le M\|f\| + \int_0^t \Theta(t-s)[S(s)f]_0\, ds.$$

From hypothesis (i) and the semigroup property, we also have

$$[S_t f]_0 \ge [S_s f]_0, \quad \forall\, t \ge s \ge 0.$$

The two above estimates together imply

$$\begin{aligned} \|\Phi_{T_0} f\| &\le \frac{M\|f\|}{[S_{T_0} f]_0} + \int_0^{T_0} \Theta(T_0 - s) \frac{[S_s f]_0}{[S_{T_0} f]_0}\, ds \\ &\le \frac{1}{2}\|f\| + \|\Theta\|_{L^1} \le R, \end{aligned}$$

for any $f \in \mathcal{C}$. We have thus proved $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$. Thanks to the Tykonov fixed point Theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0} f_{T_0} = f_{T_0}$. In other words, we have built a pair of "almost eigenvalue and eigenfunction"

$$f_{T_0} \ge 0, \quad [f_{T_0}]_0 = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0},$$

with $e^{\lambda_1 T_0} = [S_{T_0} f]_0$ and thus $\lambda_1 \in [0, \kappa_1]$. We conclude to the existence of $f_1 \in \mathcal{C}$ such that $\mathcal{L} f_1 = \lambda_1 f_1$ really similarly as in Step 2 of the Second proof of Theorem 3.1.

Step 3. We assume that the second term **(2)** of the alternative holds true. In that case, for any $n \ge 1$, there exists $f_n \in \mathcal{C}$ such that $[S(n) f_n]_0 \le 2M$. By compactness, there exists $f_0 \in \mathcal{C}$ and a subsequence (f_{n_k}) such that $f_{n_k} \rightharpoonup f_0 \in \mathcal{C}$ and

$$\forall\, t \ge 0,\ \forall\, k\, (n_k \ge t), \quad [S(t) f_{n_k}]_0 \le [S(n_k) f_{n_k}]_0 \le 2M,$$

so that

$$\forall\, t \ge 0, \quad [S(t) f_0]_0 \le 2M. \tag{3.43}$$

Using this particular initial datum, we argue similarly as in [81, proof of Theorem 6.1], and we conclude to the existence of a stationary state. More precisely, we come back to the splitting structure (3.39) of the semigroup S and we introduce the associated Cesàro means

$$U_T := \frac{1}{T}\int_0^T S(t)\, dt, \quad V_T := \frac{1}{T}\int_0^T v(t)\, dt, \quad K_T := \frac{1}{T}\int_0^T (w * S)(t)\, dt, \tag{3.44}$$

for any $T > 0$. We obviously have

$$\|V_T\|_{\mathscr{B}(X)} \le \frac{1}{T}\int_0^T \|v(t)\|_{\mathscr{B}(X)}\, dt \le M.$$

On the other hand, we have

$$\int_0^T (w * S)(t)\, dt = \int_0^T \int_s^T w(t-s) dt S(s)\, ds \le \int_0^T w(\tau) d\tau \int_0^T S(s)\, ds,$$

thanks to the Fubini theorem and the positivity of the two operators involved in this integral formula. We deduce

$$\begin{aligned} \|K_T f_0\| &\le \Big\| \int_0^T w(\tau) d\tau \frac{1}{T}\int_0^T S(s)\, f_0\, ds \Big\| \\ &\le \int_0^\infty \big\|w(\tau)\big\|_{\mathscr{B}(\mathcal{X},X)} d\tau \Big[\frac{1}{T}\int_0^T S(s) f_0\, ds\Big]_0 = \|\Theta\|_{L^1}\, [U_T f_0]_0, \end{aligned}$$

thanks to assumption (ii), so that $K_T f_0$ is uniformly bounded in X thanks to (3.43) and the elementary estimate $[U_T f_0]_0 \le [S_T f_0]_0$. We then deduce that $U_T = V_T + K_T$ satisfies

$$\|U_T f_0\| \le M\|f_0\| + 2M\|\Theta\|_{L^1} \quad \text{and} \quad 1 \le [S_T f_0]_0 \le 2M,$$

for any $T > 0$. By compactness, there exists $T_k \to +\infty$ and $f_1 \in X_+$ such that $U_{T_k} f \rightharpoonup f_1$ weakly $*$ in X. Thanks to the second inequality, we have $[f_1]_0 \ge 1$. We then argue thanks to the usual mean ergodic theorem trick. For any fixed $s > 0$, we observe that

$$\begin{aligned} S(s) f_1 - f_1 &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k}\int_0^{T_k} S(s)S(t) f_0 dt - \frac{1}{T_k}\int_0^{T_k} S(t) f_0\, dt \Big\} \\ &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k}\int_{T_k}^{T_k+s} S(t) f_0 dt - \frac{1}{T_k}\int_0^{s} S(t) f_0\, dt \Big\} \end{aligned}$$

weakly $*$ in X. By the lower semicontinuous property of the norm $[\cdot]_0$, we deduce

$$[S(s) f_1 - f_1]_0 \le \liminf_{k\to\infty} \Big\{ \frac{1}{T_k}\int_{T_k}^{T_k+s} [S(t) f_0]_0 dt + \frac{1}{T_k}\int_0^{s} [S(t) f_0]_0\, dt \Big\} = 0,$$

so that f_1 is a stationary solution, and thus f_1 is an eigenfunction associated to the eigenvalue $\lambda_1 = 0$. □

4. Irreducibility and geometry of the first eigenvalue

In this section, we are concerned with the geometric part of the Krein-Rutman theorem for an unbounded operator $\mathcal{L}$ on a Banach lattice X. We assume that the conclusions of the existence part are achieved, namely

(C1) the first primal and dual eigenvalue problem has a solution (λ_1, f_1, ϕ_1): there exist $\lambda_1 \in \mathbb{R}$, $f_1 \in X_+ \cap D(\mathcal{L})\backslash\{0\}$, $\phi_1 \in Y_+ \cap D(\mathcal{L}^*)\backslash\{0\}$ such that

$$\mathcal{L} f_1 = \lambda_1 f_1 \qquad \text{and} \qquad \mathcal{L}^* \phi_1 = \lambda_1 \phi_1. \tag{4.1}$$

By construction, we also have $\Sigma(\mathcal{L}) \subset \{z \in \mathbb{C},\ \Re e(z) \le \lambda_1\}$.
Assuming that S is positive as for the existence part and an additional strong maximum principle property, we analyze the first eigenvalue problem.

4.1. **More about positivity.** For further references, we introduce several notions which are strongly related to the positivity property for semigroups.

The signum operator sign. In a real Banach lattice X, we say that $\mathrm{sign}\, f \in \mathscr{B}(X, X'')$ is a signum operator for $f \in X$, if it satisfies the following properties

$$(\mathrm{sign}\, f)\, f = |f|, \tag{4.2}$$

$$(\mathrm{sign}\, f)\, g \le |g|, \quad \forall\, g \in X. \tag{4.3}$$

In the sequel, we will always assume that such an operator exists. We refer to [15, Sec. C.I & C.II] for a general introduction to the topic. In practice, we will only need a weak formulation of the sign operator (see below) which may be defined only in some subspace $\mathcal{X} \subset X$. We always additionally assume that the signum operator satisfies

$$(\mathrm{sign}\,(-f))\,(-g) = (\mathrm{sign}\, f) g, \quad \forall\, g \in X,$$
$$(\mathrm{sign}\, f)\, g = g, \quad \forall\, g \in X, \text{ if } f \in X_+,$$

We also define

$$\mathrm{sign}_+ f := \frac{1}{2}\big(I + \mathrm{sign} f\big).$$

• When X is a space of functions, the sign operator $\mathrm{sign}\, f$ associated to $f \in X$ corresponds to the multiplication by the function $\mathrm{sign}\, f := \mathbf{1}_{f>0} - \mathbf{1}_{f<0}$. When $X := L^p(E)$, we obviously see that $\mathrm{sign}\, f \in \mathscr{B}(L^p(E))$ for any $f \in L^p(E)$. On the other hand, when $X := C_0(E)$, we only have $\mathrm{sign}\, f \in \mathscr{B}(C_0(E); \mathcal{M}^\infty(E))$, where $\mathcal{M}^\infty(E)$ denotes the space of uniformly bounded measurable functions, so that $\mathcal{M}^\infty(E) \subset (C_0(E))''$. In a space of bounded measures $X = M^1(E)$, we may define the sign operator by means of the Radon-Nikodym theorem. For a given $f \in M^1(E)$, using Hahn decomposition, there exists indeed a measurable function $\alpha : E \to \{-1, 1\}$ such that $f = \alpha |f|$, and we then define $(\mathrm{sign}\, f) g = \alpha g$ for any $g \in M^1(E)$.

• When X is σ-order complete, in the sense that any increasing and upper bounded sequence has a supremum (a common least upper bound), the operator sign exists and is more regular, namely $\mathrm{sign}\, f \in \mathscr{B}(X)$ for any $f \in X$, see [296] and also [15, Sec. C.I.8]. We recover in particular that $\mathrm{sign}\, f \in \mathscr{B}(L^p(E))$ for any $f \in L^p(E)$.

Weak principle maximum and Kato's inequality. We introduce now two definitions formulated on an operator $\mathcal{L}$ which are almost equivalent to the positivity property of the semigroup S when $\mathcal{L}$ is the generator of S.

• We say that the operator $\mathcal{L}$ satisfies the *weak maximum principle* when

$$\kappa \in \mathbb{R},\ f \in D(\mathcal{L}) \text{ and } (\kappa - \mathcal{L}) f \ge 0 \quad \text{imply} \quad f \ge 0; \tag{4.4}$$

• We say that the operator $\mathcal{L}$ satisfies *Kato's inequality* when

$$\forall\, f \in D(\mathcal{L}), \quad \mathcal{L}|f| \ge (\mathrm{sign}\, f) \mathcal{L} f. \tag{4.5}$$

Since $|f|$ does not necessarily belong to $D(\mathcal{L})$, the correct way to understand Kato's inequality is

$$\forall\, f \in D(\mathcal{L}),\ \forall\, \psi \in D(\mathcal{L}^*) \cap Y_+, \quad \langle |f|, \mathcal{L}^* \psi \rangle \ge \langle (\mathrm{sign}\, f) \mathcal{L} f, \psi \rangle. \tag{4.6}$$

We immediately see from the definitions that (4.5) is equivalent to assume

$$\forall\, f \in D(\mathcal{L}), \quad \mathcal{L} f_+ \ge (\mathrm{sign}_+ f) \mathcal{L} f. \tag{4.7}$$

Remark 4.1. *We complement Lemma 2.1, by claiming that for a semigroup $S = S_\mathcal{L}$ on a Banach lattice X, there is equivalence between the fact that S is positive and $\kappa - \mathcal{L}$ satisfies the weak maximum principle for any $\kappa > \omega(\mathcal{L})$, what is straightforward using that these properties are equivalent to the fact that $\mathcal{R}_\mathcal{L}(\kappa) \ge 0$ for any $\kappa > \omega(\mathcal{L})$. These properties also imply that Kato's inequality holds true, see* [296, 12], [13, Prop. 1.1], [11, Rk. 3.10] *and the textbook* [15, Thms C.II.2.4, C.II.2.6 and Rk. C-II.3.12]. *When for instance $f, |f| \in D(\mathcal{L})$, we may indeed compute*

$$(\mathrm{sign}\, f) \mathcal{L} f = (\mathrm{sign}\, f) \lim_{t \to 0} \frac{S_t f - f}{t} \le \lim_{t \to 0} \frac{S_t |f| - |f|}{t} = \mathcal{L}|f|,$$

where we have used the very definition of the generator $\mathcal{L}$ and the properties (4.2)-(4.3) *of* $\mathrm{sign}\, f$ *in the inequality.*

We end this section by introducing other notions of positivity which strengthen the previously defined positivity condition.

Strict order. We may define a first stronger order $>$ (or $<$) on X by writing for $f \in X$

$$f > 0 \quad \text{if} \quad f \in X_+ \backslash \{0\}$$

and similarly a stronger order $>$ (or $<$) on X' by writing for $\phi \in X'$

$$\phi > 0 \quad \text{if} \quad \phi \in X'_+ \backslash \{0\}.$$

We may next define the strict (and stronger) order $\gg$ (or $\ll$) on X by writing for $f \in X$

$$f \gg 0 \text{ or } f \in X_{++} \quad \text{iff} \quad \forall \psi \in X'_+ \backslash \{0\}, \ \langle \psi, f \rangle > 0, \tag{4.8}$$

and similarly the strict order $\gg$ (or $\ll$) on X' by writing for $\phi \in X'$

$$\phi \gg 0 \text{ or } \phi \in X'_{++} \quad \text{iff} \quad \forall g \in X_+ \backslash \{0\}, \ \langle \phi, g \rangle > 0. \tag{4.9}$$

On the two Banach lattices X and Y, we thus have three positivity notions with $\gg$ (associated to X_{++} and Y_{++}) stronger than $>$ (associated to $X_+ \backslash \{0\}$ and $Y_+ \backslash \{0\}$) which itself is stronger than $\geq$ (associated to X_+ and Y_+).

Let us comment on the notion of strict positivity.

Remark 4.2. *When $X = Y'$ for instance, there are two possible strict positivity notions on X given by* (4.8) *for the space X (namely $\phi \in X$ is stricly positive when $\langle \xi, \psi \rangle > 0$ for any $\xi \in X'_+ \backslash \{0\}$) and by* (4.9) *for the space Y (namely $\phi \in X$ is stricly positive when $\langle \psi, g \rangle > 0$ for any $g \in Y_+ \backslash \{0\}$). They clearly coincide when X is reflexive, but in general the first one is stronger than the second one. In that situation, we will always consider that X is endowed with the weakest "dual" strict order* (4.9).

Examples 4.3. *In the space $C_0(E)$, the strict order is defined by $f \gg 0$ iff $f(x) > 0$ for any $x \in E$. In a space $L^p(E, \mathscr{E}, \mu)$, $1 \leq p \leq \infty$, the strict order is defined by $f \gg 0$ iff $f(x) > 0$ for μ-a.e. $x \in E$. In the space $M^1(E) = C_0(E)'$, the strict order is defined by duality by $f \gg 0$ iff $\langle f, \varphi \rangle > 0$ for any $\varphi \in C_0(E)$, $\varphi \geq 0$, $\varphi \not\equiv 0$.*

Remark 4.4. *In a Banach lattice X such that $\operatorname{int} X_+ \neq \emptyset$, the common definition of the strict order is $X_{++} := \operatorname{int} X_+$. In particular, in the case when E is compact and $X = C_0(E) = C(E)$, we have $\operatorname{int} X_+ \neq \emptyset$ and the definition of X_{++} introduced in Examples 4.3 coincides with $\operatorname{int} X_+$. In all the other examples considered, we have $\operatorname{int} X_+ = \emptyset$, and thus our definition of the strict order does not coincide with the one defined through the set $\operatorname{int} X_+$.*

Remark 4.5. *Another notion of strict order can be defined through the notions of ideals and quasi-interior points as briefly explained now, see* [15] *or* [41, Chap. 10] *and the references therein for details. Defining the segment $[g_1, g_2]$ and the set I_f for $g_1, g_2 \in X$ and $f \in X_+ \backslash \{0\}$ by*

$$[g_1, g_2] := \{g \in X; \ g_1 \leq g \leq g_2\}, \quad I_f := \bigcup_{k \geq 0} [-kf, kf] = \operatorname{Span}[0, f],$$

one shows that I_f is an ideal in the sense that $g \in I_f$ implies $|g| \in I_f$ and $0 \leq g \leq f$ implies $g \in I_f$. We say that f is an order unit if $I_f = X$. When $\operatorname{int} X_+ \neq \emptyset$, we find that f is an order unit iff $f \in \operatorname{int} X_+$ from Lemma 2.23, so that we recover the notion of strict positivity defined above. On the other hand, we say that f is a quasi-interior point if $\bar{I}_f = X$. It can be shown that f is a quasi-interior point iff f is strictly positive in the sense of the direct strict order (4.8)*, see for instance* [337, Thm. II.6.3]. *These two notions of strict positivity and quasi interior point thus coincide when X is reflexive or when $X = L^p(E, \mathscr{E}, \mu)$, $1 \leq p < \infty$, see also* [41, Examples 10.16] *when μ is a σ-finite diffuse (or atomless) measure. On the other hand, it is important to point out again that the "dual" strict order* (4.9) *considered here is a weaker notion than the quasi-interior point notion. For instance, in $X = C_0(E)' = M^1(E)$, there is no quasi-interior point but $X_{++} \neq \emptyset$.*

We finally point out the following result. For a semigroup $S = S_{\mathcal{L}}$ in a Banach lattice, under the mild assumption that there exists a strictly positive subeigenvector for the dual problem, namely

$$\exists \phi \in X'_{++}, \ \exists b \in \mathbb{R}, \quad \mathcal{L}^* \phi \leq b \, \phi,$$

then Kato's inequality (4.5) implies that S is positive, see [13, Thm. 1.6].

4.2. **Irreducibility and strong maximum principle.** We present some other material involving the strict positivity. When satisfied by $\xi \in X$ or Y, we will in particular make use of the property

$$\xi_+ \gg 0 \quad \text{implies} \quad \xi \gg 0 \quad (\text{or equivalently } \xi_- = 0). \tag{4.10}$$

For further references, we introduce some general framework for the couple of Banach lattices we will use in the sequel:

$$\textbf{(X1)} \begin{cases} \textbf{(i)} \quad X_{++} \neq \emptyset, Y_{++} \neq \emptyset \text{ and the signum operator is well define in both } X \text{ and } Y; \\ \textbf{(ii)} \quad \text{the property (4.10) holds in both spaces } X \text{ and } Y. \end{cases}$$

While **(X1)-(i)** is always satisfied in the applications, it is not the case for **(X1)-(ii)**.

Lemma 4.6. *The property* (4.10) *holds true in a space X endowed with the direct strict order* (4.8)*, in particular in $X = L^p$, $p \in [1, \infty)$, and $X = C_0$, and also in the space $L^\infty = (L^1)'$ endowed with the dual strict order* (4.9).

Proof of Lemma 4.6. We start recalling the proof of (4.10) in a general space X endowed with the strict order (4.8). Consider an element f of the Banach lattice X and assume that $f_+ \gg 0$. The vectors f_+ and f_- are disjoint, *i.e.* $f_+ \wedge f_- = 0$, see for instance [264, Thm. 1.1.1 iv)]. On the one hand, since $f_+ \gg 0$, we have that

$$(nf_+) \wedge f_- \to f_-$$

with respect to the norm as $n \to \infty$, see [337, Thm. II.6.3]. On the other hand, we have

$$0 \le (nf_+) \wedge f_- \le (nf_+) \wedge (nf_-) = n(f_+ \wedge f_-) = 0,$$

for every integer $n \ge 1$, where the last equality follows from the fact that f_+ and f_- are disjoint. We deduce by passing to the limit $n \to \infty$ that $f_- = 0$. Thus, $f = f_+ \gg 0$.

We now establish (4.10) when $X = L^\infty$ (notice that exactly the same arguments may be used when $X = L^p$ and $X = C_0$, what provides an elementary proof in these cases too). Take $f \in L^\infty$ such that $f_+ \gg 0$. From the definition of the strict order made explicit in Examples 4.3, we have $f_+(x) = \max(f(x), 0) > 0$ a.e., so that $f(x) > 0$ a.e. and finally $f_-(x) = 0$ a.e.. ☐

We give now a counter-example in the Radon measures space case.

Example 4.7. *Consider $M^1([0,1]) = C([0,1])'$ endowed with the dual strict order* (4.9)*. Let (q_n) be an enumeration of the rational numbers in $[0,1]$ and let r be an irrational number in $[0,1]$. The functional ϕ given by $\langle \phi, f \rangle := \sum_{n=1}^\infty 1/2^n f(q_n) - f(r)$ satisfies $\phi_+ \gg 0$, but ϕ itself is not positive.*

For an operator $A \in \mathscr{B}(X)$, we have yet formalized a positivity condition in section 2.1, by
(P1) $A \ge 0$ if $A : X_+ \to X_+$.
Other possible definition of positivity may be
(P2) $A : X_+ \backslash \{0\} \to X_+ \backslash \{0\}$;
(P3) $A : X_{++} \to X_{++}$.
We now define a stronger notion of positivity, named as strong positivity condition, by
(P4) $A > 0$ if $A : X_+ \backslash \{0\} \to X_{++}$.
We list without proof some elementary properties about these different notions and also refer to Section 6.1 for further discussion. We have (P2) $\Rightarrow$ (P1), (P3) $\Rightarrow$ (P1) as well as (P4) $\Rightarrow$ ((P3), (P2)). We also have $A : X_+ \to X_+$ iff $A^* : Y_+ \to Y_+$; $A : X_{++} \to X_{++}$ iff $A^* : Y_{++} \to Y_{++}$; $A : X_+ \backslash \{0\} \to X_{++}$ iff $A^* : Y_+ \backslash \{0\} \to Y_{++}$.

We say that $\lambda - \mathcal{L}$ satisfies the strong maximum principle if

$$f \in X_+ \cap D(\mathcal{L}),\ (\lambda - \mathcal{L}) f \ge 0 \quad \text{imply} \quad f \gg 0 \text{ or } f = 0. \tag{4.11}$$

It is worth emphasizing that if $\lambda - \mathcal{L}$ satisfies the strong maximum principle for some $\lambda \in \mathbb{R}$ then $\lambda' - \mathcal{L}$ satisfies the strong maximum principle for any $\lambda' \le \lambda$.

We say that a positive semigroup S is irreducible if

$$\forall f \in X_+ \backslash \{0\}, \forall \phi \in Y_+ \backslash \{0\},\ \exists \tau > 0 \quad \langle S_\tau f, \phi \rangle > 0. \tag{4.12}$$

A semigroup S is classically said to be irreducible and aperiodic if the above positivity condition holds for all sufficiently large times, namely

$$\forall f \in X_+ \backslash \{0\}, \forall \phi \in Y_+ \backslash \{0\},\ \exists T > 0, \forall \tau \ge T \quad \langle S_\tau f, \phi \rangle > 0. \tag{4.13}$$

Other notions of strong positivity for the semigroup S are

$$\exists T > 0, \quad S_T : X_+\backslash\{0\} \to X_{++}, \tag{4.14}$$

$$\exists T > 0, \quad \int_0^T S(t)dt : X_+\backslash\{0\} \to X_{++}. \tag{4.15}$$

We summarize some possible implications between the previous positivity notions.

Lemma 4.8. *For a positive semigroup S, the following hold:*
(1) *The pointwise strong positivity condition* (4.14) *implies the condition* (4.15)*;*
(2) *The integral strong positivity condition* (4.15) *implies the irreducibility condition* (4.12)*, but the reverse implication is false. Similarly, the irreducibility and aperiodicity condition* (4.13) *implies the irreducibility condition* (4.12)*, but the reverse implication is false;*
(3) *The irreducibility condition* (4.12) *is equivalent to the fact that $\mathcal{R}_\mathcal{L}(\lambda) : X_+\backslash\{0\} \to X_{++}$, for any $\lambda > \lambda_1$, as well as to the fact that $\lambda - \mathcal{L}$ satisfies the strong maximum principle* (4.11) *for any $\lambda \in \mathbb{R}$.*

The result is very classic, at least for strongly positive semigroup, see e.g. [15, Definition C.3.1] or [41, Prop. 14.10]. For the sake of completeness, we however present a short proof.

Proof of Lemma 4.8. We prove **(1)**. We assume (4.14) and we fix $g \in X_+\backslash\{0\}$, $\phi \in Y_+\backslash\{0\}$, so that $\langle S(T)g, \phi\rangle > 0$. Observing that the function $t \mapsto \langle S(t)g, \phi\rangle$ is continuous, there exists $\varepsilon > 0$ such that $\langle S(t)g, \phi\rangle > 0$ for any $t \in [T - \varepsilon, T]$, so that

$$\Big\langle \int_0^T S(t)dtg, \phi \Big\rangle = \int_0^T \langle S(t)g, \phi\rangle dt > 0.$$

Because $\phi \in Y_+\backslash\{0\}$ may be chosen arbitrary, we deduce (4.15).
We prove **(2)**. We assume now (4.15) and we fix $g \in X_+\backslash\{0\}$, $\phi \in Y_+\backslash\{0\}$, so that

$$\int_0^T \langle S(t)g, \phi\rangle dt = \Big\langle \int_0^T S(t)dtg, \phi \Big\rangle > 0,$$

by assumption. We get (4.12) by observing that the function $t \mapsto \langle S(t)g, \phi\rangle$ must be positive somewhere on $[0, T]$. For the reverse implication we refer to [50, 169], where is studied an example of growth-fragmentation operator associated to mitosis satisfying the irreducibility condition (4.12) but not the integral strong positivity condition (4.15) nor the irreducibility and aperiodicity condition (4.13), see also Section 9.
We prove **(3)**. We finally assume (4.12). From the above continuity argument, for any $g \in X_+\backslash\{0\}$, $\phi \in Y_+\backslash\{0\}$ there exist $\tau > \varepsilon > 0$ such that $\langle S(t)g, \phi\rangle > 0$ for any $t \in [\tau - \varepsilon, \tau + \varepsilon]$. As a consequence and thanks to the representation formula (2.13) for any fixed $\lambda > \lambda_1$ which holds thanks to Lemma 2.2-**(ii)**, we have

$$\langle \mathcal{R}_\mathcal{L}(\lambda)g, \phi\rangle = \Big\langle \int_0^\infty e^{-\lambda t} S(t)dtg, \phi \Big\rangle > 0.$$

Because $\phi \in Y_+\backslash\{0\}$ is arbitrary, we have established that $\mathcal{R}_\mathcal{L}(\lambda)g \in X_{++}$ for any $g \in X_+\backslash\{0\}$. In other words, when $\lambda > \lambda_1$ and $f \in X_+ \cap D(\mathcal{L})$ satisfy $g := (\lambda - \mathcal{L})f \geq 0$, we deduce that $f = \mathcal{R}_\mathcal{L}(\lambda)g \in X_{++}$, what is the strong maximum principle. This one is obviously equivalent to the strong positivity property $\mathcal{R}_\mathcal{L}(\lambda) : X_+\backslash\{0\} \to X_{++}$. On the other way round, writing the above identity as

$$\int_0^\infty e^{-\lambda t} \langle S(t)g, \phi\rangle dt = \langle \mathcal{R}_\mathcal{L}(\lambda)g, \phi\rangle,$$

we see that the strong maximum principle implies that the RHS term is positive for any $g \in X_+\backslash\{0\}$, $\phi \in Y_+\backslash\{0\}$. As a consequence, the LHS term is positive and there exists $\tau > 0$ such that $\big\langle S(\tau)g, \phi\big\rangle > 0$, which is nothing but the irreducibility condition (4.12).

□

We present two other elementary results about the strong maximum principle.

Lemma 4.9. *Consider $\mathcal{L}$ satisfying* **(H1)** *and $\lambda \in \mathbb{R}$. Then the following assertions are equivalent*
(1) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $f \in D(\mathcal{L}) \cap X_+$;
(2) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $f \in D(\mathcal{L}^k) \cap X_+$ for some $k \geq 1$;
(3) $\lambda - \mathcal{L}^*$ satisfies the strong maximum principle for any $\phi \in D(\mathcal{L}^*) \cap Y_+$;
(4) $\lambda - \mathcal{L}$ satisfies the strong maximum principle for any $\phi \in D((\mathcal{L}^*)^\ell) \cap Y_+$ for some $\ell \geq 1$.

Proof of Lemma 4.9. Assume that $\lambda - \mathcal{L}$ satisfies the strong maximum principle for some $\lambda \in \mathbb{R}$ and $k \geq 1$ and consider $\phi \in D(\mathcal{L}^*) \cap Y_+ \backslash \{0\}$ such that $(\lambda - \mathcal{L}^*)\phi \geq 0$. For any $\kappa > \max(\lambda, \lambda_1)$ and $g \in D(\mathcal{L}^{k-1}) \cap X_+ \backslash \{0\}$, thanks to **(H1)** and the strong maximum principle, there exists $f \in D(\mathcal{L}^k) \cap X_{++}$ such that $(\kappa - \mathcal{L})f = g$. As a consequence, we have

$$\begin{aligned} \langle \phi, g \rangle &= \langle \phi, (\kappa - \mathcal{L})f \rangle \\ &= \langle (\kappa - \mathcal{L}^*)\phi, f \rangle \geq (\kappa - \lambda) \langle \phi, f \rangle > 0. \end{aligned}$$

Since $g \in D(\mathcal{L}^{k-1}) \cap X_+$ is arbitrary and $D(\mathcal{L}^{k-1}) \cap X_+$ is dense in X_+, we deduce that $\phi \gg 0$. We have proved that $\lambda - \mathcal{L}^*$ satisfies the strong maximum principle. The other implications can be proved similarly. □

Remark 4.10. *(1) In many applications, we start proving the strong maximum principle on smooth enough functions (belonging to the iterated domain) for which pointwise arguments may be used.*
(2) We may replace the condition (1) by assuming that $\lambda - \mathcal{L}$ satisfies the strong maximum principle for $f \in \mathcal{C} \cap X_+$ for a subspace $\mathcal{C} \subset D(\mathcal{L})$ such that $(\lambda - \mathcal{L})^{-1} \in \mathscr{B}(\mathcal{C})$ and $\mathcal{C}$ is dense in X.

The strong maximum principle can be seen as a consequence of the weak maximum principle together with the existence of a family of strictly positive barrier functions. We give now a typical result which can be applied (or modified in order to be applied) in many situations.

Lemma 4.11. *We assume that*
(i) the operator $\lambda - \mathcal{L}$ satisfies the weak maximum principle and Kato's inequalities;
(ii) there exists a subset $\mathscr{G} \subset X_{++} \cap \{g \in D(\mathcal{L}); (\mathcal{L} - \lambda)g \geq 0\}$ such that $\forall f \in D(\mathcal{L}) \cap X_+ \backslash \{0\}$, $\exists g \in \mathscr{G}$ such that $(g - f)_+ \in D(\mathcal{L})$.

Then $\lambda - \mathcal{L}$ satisfies the strong maximum principle.

Proof of Lemma 4.11. We consider $f \in D(\mathcal{L}) \cap X_+ \backslash \{0\}$ such that $(\lambda - \mathcal{L})f \geq 0$ and choose $g \in \mathscr{G}$ such that $h := (g - f)_+ \in D(\mathcal{L})$. We remark that from Kato's inequality (4.7), we have

$$(\mathcal{L} - \lambda)h \geq \text{sign}_+(g - f)(\mathcal{L} - \lambda)(g - f) \geq 0.$$

As a consequence of the weak maximum principle, we have $h \leq 0$. That implies $h = 0$, so that $g - f \leq 0$ and finally $f \gg 0$. □

The above barrier functions technique is also useful for obtaining the condition **(H2)** (possibly in a constructive way).

Lemma 4.12. *For an operator $\mathcal{L}$, we assume that*
(i) the condition **(H1)** *holds with a constant $\kappa_1 \in \mathbb{R}$;*
(ii) the hypothesis (ii) in Lemma 4.11 holds with $\lambda = \kappa_1$;
(iii) there exists $h_0 \in X_+ \backslash \{0\}$ such that for any $g \in \mathscr{G}$ there exists $\varepsilon > 0$ such that $g \geq \varepsilon h_0$.

Then the property **(H2)** *holds true.*

Proof of Lemma 4.12. Thanks to assumption (i), we may define $f_0 \in D(\mathcal{L}) \cap X_+ \backslash \{0\}$ as the solution to the equation $(\kappa_1 - \mathcal{L})f_0 = h_0$. From the proof of Lemma 4.11 and condition (iii), there exists $g \in \mathscr{G}$ and next $\varepsilon > 0$ such that $f_0 \geq g \geq \varepsilon h_0$. Coming back to the equation, we have

$$\mathcal{L}f_0 = \kappa_1 f_0 - h_0 \geq (\kappa_1 - \varepsilon^{-1})f_0,$$

so that condition **(H2)** holds true with $\kappa_0 := \kappa_1 - \varepsilon^{-1}$ thanks to Lemma 2.4-**(ii)**. □

4.3. **The geometry of the first eigenvalue problem.** We come back on and state a result about the geometry of the first eigenvalue.

We consider an operator $\mathcal{L}$ on X which satisfies the conclusion **(C1)** about the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem. We next assume

(H1′) the weak maximum principle

$$\lambda > \lambda_1,\ f \in D(\mathcal{L}),\ (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \geq 0 \tag{4.16}$$

and its *Kato's inequalities* counterpart

$$(\text{sign} f)\mathcal{L}f \leq \mathcal{L}|f|, \qquad (\text{sign}_+ f)\mathcal{L}f \leq \mathcal{L}f_+, \tag{4.17}$$

as well as

(H4) the strong maximum principle

$$\lambda \in \mathbb{R},\ f \in X_+ \cap D(\mathcal{L}),\ (\lambda - \mathcal{L})f \geq 0 \quad \text{imply} \quad f \gg 0 \text{ or } f = 0. \tag{4.18}$$

We also assume the same properties for the adjoint operator $\mathcal{L}^*$.

We may then state our main result in this section, where we recall that $N(A)$ denotes the null space associated to the operator A.

Theorem 4.13. *We assume that X and Y are Banach lattices satisfying* **(X1)**. *We consider an unbounded operator $\mathcal{L}$ on X which satisfies the conclusion* **(C1)** *about the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet eigenvalue problem. We also assume that $\mathcal{L}$ and $\mathcal{L}^*$ both satisfy the weak maximum principle and Kato's inequalities* **(H1′)** *as well as the strong maximum principle* **(H4)**.
Then the following hold
i) $f_1 \gg 0$, $\phi_1 \gg 0$ and λ_1 is the unique eigenvalue associated to a positive eigenvector. We next make the normalization choice

$$\|\phi_1\| = 1, \quad \langle \phi_1, f_1 \rangle = 1. \tag{4.19}$$

ii) λ_1 is algebraically simple:

$$N((\mathcal{L} - \lambda_1)^k) = \text{Span}(f_1), \quad \forall\, k \geq 1, \tag{4.20}$$

$$N((\mathcal{L}^* - \lambda_1)^k) = \text{Span}(\phi_1), \quad \forall\, k \geq 1. \tag{4.21}$$

In particular f_1 (resp. ϕ_1) is the unique positive and normalized eigenvector of $\mathcal{L}$ (resp. of $\mathcal{L}^*$) associated to λ_1. Finally, the projection on the first eigenspace (associated to λ_1) is given by

$$\Pi f := \langle f, \phi_1 \rangle f_1.$$

Remark 4.14. *(1) It is worth emphasizing again that* (4.16) *and* (4.17) *for both $\mathcal{L}$ and $\mathcal{L}^*$ are true when $\mathcal{L}$ is the generator of a positive semigroup (see Lemma 2.1 and Remark 4.1) and that* (4.18) *is true when $S_{\mathcal{L}}$ enjoys additional strong positivity (or irreducibility) condition as formulated in* (4.12), (4.13), (4.14) *or* (4.15). *As a consequence, the conclusion of Theorem 4.13 holds true when $\mathcal{L}$ is the generator of a positive semigroup which satisfies the hypotheses of the existence part of the Krein-Rutman Theorem 2.21 and one of the additional above strict positivity conditions.*

(2) Theorem 4.13 has to be compared with the seminal Krein and Rutman Theorem 1.2 ([238]*), to the many results gathered in* [15, Part C-III] *(see in particular* [15, Prop. C.3.5], [15, Thm. C.3.8] *and the original paper* [185]*) and to the more recent contributions* [278, Thm. 5.3], [41, Thm. 14.15] *and* [231, Thm. 5.1]. *Probably many of the conclusions of Theorem 4.13 are very similar (or even included) in the material of* [15, Part C-III]. *However, our assumptions slightly different since we do no make explicit reference to a positive semigroup but rather refer to the weak and strong maximum principles.*

(3) Our proof is quite direct and elementary and uses similar arguments as those used during the proof of [278, Thm. 4.3] *and* [231, Thm. 5.1]. *We learnt this kind of technique in the (less abstract and general) proof of the uniqueness part of* [313, Lem. 2.1].

(4) From ii), we deduce that $\mathcal{L}$ decomposes according to $X = X_0 \oplus X_1$ with $X_1 := \text{Span}\, f_1$ and $X_0 := (\text{Span}\,\phi_1)^\perp = \{f \in X;\ \langle f, \phi_1 \rangle = 0\}$ in the sense of [229, § III.5.6]. *More precisely, $X = X_0 \oplus X_1$ is a topological direct sum, $\mathcal{L} : X_0 \cap D(\mathcal{L}) \to X_0$ and $\mathcal{L} : X_1 \to X_1$.*

(5) One can observe from the proof below that the conclusion (i) in Theorem 4.13 holds under the sole assumptions **(X1)-(i)** *for X and Y,* **(C1)** *and* **(H4)** *for $\mathcal{L}$ and $\mathcal{L}^*$. The conclusion* (4.20) *holds when assuming furthermore that* (4.10) *holds in X and $\mathcal{L}$ satisfies* **(H1′)**. *The similar conclusion* (4.21) *holds when assuming furthermore that* (4.10) *holds in Y and $\mathcal{L}^*$ satisfies* **(H1′)**.

(6) We finally recall that under condition **(H1)**, *the strong maximum principle* **(H4)** *for $\mathcal{L}$ is equivalent to the strong maximum principle* **(H4)** *for $\mathcal{L}^*$ (see Lemma 4.9). When furthermore condition* **(H2)** *holds and λ_1 in* **(C1)** *is defined by* (2.16), *the weak maximum principle* (4.16) *for $\mathcal{L}$ is equivalent to the weak maximum principle* (4.16) *for $\mathcal{L}^*$ (see the proof of Lemma 2.3).*

The proof of Theorem 4.13 is split into the following Lemma 4.15, Lemma 4.17, Lemma 4.18 and Lemma 4.20.

Lemma 4.15. *Under assumptions* **(X1)-(i)**, **(C1)** *and* **(H4)** *for both $\mathcal{L}$ and $\mathcal{L}^*$, the solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem satisfies*

$$f_1 \gg 0 \quad \text{and} \quad \phi_1 \gg 0. \tag{4.22}$$

Proof of Lemma 4.15. We just apply the strong maximum principle to the two eigenfunctions $f_1 \in X\backslash\{0\}$ and $\phi_1 \in Y\backslash\{0\}$. ☐

Remark 4.16. *It is worth emphasizing that the same conclusion clearly holds when instead of* **(C1)** *we only assume that $f_1 \in X_+\backslash\{0\}$ and $\phi_1 \in Y_+\backslash\{0\}$ satisfy*

$$\mathcal{L} f_1 = \lambda_1 f_1, \quad \mathcal{L}^* \phi_1 = \lambda_1^* \phi_1. \tag{4.23}$$

In that case, we deduce that $\lambda_1^* = \lambda_1$ by writing

$$\lambda_1 \langle f_1, \phi_1 \rangle = \langle \mathcal{L} f_1, \phi_1 \rangle = \langle f_1, \mathcal{L}^* \phi_1 \rangle = \lambda_1^* \langle f_1, \phi_1 \rangle,$$

and observing that $\langle f_1, \phi_1 \rangle \neq 0$.

Lemma 4.17. *Under assumptions* **(X1)-(i)**, **(C1)** *and* **(H4)** *for $\mathcal{L}^*$ (resp. $\mathcal{L}$), λ_1 is the unique eigenvalue associated to a positive eigenvector for $\mathcal{L}$ (resp. for $\mathcal{L}^*$).*

Proof of Lemma 4.17. Consider $\lambda \in \mathbb{C}$ and $f \in X_+\backslash\{0\}$ such that $\mathcal{L} f = \lambda f$ and observe that from the proof of Lemma 4.15, we have $\phi_1 \gg 0$. We compute

$$0 = \langle (\lambda - \mathcal{L}) f, \phi_1 \rangle = \langle f, (\lambda - \mathcal{L}^*) \phi_1 \rangle = (\lambda - \lambda_1) \langle f, \phi_1 \rangle,$$

and thus $\lambda = \lambda_1$ since $\langle f, \phi_1 \rangle > 0$. The same proof applies to the dual problem. ☐

Lemma 4.18. *We assume again* **(X1)-(i)**, **(C1)** *and* **(H4)** *for both $\mathcal{L}$ and $\mathcal{L}^*$. Under the additional condition* **(H1′)** *for $\mathcal{L}$ and* (4.10) *in X (resp.* **(H1′)** *for $\mathcal{L}^*$ and* (4.10) *in Y), we have $N(\mathcal{L} - \lambda_1) = \mathrm{Span}(f_1)$ (resp. $N(\mathcal{L}^* - \lambda_1) = \mathrm{Span}(\phi_1)$). In particular, f_1 (resp. ϕ_1) is unique (because of the positivity and the normalization conditions).*

Proof of Lemma 4.18. Consider a eigenfunction $f \in X\backslash\{0\}$ associated to the eigenvalue λ_1. First, we observe from Kato's inequality that

$$\lambda_1 |f| = \lambda_1 \mathrm{sign}(f) f = \mathrm{sign}(f) \mathcal{L} f \le \mathcal{L} |f|.$$

That inequality is in fact an equality, otherwise we would have

$$\lambda_1 \langle |f|, \phi_1 \rangle \neq \langle \mathcal{L} |f|, \phi_1 \rangle = \langle |f|, \mathcal{L}^* \phi_1 \rangle = \lambda_1 \langle |f|, \phi_1 \rangle,$$

and a contradiction. As a consequence, $|f|$ is a solution to the eigenvalue problem $\lambda_1 |f| = \mathcal{L} |f|$, so that

$$\lambda_1 f_\pm = \mathcal{L} f_\pm,$$

by writing $f_\pm = (|f| \pm f)/2$. The strong maximum principle assumption implies $f_\pm \gg 0$ or $f_\pm = 0$, and thus $f_+ \gg 0$ or $f_- \gg 0$ since $f \neq 0$. Without loss of generality we may assume $f_+ \gg 0$. From (4.10), we then deduce $f \gg 0$. We introduce the normalized eigenfunctions $\tilde f := r f$ and $\tilde f_1 = r_1 f_1$ with

$$\langle \tilde f, \phi_1 \rangle = \langle \tilde f_1, \phi_1 \rangle = 1. \tag{4.24}$$

Now, thanks to Kato's inequality again, we write

$$\lambda_1 (\tilde f - \tilde f_1)_+ = \mathrm{sign}_+(\tilde f - \tilde f_1) \mathcal{L} (\tilde f - \tilde f_1) \le \mathcal{L} (\tilde f - \tilde f_1)_+,$$

and for the same reason as above that last inequality is in fact an inequality. The strong maximum principle implies that either $(\tilde f - \tilde f_1)_+ = 0$, which also reads $\tilde f \le \tilde f_1$, or $(\tilde f - \tilde f_1)_+ \gg 0$, which implies that $\tilde f \gg \tilde f_1$ by using again (4.10). Because of the identity (4.24) and the fact that $\phi_1 \in X'_+\backslash\{0\}$, the second case in the above alternative is not possible. Repeating the same argument with $(\tilde f_1 - \tilde f)_+$ we get that $\tilde f_1 \le \tilde f$ and we conclude with $\tilde f = \tilde f_1$. The same proof applies to the dual problem. □

Remark 4.19. *Under the same hypotheses as in Lemma 4.18, we have $\psi \in \operatorname{span}(\phi_1)$ if $\psi \in Y_+$ satisfies $\mathcal{L}^*\psi \ge \lambda_1 \psi$ and $g \in \operatorname{span}(f_1)$ if $g \in X_+$ satisfies $\mathcal{L} g \ge \lambda_1 g$. In the second case, we indeed cannot have $\mathcal{L}^* g - \lambda_1 g \in X_+\backslash\{0\}$, since this would implies*

$$\langle \mathcal{L} g - \lambda_1 g, \phi_1 \rangle > 0,$$

and this would be in contradiction with the fact that

$$\langle \mathcal{L} g - \lambda_1 g, \phi_1 \rangle = \langle g, \mathcal{L}^*\phi_1 - \lambda_1 \phi_1 \rangle = 0.$$

We thus must have $\mathcal{L} g - \lambda_1 g = 0$ and we conclude thanks to Lemma 4.18. The same proof applies to the dual problem.

Lemma 4.20. *Under the same assumptions as in Lemma 4.18, λ_1 is algebraically simple for $\mathcal{L}$ (resp. for $\mathcal{L}^*$).*

Proof of Lemma 4.20. We use an induction argument. We have already proved that $N((\mathcal{L} - \lambda_1)^k) = \operatorname{Span}(f_1)$ for $k = 1$. Assume then the result proved for any ℓ, $1 \le \ell \le k$, and consider $f \in N((\mathcal{L} - \lambda_1)^{k+1})$. That means that $(\mathcal{L} - \lambda_1) f \in N((\mathcal{L} - \lambda_1)^k)$, and thus $(\mathcal{L} - \lambda_1) f = r f_1$, with $r \in \mathbb{R}$, thanks to the induction hypothesis. If $r = 0$, then $f \in N(\mathcal{L} - \lambda_1) = \operatorname{Span}(f_1)$. Otherwise, $r \ne 0$, and then

$$\lambda_1 \langle f, \phi_1 \rangle = \langle f, \mathcal{L}^* \phi_1 \rangle = \langle \mathcal{L} f, \phi_1 \rangle = \langle \lambda_1 f + r f_1, \phi_1 \rangle,$$

which in turn implies $r\langle f_1, \phi_1 \rangle = 0$ and a contradiction. That concludes the proof. □

4.4. **Mean ergodicity.** We deduce from the above analysis a first classical and general but rough information about the long-time behaviour of the trajectories associated to a semigroup.
More precisely, assuming the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) for the generator $\mathcal{L}$ of a semigroup S and introducing the rescaled semigroup $\widetilde S_t := e^{-\lambda_1 t} S(t)$, we wish to establish the following mean ergodic property

(E1) for any $f \in X$, there holds

$$\frac{1}{T} \int_0^T \widetilde S_t f dt \to \langle f, \phi_1 \rangle f_1, \quad \text{as} \quad T \to \infty, \tag{4.25}$$

in a sense to be specified.

We start with a general result, taken from [152, Thm. V.4.5], which states that, under the conclusions of Theorem 4.13, **(E1)** holds for the strong topology if the semigroup $(\widetilde S_t)$ is bounded.

Theorem 4.21. *Consider a positive semigroup S on a Banach lattice X and assume that its generator $\mathcal{L}$ satisfies the conclusions of Theorem 4.13 about the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1). Assume furthermore that $(\widetilde S_t)_{t \ge 0}$ is bounded. Then, the above mean ergodic property* **(E1)** *holds for the strong topology.*

Proof of Theorem 4.21. Following the proof of [152, Thm. V.4.5], we consider the subspace

$$X_0 := \operatorname{Span} f_1 \oplus \operatorname{Span}\{f - \widetilde S(t) f \,:\, f \in X, t \ge 0\}$$

of X and we take $\phi \in Y$ which vanishes on X_0. Since ϕ vanishes on each element of the form $f - S(t) f$, this implies that $\widetilde S^*(t)\phi = \phi$ for all $t \ge 0$. We deduce that $\mathcal{L}^*\phi = \lambda_1 \phi$, and consequently $\phi \in \operatorname{Span} \phi_1$ due to the point *ii)* in Theorem 4.13. Since we also have $\langle \phi, f_1 \rangle = 0$, we deduce that $\phi = 0$ and therefore $\overline{X_0} = X$. We observe now

$$\Big(\int_0^T \widetilde S(s) ds \Big) \big(I - \widetilde S(t) \big) = \big(I - \widetilde S(T) \big) \int_0^t \widetilde S(s) ds$$

for all $t, T \ge 0$, which is an immediate consequence of the semigroup property. The above relation and the boundedness assumption on $(\widetilde S_T)_{T \ge 0}$ imply that the convergence (4.25) holds for $f =$

$g - \widetilde{S}(t)g$ with $g \in X$, $t \geq 0$, and thus for any $f \in X_0$. Finally, since X_0 is dense in X and using again the fact that $(\widetilde{S}_t)_{t\geq 0}$ is bounded, we can readily extend the validity of (4.25) to any $f \in X$. □

We will now give weaker versions of Theorem 4.21 with proofs which are based on compactness arguments. The motivation for providing such alternative proofs that require stronger assumptions is that, unlike the proof of Theorem 4.21, the methods can be adapted to derive stronger ergodicity results, namely without averaging in time, see Section 5.5.

Theorem 4.22. *Consider a positive semigroup S on a Banach lattice X and assume that its generator $\mathcal{L}$ satisfies the conclusions of Theorem 4.13 about the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1). With the above notations, we assume furthermore that*
(1) *$(\widetilde{S}_t)$ is bounded;*
(2) *B_X is weakly compact for a topology which makes $f \mapsto \langle f, \phi_1\rangle$ continuous.*
Then, the above mean ergodic property **(E1)** *holds for the topology introduced in* **(2)**.

Proof of Theorem 4.22. Fix $f \in X$ and define

$$u_T := \frac{1}{T}\int_0^T \widetilde{S}_t f\, dt.$$

From **(1)**, we have

$$\|u_T\| \leq \frac{1}{T}\int_0^T \|\widetilde{S}_t f\|\, dt \leq M\,\|f\|, \quad \forall\, T > 0.$$

We also compute

$$\langle u_T, \phi_1\rangle = \frac{1}{T}\int_0^T \langle \widetilde{S}_t f, \phi_1\rangle\, dt = \langle f, \phi_1\rangle, \quad \forall\, T > 0.$$

Thanks to assumption **(2)**, we deduce that there exists $f^* \in X$ and a sequence (T_k) such that

$$u_{T_k} \to f^* \quad \text{and} \quad \langle f^*, \phi_1\rangle = \langle f, \phi_1\rangle.$$

Because $(\widetilde{S}_t f)$ is bounded, we may use the usual ergodicity trick as in the second proof of Theorem 3.1 and for any $t > 0$, we have

$$\widetilde{S}_t f^* - f^* = \lim_{k\to\infty} \frac{1}{T_k}\Big\{\int_{T_k}^{T_k+t} \widetilde{S}_s f ds - \int_0^t \widetilde{S}_s f ds\Big\} = 0.$$

We have established $(\mathcal{L} - \lambda_1)f^* = 0$, so that $f^* \in \mathrm{Span}(f_1)$ and more precisely $f^* = \langle f, \phi_1\rangle f_1$. By uniqueness of the limit, it is the whole family (u_T) which converges to f^*. □

We present a variant of the previous result in which we see that in a very general framework (including all the applications we present in the second part of this work) the above hypotheses **(1)** and **(2)** are not needed (or more precisely are automatically satisfied).

Theorem 4.23. *(1) Consider a Banach lattice $X \subset L^1_{\rm loc}(E, \mathscr{E}, \mu)$ and $Y \subset L^1_{\rm loc}(E, \mathscr{E}, \mu)$ (so that in particular $\phi_1 \in L^1_{loc}$ and $L^1_{\phi_1}$ is well-defined) and a positive semigroup S on X such that its generator $\mathcal{L}$ satisfies the conclusions of Theorem 4.13 about the existence, positivity and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1). Then the mean ergodic convergence* **(E1)** *holds for the weak topology of $L^1_{\phi_1}$.*
(2) Assuming additionally that S is strongly continuous and that

(4.26) $$\mathcal{X}^k := (D(\mathcal{L}^k), \|\cdot\|_{\mathcal{X}^k}) \subset L^1_{\rm loc} \text{ with strong compact embedding for some } k \geq 1,$$

where

$$\|f\|_{\mathcal{X}^k} := \|f\|_{L^1_{\phi_1}} + \cdots + \|\mathcal{L}^k f\|_{L^1_{\phi_1}}, \quad \forall\, f \in D(\mathcal{L}^k),$$

then the mean ergodic convergence **(E1)** *holds for the strong topology of $L^1_{\phi_1}$.*

Proof of Theorem 4.23. Step 1. We first recall a very classical result about conservative semigroups. Denoting $\widetilde{S}_t := e^{-\lambda_1 t}\, S(t)$, we observe that this rescaled semigroup satisfies

- **(i)** $\widetilde{S}_t \geq 0$;
- **(ii)** $\widetilde{S}_t f_1 = f_1$ for any $t \geq 0$;

(iii) $\langle \widetilde{S}_t g, \phi_1 \rangle = \langle g, \phi_1 \rangle$ for any $g \in X$ and $t \geq 0$.

We denote $[f]_1 := \langle |f|, \phi_1 \rangle$ which is a norm on X (we use here that $\phi_1 \gg 0$) and $\widetilde{S}_t$ is obviously a contraction for this one. Indeed, for any $f \in X$, there holds

$$|\widetilde{S}_t f| = |\widetilde{S}_t f_+ - \widetilde{S}_t f_-| \leq \widetilde{S}_t f_+ + \widetilde{S}_t f_- = \widetilde{S}_t |f|,$$

using **(i)** in the inequality, and next

$$[\widetilde{S}_t f]_1 = \langle |\widetilde{S}_t f|, \phi_1 \rangle \leq \langle \widetilde{S}_t |f|, \phi_1 \rangle = [f]_1, \tag{4.27}$$

using **(iii)** in the last equality. Abusing notations, we also denote by $\mathcal{X}$ the completion of X for the $L^1_{\phi_1}$ norm (so that we may identify $\mathcal{X}$ to a closed subspace of $L^1_{\phi_1}$). We may then extend $\widetilde{S}_t$ to $\mathcal{X}$ by uniform continuity and this extension still satisfies the properties **(i)-(ii)-(iii)** on $\mathcal{X}$. Consider now $f \in X$ such that $H(f/f_1)f_1 \in X$ for some convex function $H : \mathbb{R} \to \mathbb{R}$, where we use here that $X \subset L^1_{\rm loc}$, and thus in particular $f_1 > 0$ a.e. on E, in order to give a sense to the term $H(f/f_1)f_1$. From **(ii)**, we have

$$\ell[(\widetilde{S}_t f)/f_1] f_1 = \widetilde{S}_t[\ell(f/f_1)f_1],$$

for any real affine function ℓ. Next from **(i)** and (2.7), we have

$$H[(\widetilde{S}_t f)/f_1] f_1 \leq \widetilde{S}_t[H(f/f_1)f_1],$$

because of $H = \sup_{\ell \leq H} \ell$ and the supremum can be taken on a numerable set of affine functions. Thanks to **(iii)**, we conclude that

$$\langle H[(\widetilde{S}_t f)/f_1] f_1, \phi_1 \rangle \leq \langle H[f/f_1] f_1, \phi_1 \rangle, \quad \forall t \geq 0. \tag{4.28}$$

Step 2. We normalize $\langle f_1, \phi_1 \rangle = 1$. For $f \in \mathcal{X} \subset L^1_{\phi_1}$ so that $f\phi_1 = (f/f_1)f_1\phi_1 \in L^1$, the de la Vallée Poussin theorem tells us that there exists an even and convex function $H : \mathbb{R} \to \mathbb{R}_+$ such that $H(s)/s \to +\infty$ as $s \to \infty$ and $H(f/f_1)f_1\phi_1 \in L^1$. Using the notations of the proof of Theorem 4.22, the Jensen inequality and the above estimate (4.28), we deduce

$$\int_E H(u_T/f_1) f_1 \phi_1 d\mu \leq \frac{1}{T} \int_0^T \int_E H[(\widetilde{S}_t f)/f_1] f_1 \phi_1 d\mu dt \leq \int_E H(f/f_1) f_1 \phi_1 d\mu,$$

for any $T > 0$. Now, for any $A \in \mathscr{E}$ and $T, K > 0$, we have

$$\begin{aligned}
\int_E u_T \mathbf{1}_A \phi_1 d\mu &= \int_E \frac{u_T}{f_1} \mathbf{1}_{\frac{|u_T|}{f_1} > K} \mathbf{1}_A f_1 \phi_1 d\mu + \int_E \frac{u_T}{f_1} \mathbf{1}_{\frac{|u_T|}{f_1} \leq K} \mathbf{1}_A f_1 \phi_1 d\mu \\
&\leq \frac{K}{H(K)} \int_E H(u_T/f_1) f_1 \phi_1 d\mu + K \int_E \mathbf{1}_A f_1 \phi_1 d\mu \\
&\leq \frac{K}{H(K)} \int_E H(f/f_1) f_1 \phi_1 d\mu + K \int_E \mathbf{1}_A f_1 \phi_1 d\mu,
\end{aligned}$$

from what we immediately deduce that (u_T) belongs to a weak compact set of $L^1_{\phi_1}$. We conclude that (4.25) holds for the weak convergence in $L^1_{\phi_1}$ as in the proof of Theorem 4.22.

Step 3. We now additionally assume that (4.26) holds with strong compact embedding for some $k \geq 1$. Taking $f \in D(\mathcal{L}^k)$, we compute

$$\langle |\mathcal{L}^j(\widetilde{S}_t f)|, \phi_1 \rangle = \langle |\widetilde{S}_t(\mathcal{L}^j f)|, \phi_1 \rangle \leq \langle |\mathcal{L}^j f|, \phi_1 \rangle,$$

for any $j \leq k$ and any $t \geq 0$, and thus the same bound holds for (u_T). From (4.26), we deduce that up to the extraction of a subsequence, (u_T) converges a.e. on E. Together with the weak convergence in $L^1_{\phi_1}$ yet established, we classically deduce that the whole family (u_T) converges for the strong topology in $L^1_{\phi_1}$. We conclude that the same holds for any $f \in X$ by taking advantage of the fact that $D(\mathcal{L}^k)$ is dense in X for the strong topology of X, and thus for the strong topology of $\mathcal{X}$, and of the estimate of contraction (4.27). □

Remark 4.24. *(1) A similar conclusion holds as in Theorem 4.23 when we assume $X \subset M^1_{\rm loc}$, $D(\mathcal{L}^k) \subset L^1_{\rm loc}$ and $D(\mathcal{L}^{*k}) \subset L^1_{\rm loc}$ for some $k \geq 1$ instead of $X, Y \subset L^1_{\rm loc}$. For $f \in D(\mathcal{L}^k) \subset L^1_{\rm loc}$, we may indeed repeat the proof of Theorem 4.23 and we obtain the same conclusion. We next define $\mathcal{X}$ as the closure of $D(\mathcal{L}^k)$ for the norm $[\cdot]_1$. We conclude that (4.25) holds weakly in $L^1_{\phi_1}$ for any $f \in \mathcal{X}$ by a density argument.*

(2) The proof of Theorem 4.23 is based on so-called General Relative Entropy (GRE) techniques as developed for instance in [258], [269] *and* [49]. *These ones are known to be useful for some classes of PDEs and for stochastic semigroups in order to establish uniform in time estimates and longtime convergence results.*

The main interest of the two previous results is that they do not ask any new information on the semigroup but they are just based on the eigentriplet stationary problem. The shortcoming is that they are formulated in terms of the norm $[\cdot]_1$ instead of the norm of X. We present a second variant of Theorem 4.22 which is well adapted to the splitting framework developed in Sections 2 and 3 and is precisely formulated in a weak or strong topology of a space $X_0 \supset X$.

Theorem 4.25. *Consider a positive semigroup $S = S_{\mathcal{L}}$ such that $\mathcal{L}$ satisfies the conclusions of Theorem 4.13 about the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1). Assume furthermore that S satisfies the splitting structure introduced in* **(HS2)** *in section 3.2 or* **(HS3)** *in Section 3.2, or more precisely, there exist two families of operators $(V(t))$ and $(W(t))$ such that*

$$S = V + W * S,$$

a real number $\kappa \le \lambda_1$ and some decaying functions $\Theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ with $\Theta_1(t) \to 0$ as $t \to \infty$, $\Theta_2 \in L^1(\mathbb{R}_+)$ such that the following estimates hold

$$\|V(t)e^{-\kappa t}\|_{\mathscr{B}(X)} = \mathcal{O}(1), \quad \|V(t)e^{-\kappa t}\|_{\mathscr{B}(X,X_0)} = \mathcal{O}(\Theta_1), \tag{4.29}$$

$$\|W(t)e^{-\kappa t}\|_{\mathscr{B}(\mathcal{X}_0,\mathcal{X}_1)} = \mathcal{O}(\Theta_2), \tag{4.30}$$

with $\mathcal{X}_1 \subset X_0 \subset \mathcal{X}_0$, where $\mathcal{X}_0$ is the space X endowed with the norm $[g]_1 := \langle |g|, \phi_1\rangle$.

(1) Assume furthermore that $\mathcal{X}_1 \subset X_0$ with compact embedding for the weak or the strong topology in X_0 and this topology makes $f \mapsto \langle f, \phi_1\rangle$ continuous. Then the mean ergodic convergence **(E1)** *holds true for the above strong or weak topology.*

(2) Assume furthermore that $X \subset L^1_{\text{loc}}$, S is strongly continuous, and that the space $\mathcal{X}^k$ defined by (4.26) is strongly compact embedded in L^1_{loc} for some $k \ge 1$. Then the mean ergodic convergence **(E1)** *holds true for the strong topology of X_0.*

Proof of Theorem 4.25. We define

$$\widetilde{V}(t) := V(t)e^{-\lambda_1 t}, \quad \widetilde{W}(t) := W(t)e^{-\lambda_1 t},$$

so that

$$\widetilde{S} = \widetilde{V} + \widetilde{W} * \widetilde{S},$$

and

$$M := \sup_{t\ge 0} \|\widetilde{V}(t)\|_{\mathscr{B}(X)} < \infty, \quad \|\widetilde{V}\|_{\mathscr{B}(X,X_0)} \lesssim \widetilde{\Theta}_1 \in C_0(\mathbb{R}_+), \tag{4.31}$$

$$\widetilde{\Theta}_2(t) := \|\widetilde{W}(t)\|_{\mathscr{B}(\mathcal{X}_0,\mathcal{X}_1)} \in L^1(\mathbb{R}_+).$$

Step 1. We furthermore assume (1) and that the weak topology of X_0 makes $f \mapsto \langle f, \phi_1\rangle$ continuous. We denote by $\mathscr{T}$ the weak or the strong topology X_0 (depending of the assumption made on the embedding $\mathcal{X}_1 \subset X_0$). For $f_0 \in X$, we split

$$f(t) := \widetilde{S}_t f_0 = v(t) + k(t), \quad v(t) := \widetilde{V}(t) f_0, \quad k(t) := (\widetilde{W} * \widetilde{S})(t) f_0,$$

and we observe that $\|v(t)\|_{X_0} \to 0$ as $t \to \infty$ from the second estimate in (4.31). On the other hand, we have

$$\sup_{t\ge 0} \|k(t)\|_{\mathcal{X}_1} \le \|\widetilde{W}\|_{L^1} \sup_{t\ge 0} \|\widetilde{S}_t f_0\|_{\mathcal{X}_0} \le \|\widetilde{W}\|_{L^1} \|f_0\|_{\mathcal{X}_0},$$

from (4.27). In particular, $k(t)$ belongs to a compact set of $\mathscr{T}$, so that $(f(t))_{t\ge 0}$ also belongs to a compact set for the same topology $\mathscr{T}$. The same argument used on the Cesàro function (u_T) defined during the proof of Theorem 4.22 implies that there exist $f^* \in X$ and a sequence (T_k) such that

$$u_{T_k} \to f^* \text{ in the sense of } \mathscr{T} \quad \text{and} \quad \langle f^*, \phi_1\rangle = \langle f, \phi_1\rangle,$$

the last identity following from the assumption that $f \mapsto \langle f, \phi_1\rangle$ continuous for $\mathscr{T}$. We may then conclude as in the proof of Theorem 4.22.

Step 2. We furthermore assume (2), and by linearity we may assume $f_0 \in X$, $\langle f_0, \phi_1\rangle = 0$. We recall that (4.25) holds for the strong topology of $L^1_{\phi_1}$ from Theorem 4.23 and that $\|v(t)\|_{X_0} \to 0$ as $t \to \infty$ from Step 1. Arguing as in Step 3 of the proof of Theorem 3.4, we have

$$K(T) := \frac{1}{T}\int_0^T (W * \widetilde{S})(t)\,dt \;=\; \frac{1}{T}\int_0^T W(s)\int_0^{T-s}\widetilde{S}(u)\,duds$$
$$= \int_0^T W(s)\frac{T-s}{T}\widetilde{U}(T-s)\,ds,$$

where $\widetilde{U}_T := U_T e^{-\lambda_1 T}$, U_t is defined by (3.44), so that $u_T = \widetilde{U}_T f_0$ and $[u_T]_1 \to 0$ as $T \to \infty$ from Theorem 4.23. As a consequence, we have

$$\|K(T)f_0\|_{\mathcal{X}_1} \;\le\; \int_0^{T/2}\Theta_2(s)[\widetilde{U}(T-s)]_0\,ds + \int_{T/2}^T \Theta(s)[\widetilde{U}(T-s)]_0\,ds$$
$$\le\; \|\widetilde{\Theta}_2\|_{L^1}\sup_{t\ge T/2}[\widetilde{U}(t)]_0 + \int_{T/2}^\infty \widetilde{\Theta}_2(s)ds\sup_{t\ge 0}[\widetilde{U}(t)]_0 \to 0,$$

as $T \to \infty$. All together, we have established that $\|u_T\|_{X_0} \to 0$ as $T \to \infty$. □

5. The geometry of the boundary point spectrum

We summarize the results established up to now by assuming that the main conclusions in the previous sections are achieved, namely

(C2) the first eigentriplet problem (4.1) has a unique solution (λ_1, f_1, ϕ_1), and furthermore, $f_1 \gg 0$ and $\phi_1 \gg 0$. In that situation, we make the usual normalization (4.19).

In this section, we aim to describe one step further the geometry of the spectrum and more precisely to get some some information on the boundary point spectrum

$$\Sigma_P^+(\mathcal{L}) := \Sigma_P(\mathcal{L}) \cap \overline{\Delta}_{\lambda_1} = \Sigma_P(\mathcal{L}) \cap \Sigma_+(\mathcal{L}).$$

That will be possible by introducing first a suitable and usual complexification framework and next by assuming a stronger positivity property on $\mathcal{L}$ or on the associated semigroup. Here and for further references below, we recall that we define the sets

$$\Sigma_d(\mathcal{L}) \subset \Sigma_P(\mathcal{L}) \subset \Sigma(\mathcal{L}),$$

where the point spectrum set $\Sigma_P(\mathcal{L})$ is the set of eigenvalues, namely $\lambda \in \Sigma_P(\mathcal{L})$ if $N(\mathcal{L}-\lambda) \neq \{0\}$, and the discret spectrum set $\Sigma_d(\mathcal{L})$ is the set of eigenvalues which are isolated and have finite algebraic multiplicity.

5.1. Complexification and the sign operator.

We present some materials, most of them being very classical, about the sign operator in a complex Banach lattice and we refer to [337, 15] for more details.

Complexification. The complexification space $X_\mathbb{C}$ associated to a real Banach lattice X is defined by $X_\mathbb{C} := X + iX$ so that $f \in X_\mathbb{C}$ if $f = g + ih$, $g, h \in X$. In general, we just write X without mentioning the field, although when we need to specify it, we write $X_\mathbb{C}$ or $X_\mathbb{R}$. We extend on $X_\mathbb{C}$ the order defined on $X_\mathbb{R}$ by writing

$$f = g + ih \ge 0 \quad \text{if} \quad g \ge 0 \text{ and } h = 0.$$

The conjugate $\bar f$ of a complex vector $f = g + ih$ is classically defined by $\bar f = g - ih$. We then define the modulus

$$|f| := \sup_{\theta\in[0,2\pi]} (g\cos\theta + h\sin\theta), \tag{5.1}$$

which indeed exists for such a family of vectors. One checks the usual modulus properties:

$$|f| \ge 0, \quad |f| = 0 \text{ iff } f = 0, \quad |\lambda f| = |\lambda|\,|f|, \quad |f+g| \le |f| + |g|,$$

for any $f, g \in X$ and $\lambda \in \mathbb{C}$. We finally define the norm on $X_\mathbb{C}$ by

$$\|f\| := \||g + ih|\|_{X_\mathbb{R}},$$

and we observe that $X_\mathbb{C}$ has a complex Banach lattice structure. We extend the definition of $A \in \mathcal{B}(X_\mathbb{R})$ to $X_\mathbb{C}$ by setting

$$A(g+ih) = Ag + iAh, \quad \forall\, g+ih \in X_\mathbb{C}.$$

The operator sign**.** We classically extend the sign operator defined in Section 4.1 to the present complex Banach lattice framework. Instead of dealing with the most general case, we will use some regularity assumption on the Banach lattice X which is suitable for our purpose and that we present below. Similarly as in Remark 4.5, for $f \in X$, we define

$$X_f := \bigcup_n \{g \in X;\ |g| \le n|f|\},$$

and next, similarly as in Theorem 2.24, we define

$$A_f[g] := \inf\{C > 0;\ |g| \le C|f|\}, \quad \forall\, g \in X_f.$$

We summarize the regularity conditions we need on the Banach lattice X by assuming :

(X2) For any $f \in X$ such that $|f| \in X_{++}$, there exists a sign operator $\operatorname{sign} f \in \mathscr{B}(X)$, with the following properties

$$\operatorname{sign} f \circ \operatorname{sign} \bar f = I, \quad (\operatorname{sign} f) f = |f|, \tag{5.2}$$

$$(\operatorname{sign} f)\, g = (\operatorname{sign}(uf))\,(ug), \quad |(\operatorname{sign} f)\, g| \le |g|, \quad \forall\, g \in X,\ \forall\, u \in \mathbb{S}^1. \tag{5.3}$$

and furthermore

(X3) for any $f \in X$ such that $|f| \in X_{++}$, the inclusion $X_f \subset X$ is dense for the strong, the weak, or the weak-$*$ topology, and for all $f \in X$ and $g \in X_f$

$$\big(g \in X_\mathbb{R} \text{ and } |g| \le C|f| \big) \ \Leftrightarrow\ A_f[g - ir|f|] \le \sqrt{C^2 + r^2},\ \forall\, r \in \mathbb{R}. \tag{5.4}$$

For a space of functions, the sign operator is defined as the multiplication by (abusing notations)

$$\operatorname{sign} f := \bar f / |f|, \quad \forall\, f \in X,\ |f| \in X_{++}. \tag{5.5}$$

Lemma 5.1. *With* (5.5)*, the properties* **(X2)** *and* **(X3)** *hold when* $X = L^p(E, \mathscr{E}, \mu)$ *or* $X = C_0(E)$*.*

Proof of Lemma 5.1. For $f \in X$, $|f| \in X_{++}$, we just indicate the proof of $\overline{X}_f = X$, the other algebraic properties being clear from the definition (5.5). When $f \in L^p$ such that $|f| > 0$ μ-a.e. and $0 \le g \in L^p$, we set $g_n := g \wedge (n|f|)$. We have $0 \le g_n \le g$ and $g_n \to g$ strongly L^p if $p < \infty$ and weakly-$*$ L^∞ if $p = \infty$. The general case $g \in L^p$ is dealt in the usual way by introducing positive and negative parts and next real and imaginary part. That concludes the proof of $\overline{X}_f = L^p$. The proof of $\overline{X}_f = C_0(E)$ is similar. ☐

A sign operator satisfying **(X2)** and **(X3)** can actually be built by using Kakutani's theorem in general Banach lattices whenever $|f|$ is a quasi-interior point, see for instance [41, Chapter 14.3]. In $X = L^\infty(E, \mathscr{E}, \mu)$, being a quasi-interior point is more demanding than belonging to X_{++}, and our framework is thus more general in that case. In $X = M^1(E)$, the situation is even worst since there is no quasi-interior point, so the approach via Kakutani's theorem does not provide any sign operator. However, we can associate to $f \in M^1(E)$ such that $|f| \gg 0$ a sign operator by means of the Radon-Nikodym theorem. Denoting $\alpha : E \to \mathbb{S}^1$ the measurable function such that $f = \alpha|f|$, the multiplication by $\bar\alpha/|\alpha|$ defines a sign operator $\operatorname{sign} f \in \mathcal{B}(X)$, or in other words (abusing notations)

$$\operatorname{sign} f := \bar\alpha / |\alpha|, \quad \forall\, f = \alpha|f| \in M^1,\ |f| \in M^1_{++}. \tag{5.6}$$

Lemma 5.2. *With the definition* (5.6)*,* $X = M^1(E)$ *enjoys the properties* **(X2)** *and* **(X3)***.*

Proof of Lemma 5.2. As for Lemma 5.1, we only sketch the proof of the density property $\overline{X}_f = X$, which holds here for the weak-$*$ topology, the other algebraic properties being clear from the definition (5.6). Without loss of generality, we may take $f \in X_{++}$, meaning that $f(\mathcal{O}) > 0$ for any open set $\mathcal{O} \subset E$. For $\varepsilon > 0$ and $\varphi \in C_0(E)$, we can find a partition $E_1, \dots, E_n$ of E and some elements $x_1, \dots, x_n$ of E such that for any $i \in \{1, \cdots, n\}$:

$$f(E_i) > 0, \quad x_i \in E_i \quad \text{and} \quad \sup_{x \in E_i} |\varphi(x) - \varphi(x_i)| < \varepsilon.$$

For $g \in X$ and $\varepsilon > 0$, defining g_ε by

$$g_\varepsilon := \sum_{i=1}^n \frac{g(E_i)}{f(E_i)} f_{|E_i} \in X_f,$$

we have

$$\left| \langle g_\varepsilon, \varphi \rangle - \sum_{i=1}^n \varphi(x_i) g(E_i) \right| \le \sum_{i=1}^n |g(E_i)| \int_{E_i} |\varphi(x) - \varphi(x_i)| \frac{f(dx)}{f(E_i)} \le \varepsilon \|g\|_X,$$

as well as

$$\left| \langle g, \varphi \rangle - \sum_{i=1}^n \varphi(x_i) g(E_i) \right| \le \sum_{i=1}^n \int_{E_i} |\varphi(x) - \varphi(x_i)| \, |g|(dx) \le \varepsilon \|g\|_X.$$

We have established that $|\langle g_\varepsilon - g, \varphi \rangle| \le 2\varepsilon \|g\|$ for any $\varepsilon > 0$, from what we deduce that g belongs to the weak-$*$ closure of X_f. □

Lemma 5.3. *Assume* **(X2)**-**(X3)***, and $f \in X_{++}$. Consider a linear operator $\mathcal{Q} : X_f \to X_f$ such that $\mathcal{Q} f = f$ and $A_f(\mathcal{Q} g) \le A_f(g)$ for any $g \in X_f$. Then $\mathcal{Q} \ge 0$.*

Proof of Lemma 5.3. Take $0 \le g \in X_f$ such that $g \le 2Cf$, $C > 0$, and observe that

$$-Cf \le g - Cf \le Cf.$$

For any $r \in \mathbb{R}$, we compute

$$\begin{aligned} A_f[(\mathcal{Q} g) - Cf - irf] &= A_f[\mathcal{Q}(g - Cf - irf)] \\ &\le A_f[g - Cf - irf] \\ &\le \sqrt{C^2 + r^2}, \end{aligned}$$

by using the non expansion property of $\mathcal{Q}$ and the claim (5.4). Using again (5.4), we deduce $-Cf \le (\mathcal{Q} g) - Cf \le Cf$ and the conclusion. □

We generalize Kato's inequality (4.5) to the present complex framework by saying that an operator $\mathcal{L}$ on X satisfies (the complex) Kato's inequality if

$$\forall\, f \in D(\mathcal{L}), \quad \Re e(\mathrm{sign} f) \mathcal{L} f \le \mathcal{L} |f|, \tag{5.7}$$

possibly in a dual sense as in (4.6). As for the real Kato's inequality, when $\mathcal{L}$ is the generator of a semigroup, Kato's inequality (5.7) is a consequence of the positivity of the semigroup, and we refer to Remark 4.1 for references about this claim.

5.2. On the subgroup and discrete structure of the boundary point spectrum.

In this section, we establish that the boundary point spectrum enjoys a subgroup structure under the same kind of hypotheses as considered in the previous sections.

Lemma 5.4. *Under assumptions* **(C2)**, **(X2)** *and the complex Kato's inequality* (5.7)*, for any $\lambda \in \Sigma_P^+(\mathcal{L}) \backslash \{0\}$ the associated normalized eigenfunction f satisfies $|f| = f_1$.*

Proof of Lemma 5.4. By definition $\mathcal{L} f = \lambda f$ and $f \in D(\mathcal{L})$. By linearity of the operator sign and thanks to (5.2) and Kato's inequality (5.7), we have

$$\lambda_1 |f| = \Re e[\lambda (\mathrm{sign} f) f] = \Re e(\mathrm{sign} f)(\lambda f) = \Re e(\mathrm{sign} f) \mathcal{L} f \le \mathcal{L} |f|.$$

By the duality argument introduced during the proof of Lemma 4.18, we must have $\lambda_1 |f| = \mathcal{L} |f|$ and the conclusion. □

Theorem 5.5. *Assume* **(C2)**, **(X2)**, **(X3)** *and that the complex Kato's inequality* (5.7) *holds true. Denoting $\widetilde{\mathcal{L}} = \mathcal{L} - \lambda_1$, the set $\mathcal{S} := \Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R}$ is an additive subgroup and* $\mathrm{dim} N(\widetilde{\mathcal{L}} - i\alpha)^k = 1$ *for any $i\alpha \in \mathcal{S}$ and $k \ge 1$.*

Theorem 5.5 is similar but more general than [15, C-III, Cor. 2.12] and [41, Prop. 14.15]. Our proof is also very similar to the proof of [41, Prop. 14.15]. However, it is more direct and avoid the use of the $C(K)$ algebra and Kakutani's Theorem [227] (see also [264, Thm. 2.1.3]).

Proof of Theorem 5.5. We split the proof into three steps.

Step 1. We consider f associated to an eigenvalue $i\alpha \in \Sigma_P(\widetilde{\mathcal{L}})\backslash\{0\}$, and we define

$$T(t) := (\operatorname{sign} f) e^{-i\alpha t} \widetilde{S}(t) (\operatorname{sign} \bar{f}).$$

Observing that $\widetilde{S}(t) f = e^{i\alpha t} f$, we have

$$T(t)|f| = (\operatorname{sign} f) e^{-i\alpha t} \widetilde{S}(t) f = (\operatorname{sign} f) f = |f| = \widetilde{S}(t)|f|.$$

On the other hand, we have

$$|T(t)g| \leq |\widetilde{S}(t)(\operatorname{sign} \bar{f})g| \leq \widetilde{S}(t)|g|, \quad \forall\, g \in X,$$

which, by positivity of $\widetilde{S}(t)$, yields

$$|T(t)g| \leq A_f(g) \widetilde{S}(t)|f| = A_f(g)|f|, \quad \forall\, g \in X_f.$$

Because $|f| = f_1 \gg 0$ from Lemma 5.4, we can apply Lemma 5.3 to $|f|$ and $\mathcal{Q} := T(t)$. We deduce that $T(t) \geq 0$ on $X_{|f|} = X_f$, and then on $X = \overline{X}_f$. As a consequence, $0 \leq T(t)g = |T(t)g| \leq \widetilde{S}(t)g$ for any $g \geq 0$. In other words, we have $0 \leq \widetilde{S}(t) - T(t)$ and then $0 \leq \widetilde{S}(t)^* - T(t)^*$. We must have $\widetilde{S}(t)^* - T(t)^* = 0$. Otherwise, there would exist $\psi \in Y_+\backslash\{0\}$ such that $(\widetilde{S}(t)^* - T(t)^*)\psi \in Y_+\backslash\{0\}$, and we find a contradiction by computing

$$0 < \langle (\widetilde{S}(t)^* - T(t)^*)\psi, f_1 \rangle = \langle \psi, (\widetilde{S}(t) - T(t)) f_1 \rangle = 0.$$

We have thus established that $\widetilde{S}(t) = T(t)$.

Step 2. Consider $\alpha, \beta \in \mathbb{R}$ and $f, g \in X\backslash\{0\}$ such that $\widetilde{\mathcal{L}} f = i\alpha f$ and $\widetilde{\mathcal{L}} g = i\beta g$, and suppose first that $(\operatorname{sign} \bar{f}) : D(\mathcal{L}) \to D(\mathcal{L})$. From Step 1 and the fact that $(\operatorname{sign} \bar{f}) \circ \operatorname{sign} f = I$, for any $h \in D(\mathcal{L})$, we may compute

$$\begin{aligned} \widetilde{\mathcal{L}} h &= \lim_{t\to 0} \frac{1}{t} (\widetilde{S}(t)h - h) \\ &= (\operatorname{sign} f) \lim_{t\to 0} \frac{1}{t} (e^{-i\alpha t} \widetilde{S}(t)(\operatorname{sign} \bar{f})h - (\operatorname{sign} \bar{f})h) \\ &= (\operatorname{sign} f)(\widetilde{\mathcal{L}} - i\alpha)(\operatorname{sign} \bar{f})h, \end{aligned}$$

or in other words $\widetilde{\mathcal{L}} - i\alpha = (\operatorname{sign} \bar{f}) \widetilde{\mathcal{L}} (\operatorname{sign} f)$. We have similarly $\widetilde{\mathcal{L}} - i\beta = (\operatorname{sign} \bar{g}) \widetilde{\mathcal{L}} (\operatorname{sign} g)$. Both equations together imply

$$\widetilde{\mathcal{L}} - i(\alpha + \beta) = (\operatorname{sign} \bar{f})(\operatorname{sign} \bar{g}) \widetilde{\mathcal{L}} (\operatorname{sign} g)(\operatorname{sign} f).$$

Defining $h := (\operatorname{sign} \bar{f})(\operatorname{sign} \bar{g}) f_1$, so that $(\operatorname{sign} g)(\operatorname{sign} f) h = f_1$, we get $\widetilde{\mathcal{L}} h = i(\alpha+\beta) h$, and finally $i(\alpha + \beta) \in \mathcal{S}$, so that the additive subgroup structure is established.

When the condition $(\operatorname{sign} \bar{f}) : D(\mathcal{L}) \to D(\mathcal{L})$ is not granted, we modify the above argument by using a resolvent approach. For some $\lambda > 0$, we compute thanks to (2.13)

$$\begin{aligned} (\lambda - \widetilde{\mathcal{L}})^{-1} &= \int_0^\infty e^{-\lambda t} \widetilde{S}(t)\, dt \\ &= (\operatorname{sign} f) \int_0^\infty e^{-(\lambda + i\alpha)t} \widetilde{S}(t)\, dt\, (\operatorname{sign} \bar{f}) \\ &= (\operatorname{sign} f)(\lambda + i\alpha - \widetilde{\mathcal{L}})^{-1} (\operatorname{sign} \bar{f}). \end{aligned}$$

Repeating the argument, we obtain

$$(\lambda + i(\alpha+\beta) - \widetilde{\mathcal{L}})^{-1} = (\operatorname{sign} \bar{f})(\operatorname{sign} \bar{g})(\lambda - \widetilde{\mathcal{L}})^{-1} (\operatorname{sign} g)(\operatorname{sign} f).$$

Applying that last identity to the vector $h = (\operatorname{sign} \bar{f})(\operatorname{sign} \bar{g}) f_1$ and using that $(\lambda - \widetilde{\mathcal{L}})^{-1} f_1 = \lambda^{-1} f_1$, we deduce $(\lambda + i(\alpha+\beta) - \widetilde{\mathcal{L}})^{-1} h = \lambda^{-1} h$. In other words, we have again $\widetilde{\mathcal{L}} h = i(\alpha+\beta) h$, and we conclude as above.

Step 3. From the fact that $(\operatorname{sign} f)$ is an invertible operator and the equation

$$(\widetilde{\mathcal{L}} - i\alpha)^k = (\operatorname{sign} f)^{-1} (\widetilde{\mathcal{L}})^k (\operatorname{sign} f),$$

we see from Theorem 4.13-(ii) that $N(\widetilde{\mathcal{L}} - i\alpha)^k = (\operatorname{sign} f)^{-1} N(\widetilde{\mathcal{L}})^k = (\operatorname{sign} f)^{-1} \mathrm{Span} f_1$ for any $k \geq 1$, so that its dimension is one. □

Making an additional splitting structure hypothesis as yet introduced in Section 2.2, we may significantly improve the conclusion. We first recall a classical result on the spectrum of an operator which holds under some power compactness assumption on the resolvent.

Theorem 5.6. *We assume that $\mathcal{L}$ satisfies the splitting structure* **(HS1)** *introduced in Section 2.2 with $\mathcal{W}(z) \in \mathscr{K}(X)$ for some $N \geq 1$ and any $z \in \Delta_{\kappa_0}$. Then $\Sigma(\mathcal{L}) \cap \Delta_{\kappa_0} \subset \Sigma_d(\mathcal{L})$.*

Theorem 5.6 is a consequence of [355, Cor. 1.1]. We also refer to [278, proof of Thm. 3.1] for a possible elementary proof.

A sketch of the proof of Theorem 5.6. Iterating the formula $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} + \mathcal{R}_{\mathcal{B}}\mathcal{A}\mathcal{R}_{\mathcal{L}}$, we deduce

$$\mathcal{J}(z)R_{\mathcal{L}}(z) = \mathcal{V}(z)$$

with $\mathcal{J} := I - (\mathcal{A}\mathcal{R}_{\mathcal{B}})^N$ and $\mathcal{V} := \mathcal{R}_{\mathcal{B}} + \cdots + \mathcal{R}_{\mathcal{B}}(\mathcal{A}\mathcal{R}_{\mathcal{B}})^{N-1}$. Because $\mathcal{J}$ is holomorphic on Δ_{κ_0}, it is a compact perturbation of the identity and $\mathcal{J}(z) \to I$ when $\Re e z \to \infty$, one may use the theory of *degenerate-meromorphic functions* of Ribarič and Vidav [331] (also established independently by Steinberg, see in particular [341, Cor. 1]), and conclude that $\mathcal{J}(z)$ is invertible outside of a discrete set $\mathcal{D}$ of Δ_{κ_0}. That implies that $\Sigma(\mathcal{L}) \cap \Delta_{\kappa_0} = \mathcal{D}$ is a discrete set of Δ_{κ_0}. On the other hand, thanks to the Fredholm alternative [164], one deduces that the eigenspace associated to each spectral value $\lambda \in \mathcal{D}$ is non zero and finite dimensional, so that $\lambda \in \Sigma_d(\mathcal{L})$. See also [342, 357] for pioneering works in the subject. □

We end this section by a result which gives a more accurate description of the geometry of the boundary spectrum, and is a variant of the classical results [15, C-III, Thm. 3.12], [152, Thm. VI.1.12], [41, Thm. 14.17].

Theorem 5.7. *Assume* **(C2)**, **(X2)**, **(X3)**, *that the complex Kato's inequality* (5.7) *holds true and additionally that the splitting structure* **(HS1)** *holds with $\mathcal{W}(z) \in \mathscr{K}(X)$ for some $N \geq 1$ and any $z \in \Delta_{\kappa_0}$. Then the set $\Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R}$ is a* **discrete** *additive subgroup of $i\mathbb{R}$ and any of its elements is an* **algebraically simple** *eigenvalue. More precisely,*
- either $\Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R} = \{0\}$ and the projection on the first eigenspace (associated to λ_1) writes

$$\Pi f := \langle f, \phi_1 \rangle f_1;$$

- or $\Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R} = i\alpha\mathbb{Z}$ for some $\alpha > 0$ and there exists a sequence $(g_k, \psi_k)_{k\in\mathbb{Z}}$ such that $\mathcal{L}g_k = (\lambda_1 + ik\alpha)g_k$, $\mathcal{L}^*\psi_k = (\lambda_1 + ik\alpha)\psi_k$, and $\langle g_k, \psi_\ell \rangle = \delta_{k\ell}$.

Proof of Theorem 5.7. Combining Theorem 5.5 and Theorem 5.6, we immediately get that the subgroup $\mathcal{S} := \Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R}$ satisfies $\mathcal{S} \subset \Sigma_d(\mathcal{L})$, it is thus discrete and made of algebraically simple eigenvalues. The first case $\Sigma_P(\widetilde{\mathcal{L}}) \cap i\mathbb{R} = \{0\}$ falls yet in the conclusions of Theorem 4.13. □

In the second case, where the boundary spectrum is not trivial, the existence of a projection on the boundary eigenspace $\overline{\mathrm{Span}}(g_k)_{k\in\mathbb{Z}}$ is ensured by the Jacobs–de Leeuw–Glicksberg theorem provided that $\mathcal{L}$ is the generator of a relatively compact semigroup, see for instance [41, Thm. A.39 and Prop. A.40] and the references therein. We also refer to [229, paragraphs III.6.4 and III.6.5] for very classical results on the projector on the direct sum of eigenspaces associated to eigenvalues belonging to a subset of the spectrum. We can even give an explicit expression of this projection in terms of (g_k) and (ψ_k) under the form of a Fejér type sum, see Theorem 5.25.

5.3. Stronger positivity.

In order to go one step further and establish the triviality of the boundary point spectrum, we need to reinforce the positivity of the semigroup or its generator. One possible condition is based on the following notion.

The reverse strong positivity condition

For $A \geq 0$, we recall that from (2.6), we have

$$|Af| \leq A|f|, \quad \forall f \in X, \tag{5.8}$$

and we observe that the above inequality is an equality when $Af = uA|f|$ for some $u \in \mathbb{S}^1$. We focus now on the case of equality in (5.8).

Definition 5.8. *We say that A satisfies the "reverse strong positivity condition" for a subclass of vectors $\mathcal{C} \subset X$ if for any $f \in \mathcal{C}$*

$$|Af| = A|f| \quad \textit{implies} \quad \exists\, u \in \mathbb{S}^1,\ Af = uA|f|. \tag{5.9}$$

We start observing that $A > 0$ (as defined in Section 4.2) implies the strict positivity for non-signed vectors in $X_\mathbb{R}$.

Lemma 5.9. *Consider an operator $A > 0$ and assume X is reflexive. For $f \in X_\mathbb{R}$ such that $\pm f \notin X_+$, there holds*

$$|Af| \ll A|f|.$$

Proof of Lemma 5.9. Let us consider $f \in X_\mathbb{R}$ such that $f_\pm \neq 0$. We claim that $|Af| \ll A|f|$. Observing that

$$Af_+ = Af + Af_- \geq Af,$$

we deduce $Af_+ \geq (Af)_+$, and similarly $Af_- \geq (Af)_-$. We first consider the case $(Af)_+ > 0$. For $\phi \gg 0$, we have

$$0 < \langle (Af)_+, \phi\rangle = \sup_{0 \leq \psi \leq \phi} \langle Af, \psi\rangle = \langle Af, \psi^*\rangle = \langle f, A^*\psi^*\rangle,$$

for some $0 \leq \psi^* \leq \phi$, where we have used the very definition of X_{++}, the definition of $(Af)_+$ as an element of X'' and that $B_{X'}$ is compact for the weakly $*$ topology $\sigma(X', X)$. We deduce in particular that $\psi^* \neq 0$, so that $\psi^* > 0$ and finally $A^*\psi^* \gg 0$ because $A^* > 0$ (as an elementary consequence of the fact that $A > 0$ listed in Section 4.2). We deduce

$$\langle (Af)_+, \phi\rangle = \langle f, A^*\psi^*\rangle < \langle f_+, A^*\psi^*\rangle = \langle Af_+, \psi^*\rangle \leq \langle Af_+, \phi\rangle$$

where for the strict inequality we have first used the assumption $f_- \neq 0$ and next elementary arguments. We thus have $(Af)_+ < Af_+$. Similarly, we establish $(Af)_- < Af_-$ when $(Af)_- > 0$. As a conclusion, in the three cases $Af = 0$, $(Af)_+ \neq 0$ and $(Af)_- \neq 0$, we have

$$|Af| = (Af)_+ + (Af)_- \ll Af_+ + Af_- = A|f|,$$

which is the desired strict inequality. □

We believe that a similar result also holds true for complex vectors in a general Banach lattice framework. We do not try to prove such a statement but we rather establish the corresponding complex version for our examples of concrete Banach spaces in which the definition of the absolute value $|f|$ of a vector $f \in X$ is more tractable.

Lemma 5.10. *Consider an operator $A > 0$ on $X \subset L^1_{\rm loc}(E)$ for some locally and σ-compact metric space E. For $f \in X$ such that $|f| \gg 0$, we have*

$$|Af| = A|f| \quad \textit{implies} \quad \exists\, u \in \mathbb{S}^1,\ f = u|f|,$$

and thus (5.9) *holds.*

Proof of Lemma 5.10. We assume by contradiction that $\forall v \in \mathbb{S}^1$, $|f| > \Re e(vf)$, in particular writing $f = g + ih$, $g, h \in X_\mathbb{R}$, we have $g, h \in X \backslash \{0\}$. On the one hand, because of $A > 0$ and A is linear, for any $v = e^{i\alpha} \in \mathbb{S}^1$, we have

$$A|f| \gg A\Re e(e^{i\alpha} f)) = \cos\alpha\,(Ag) - \sin\alpha\,(Ah).$$

On the other hand, in the Banach lattice we consider here, there exists $\beta : E \to \mathbb{R}$ measurable such that $|Af| = e^{i\beta} Af$ and thus

$$A|f| = |Af| = \Re e|Af| = \cos\beta\,(Ag) - \sin\beta\,(Ah),$$

and a contradiction. We have established that there exists $v \in \mathbb{S}^1$ such that $|f| \equiv \Re e(fv)$. Now, we have

$$\sqrt{(\Re e(fv))^2 + (\Im m(fv))^2} = |fv| = |f| = \Re e(fv),$$

which in turn implies $\Im m(fv) = 0$, since $\Re e(fv) \gg 0$. That is here that we use the assumption $|f| \gg 0$ and not only $f \in X_+ \backslash \{0\}$. We conclude that $|f| = fv$ and thus that $f = u|f|$, with $u := v^{-1} \in \mathbb{S}^1$. □

A similar result also holds in the Radon space of measures. For a measurable space $(E, \mathscr{E})$, we call transition kernel, a mapping $Q : E \times \mathscr{E} \to [0, \infty]$ such that

$$\begin{array}{ll} (i) & \forall\, B \in \mathscr{E},\ x \mapsto Q(x,B) \text{ is measurable;} \\ (ii) & \forall\, x \in E,\ B \mapsto Q(x,B) \text{ is a measure.} \end{array}$$

We recall the classical Markov-Riesz representation theorem which claims that for any linear and positive operator $B : C_0(E) \to C_0(E)$ there holds

$$(B\phi)(x) = \int_E \phi(x) Q(x, dy), \quad \forall\, \phi \in C_0(E),$$

for a transition kernel Q such that in the condition (i) above the mapping is furthermore continuous.

Lemma 5.11. *Consider an operator $A > 0$ in $X = M^1 = M^1(E, \mathscr{E})$, for some Borel space $(E, \mathscr{E})$ where E is a locally and σ-compact metric set. For $f \in X$ such that $|f| \gg 0$, we have* (5.9).

Proof of Lemma 5.11. By definition, the operator A is the dual of a positive operator on $C_0(E)$. Using the representation formula recalled above for that adjoint operator, we get

$$(Af)(dy) = \int_E Q(x, dy) f(dx), \quad \forall\, f \in M^1,$$

for a transition kernel Q. We deduce that

$$\langle Af, \phi \rangle = \int_{E \times E} \phi(y) Q(x, dy) f(dx),$$

for any bounded Borel function $\phi : E \to \mathbb{C}$. In particular, the strict positivity $A > 0$ translates as $Q(x, \cdot) \gg 0$ in M^1 for any $x \in E$. We fix now $\phi \in C_0(E)$ such that $\phi \gg 0$ and $f \in M^1$ such that $|f| \gg 0$, and we observe that from the Radon-Nikodym theorem, there exist two measurable functions $\alpha, \beta : E \to [0, 2\pi)$ such that $f = e^{i\alpha} |f|$ and $Af = e^{i\beta} |Af|$. We next compute

$$\begin{aligned} \langle A|f| - |Af|, \phi \rangle &= \Re e \big\{ \langle A|f|, \phi \rangle - \langle Af, e^{-i\beta} \phi \rangle \big\} \\ &= \int_{E \times E} \Re e \big\{ 1 - e^{i(\alpha(x) - \beta(y))} \big\} \phi(y) Q(x, dy) |f|(dx) \\ &= \int_{E \times E} \big\{ 1 - \cos(\alpha(x) - \beta(y)) \big\} \phi(y) Q(x, dy) |f|(dx). \end{aligned}$$

In the case of equality $A|f| = |Af|$, we must have $1 - \cos(\alpha(y) - \beta(x)) = 0$ for μ-a.e. $x \in E$ and $|f|$-a.e. $y \in \operatorname{supp} f = E$. We deduce that β is a constant function, so that $Af = e^{i\beta}|Af| = uA|f|$, for the constant $u = e^{i\beta} \in \mathbb{S}^1$. $\square$

The reverse Kato's inequality condition

We recall that it has been stated in section 4.1 that the generator $\mathcal{L}$ of a positive semigroup $S(t)$ satisfies Kato's inequality (4.5) which in a complex framework writes

$$\forall\, f \in X, \quad \Re e (\operatorname{sign} f) \mathcal{L} f \le \mathcal{L}|f|. \tag{5.10}$$

We also observe that if $f = u|f|$ for some $u \in \mathbb{S}^1$, we have

$$\Re e (\operatorname{sign} f) \mathcal{L} f = \operatorname{sign}(u^{-1} f) \mathcal{L}(u^{-1} f) = \mathcal{L}|f|,$$

which is the case of equality in Kato's inequality.

Definition 5.12. *We say that $\mathcal{L}$ satisfies a "reverse Kato's inequality condition" for a class of vectors $\mathcal{C} \subset D(\mathcal{L})$ if for any $f \in \mathcal{C}$ the case of equality in Kato's inequality*

$$\mathcal{L}|f| = \Re e (\operatorname{sign} f) \mathcal{L} f$$

implies

$$\exists\, u \in \mathbb{C},\ f = u|f|.$$

In some situation, we may prove directly that the "reverse Kato's inequality condition" holds by reasoning at the level of the operator $\mathcal{L}$, see for instance [231, Proof of Theorem 5.1]. That is also a consequence of the strong positivity of the semigroup as we see below.

Lemma 5.13. *Consider a semigroup S and its generator $\mathcal{L}$. On the set $\mathcal{C}$ of vectors $f \in X\backslash\{0\}$ such that*

$$\exists\,\lambda \in \mathbb{C}, \quad \mathcal{L}f = \lambda f, \quad \mathcal{L}|f| = (\Re e\lambda)|f|, \tag{5.11}$$

there is equivalence between:
(i) $S(t)$ satisfies the "reverse strong positivity condition" for some (and thus any) $t > 0$;
(ii) $\mathcal{L}$ satisfies the "reverse Kato's inequality condition".

Remark 5.14. *When $X \subset L^1_{\rm loc}$, the "reverse Kato's inequality condition" (ii) implies the "reverse strong positivity condition" (i) on the class $\mathcal{C}$ of vectors such that $f \in D(\mathcal{L})$, $0 \ll |f| \in D(\mathcal{L})$. Assume indeed that $\mathcal{L}$ satisfies (ii) and consider $f \in \mathcal{C}$ such that $|S_t f| = S_t|f|$ for any $t \ge 0$. By differentiating, we get*

$$(\mathrm{sign}\, f)\mathcal{L}f = \mathcal{L}|f|. \tag{5.12}$$

From the "reverse Kato's inequality condition", we deduce that $f = u|f|$ for some $u \in \mathbb{S}^1$, so that (i) holds.

Proof of Lemma 5.13. In what follows, we fix $f \in X\backslash\{0\}$ such that (5.11) holds, and we compute

$$\Re e(\mathrm{sign}\, f)\mathcal{L}f = \Re e(\mathrm{sign}\, f)(\lambda f) = (\Re e\lambda)|f| = \mathcal{L}|f|. \tag{5.13}$$

For any $t > 0$, we also have $S_t f = e^{\lambda t} f$, $S_t|f| = e^{\Re e\lambda t}|f|$, and thus

$$|S_t f| = S_t|f|. \tag{5.14}$$

Assuming the "reverse Kato's inequality condition", we deduce from (5.13) that $f = u|f|$ for some $u \in \mathbb{S}^1$, thus $S_t f = uS_t|f|$ for some $u \in \mathbb{S}^1$, which is the conclusion of the "reverse strong positivity condition" when (5.14) holds.

On the other way round, assuming the "reverse strong positivity condition" for some $T > 0$, we deduce from (5.14) for $T > 0$ that there exists $v \in \mathbb{S}^1$ such that

$$e^{\lambda T} f = S_T f = vS_T|f| = ve^{\Re e\lambda T}|f|.$$

That implies that $f = u|f|$ with $u = ve^{-i(\Im m\lambda)T}$, which is nothing but the conclusion of the "reverse Kato's inequality condition" when (5.13) holds. □

We summarize the material developed above in the following main result of the section.

Theorem 5.15. *Assume that S is a positive semigroup on X with $X \subset L^1_{\rm loc}(E)$ or $X = M^1(E)$ for some locally and σ-compact metric space E and denote by (E_k) a sequence of increasing compact sets such that $E = \lim E_k$. We furthermore assume that for any $k \ge 1$ there exists $T > 0$ such that S_T is strictly positive on E_k, in the sense that*

$$\forall\, f \in X_+\backslash\{0\},\ f_{|E_k} \not\equiv 0,\ \forall\, \phi \in X'_+\backslash\{0\},\ \mathrm{supp}\,\phi \subset E_k, \quad \langle S_T f, \phi\rangle > 0. \tag{5.15}$$

Then $\mathcal{L}$ satisfies the "reverse Kato's inequality condition" on the set $\mathcal{C}$ of eigenvectors introduced in Lemma 5.13.

Proof of Theorem 5.15. Let us consider $f \in X\backslash\{0\}$ such that (5.11) holds, so that $S_t|f| = e^{(\Re e\lambda)t}|f|$ for any $t \ge 0$. On the one hand, we may fix $k \ge 1$ such that $|f| \not\equiv 0$ on E_k. Then for any $\ell \ge k$, there exists $T_\ell > 0$ such that (5.15) holds, so that

$$e^{(\Re e\lambda)T_\ell}\langle |f|, \phi\rangle = \langle S_{T_\ell}|f|, \phi\rangle > 0,$$

for any $\phi \in Y_+\backslash\{0\}$, $\mathrm{supp}\,\phi \subset E_\ell$. That implies $\langle |f|, \phi\rangle > 0$ on for any $\phi \in Y_+\backslash\{0\}$, and thus $|f| \gg 0$. Next, as in the proof of Lemma 5.13, we observe that

$$|S_{T_\ell} f| = S_{T_\ell}|f|, \quad \forall\, \ell \ge k.$$

Repeating the proof of Lemma 5.10 and Lemma 5.11, we deduce that there exists $u_\ell \in \mathbb{S}^1$ such that $S_{T_\ell} f = u_\ell S_{T_\ell}|f|$ on E_ℓ, or equivalently there exists $v_\ell \in \mathbb{S}^1$ such that $f = v_\ell|f|$ on E_ℓ, with $v_\ell := u_\ell e^{-i(\Im m\lambda)T_\ell}$. Because $E_\ell \supset E_1$, we have established that $f = v_1|f|$ on E which is the conclusion of the "reverse Kato's inequality condition" when (5.11) holds. □

5.4. **On the triviality of the boundary spectrum.** As in section 4.3, we still assume the existence **(C1)** of a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times Y_+$ to the first eigenvalue problem (4.1) and that $\mathcal{L}$ enjoys the weak maximum principle (4.16) and Kato's inequalities (4.17) as formulated in condition **(H1′)** as well as the strong maximum principle **(H4)**. Because we deal with complex eigenvalue, we also assume that the **complex** Kato's inequality variant (5.10) holds.

We introduce a first additional assumption:

(H5) the"reverse Kato's inequality condition" (as defined in Definition 5.12) holds true for the class $\mathcal{C}$ defined in Lemma 5.13: for $f \in X\backslash\{0\}$ such that

$$\exists\, \lambda \in \mathbb{C}, \quad \mathcal{L}f = \lambda f, \quad \mathcal{L}|f| = (\Re e\lambda)|f| = \Re e(\mathrm{sign} f)\mathcal{L}f, \tag{5.16}$$

we have

$$\exists\, u \in \mathbb{C},\ f = u|f|.$$

On the other hand, we do not need the structure assumption **(X3)**.

We are then able to make a more accurate analyse of the geometry of the spectrum.

Theorem 5.16. *Consider an unbounded operator $\mathcal{L}$ on a Banach lattice X which satisfy* **(C2)***,* **(H4)***,* (4.17) *and* **(H5)***. Then the conclusion* **(S3$_2$)** *about the uniqueness of λ_1 as the eigenvalue with largest real part holds: $\Sigma^+_P(\mathcal{L}) = \{\lambda_1\}$.*

Remark 5.17. *(1) It is worth emphasizing again that* (4.17) *is true when $\mathcal{L}$ is the generator of a positive semigroup and that* **(H5)** *is true when $S_\mathcal{L}(T)$ satisfies the "reverse strong positivity condition" for some $T > 0$ as a consequence of Lemma 5.13, see also Theorem 5.15.*

(2) During the proof we use similar arguments as in [231, Thm. 5.1].

(3) Condition **(H5)** *is reminiscent of PDE arguments as we may find for instance in* [231, Proof of Thm. 5.1] *or in the discussion in* [252, 4th course] *about an uniqueness argument due to L. Tartar.*

Proof of Theorem 5.16. Consider an eigenvalue $\lambda \in \mathbb{C}$ with normalized eigenvector $f \in X\backslash\{0\}$, and more precisely $\langle |f|, \phi_1\rangle = 1$ and $\mathcal{L}f = \lambda f$. Thanks to the complex Kato's inequality (4.17), we have

$$(\Re e\lambda)|f| = \Re e\,\mathrm{sign}(f)(\lambda f) = \Re e\,\mathrm{sign}(f)(\mathcal{L}f) \le \mathcal{L}|f|.$$

We consider two cases:

When the above inequality is not an equality, we have

$$(\Re e\lambda)\langle |f|, \phi_1\rangle < \langle \mathcal{L}|f|, \phi_1\rangle = \langle |f|, \mathcal{L}^*\phi_1\rangle = \lambda_1\langle |f|, \phi_1\rangle,$$

and thus $\Re e\lambda < \lambda_1$.

When on the contrary the above inequality is an equality, then $|f|$ is a positive eigenvector associated to the eigenvalue $\Re e\lambda$. Because of **(H4)**, we have $|f| \in X_{++}$ and repeating the proof of Lemma 4.17, we get $\Re e\lambda = \lambda_1$. The condition **(C2)** implies $|f| = f_1$. On the other hand, f satisfies (5.16) and thus $f \in \mathrm{Span}(f_1)$ from assumption **(H5)**, in particular $\lambda = \lambda_1$. □

When $\mathcal{L}$ is the generator of a positive and irreducible semigroup S, we may introduce the alternative assumption:

(H5′) the semigroup S is aperiodic as defined in (4.13), namely

$$\forall\, f \in X_+\backslash\{0\}, \forall\, \phi \in Y_+\backslash\{0\},\ \exists\, T > 0, \forall \tau \ge T \quad \langle S_\tau f, \phi\rangle > 0.$$

Theorem 5.18. *Let X be Banach lattice in which the property* (4.10) *holds true. Consider a positive and irreducible semigroup S on X which satisfies the aperiodicity condition* **(H5′)** *and such that its generator $\mathcal{L}$ satisfies* **(C2)***. Then the conclusion* **(S3$_2$)** *holds: $\Sigma^+_P(\mathcal{L}) = \{\lambda_1\}$.*

Remark 5.19. *It is worth pointing out that since* **(H5′)** *is stronger than* **(H4)***, see the points (2) and (3) in Lemma 4.8, we can use Theorem 4.13 and replace in Theorem 5.18 the assumption that* **(C2)** *is satisfied by the assumption that* **(C1)** *and* **(H1′)** *for both $\mathcal{L}$ and $\mathcal{L}^*$ are satisfied, together with the structure assumption* **(X1)** *on X and Y.*

Proof of of Theorem 5.18. We introduce the notations $\widetilde{S}_t := S_t e^{-\lambda_1 t}$ and $\widetilde{\mathcal{L}} := \mathcal{L} - \lambda_1$. Assume that $f = g + ih \in X$, $g, h \in X_\mathbb{R}$, is an eigenfunction associated to the eigenvalue $\lambda = \lambda_1 + i\alpha \in \mathbb{C}$, $\alpha > 0$, so that

$$\widetilde{\mathcal{L}}(g + ih) = i\alpha(g + ih) = \frac{2\pi i}{t_0}(g + ih),$$

for some $t_0 > 0$. On the one hand, because $\alpha \neq 0$, we must have $g \neq 0$ and $h \neq 0$, and because of

$$\alpha\langle g, \phi_1\rangle = \langle \widetilde{\mathcal{L}} h, \phi_1\rangle = \langle h, \widetilde{\mathcal{L}}^* \phi_1\rangle = 0,$$

and $\phi_1 \gg 0$, we have $g_+ \neq 0$ and $g_- \neq 0$. As a consequence, and because of (4.10), there exists $\psi \in Y_+ \backslash \{0\}$ such that $\langle g_+, \psi\rangle = 0$. On the other hand, we compute

$$\widetilde{S}_{t_0}(g + ih) = e^{i\alpha t_0}(g + ih) = g + ih,$$

from what we deduce $\widetilde{S}_{t_0} g = g$, because S_t is real. On the other hand, because S_t is positive, we have

$$g_+ = (\widetilde{S}_{t_0} g)_+ \leq \widetilde{S}_{t_0} g_+,$$

and next

$$\langle \phi_1, g_+\rangle \leq \langle \phi_1, \widetilde{S}_{t_0} g_+\rangle = \langle \widetilde{S}^*_{t_0} \phi_1, g_+\rangle = \langle \phi_1, g_+\rangle,$$

so that the inequalities are equalities (remind again that $\phi_1 \gg 0$), and thus

$$\widetilde{S}_{t_0} g_+ = g_+.$$

We conclude that

$$\langle \widetilde{S}_{kt_0} g_+, \psi\rangle = \langle g_+, \psi\rangle = 0, \quad \forall\, k \geq 0,$$

what is in contradiction with **(H5′)**. We have established that $\Sigma^+_P(\mathcal{L}) = \{\lambda_1\}$. □

We end this section with a third situation where the triviality of the boundary spectrum is an immediate consequence of Theorem 5.5 and Theorem 5.6.

Theorem 5.20. *(1) We make the same assumptions as in Theorem 5.5 and also that there exists $M > 0$ large enough such that $\lambda - \mathcal{L}$ is invertible in $\mathscr{B}(X)$ for any $\lambda \in \mathbb{C}$, $\Re e\lambda = \lambda_1$, $|\lambda| \geq M$. Then λ_1 is the unique eigenvalue with largest real part as formulated in* **(S3$_2$)**.

(2) We furthermore assume that the hypothesis of Theorem 5.6 are met and that $\lambda - \mathcal{L}$ is invertible in $\mathscr{B}(X)$ for any for any $\lambda \in \mathbb{C}$, $\Re e\lambda \geq \lambda_1 - \varepsilon$, $|\lambda| \geq M$. Then a (non constructive) spectral gap **(S3$_3$)** *holds.*

We summarize the main results established in this section as follows.

(C3) the first eigentriplet problem (4.1) has a solution (λ_1, f_1, ϕ_1), furthermore this one is unique, $f_1 \gg 0$, $\phi_1 \gg 0$, λ_1 is algebraically simples (for both $\mathcal{L}$ and $\mathcal{L}^*$) and $\Sigma^+_P(\mathcal{L}) = \{\lambda_1\}$.

5.5. **Ergodicity.** Thanks to the above analyze, we are able to formulate some convergence results on the trajectories associated to a semigroup. More precisely, assuming the existence and uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) for the generator $\mathcal{L}$ of a semigroup S and still denoting the rescaled semigroup $\widetilde{S}_t := e^{-\lambda_1 t} S(t)$, we wish in particular to establish the following ergodic property

(E2) for any $f \in X$, there holds

$$\widetilde{S}_t f \to \langle f, \phi_1\rangle f_1, \quad \text{as} \quad t \to \infty, \tag{5.17}$$

in a sense to be specified.

We start with a simple result which take advantage of some dissipativity property of the semigroup formulated by a "reverse positivity condition". We next present some more involved results which use directly the spectral information. It is worth emphasizing that our results in this section do not use any spectral gap property what contrasts with the results we will present in the next section.

Theorem 5.21. *Consider a positive semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C2)** *of existence, uniqueness and strict positivity of the first eigentriplet (λ_1, f_1, ϕ_1) and let us set $\widetilde{S}_t := e^{-\lambda_1 t} S_t$. We denote $\mathcal{X}$ the space X endowed with the norm $[\cdot]$, with $[f] := \langle |f|, \phi_1\rangle$. Assume furthermore that*
(1) *for any $f \in X$, the trajectory $(\widetilde{S}_t f)_{t\ge 0}$ is continuous in $\mathcal{X}$ and belongs to a compact set of a normed space $\mathcal{X}_1$, with $\mathcal{X}_1 \subset \mathcal{X}$;*
(2) *(S_t) satisfies the reverse positivity condition for semigroups*

$$|S_t f| = S_t|f|, \ \forall\, t > 0, \quad \text{implies} \quad \exists\, T > 0, \ \exists\, u_T \in \mathbb{S}^1, \ S_T f = u_T S_T |f|. \tag{5.18}$$

Then, the ergodicity property **(E2)** *holds in the sense of the norm of $\mathcal{X}_1$.*

Let us comment on hypotheses made in the statement of Theorem 5.21. Hypothesis **(1)** can be obtained as a consequence of a Lyapunov (or growth) condition reminiscent of the structure condition **(HS3)** introduced in Section 3.3 and an irreducibility condition. Typically, we assume first

$$\|\widetilde{S}(t) f\| \le M\|f\| + K \sup_{0\le\tau\le t} [\widetilde{S}(\tau) f]_0,$$

with $[g]_0 := \langle |g|, \psi_0\rangle$, $\psi_0 \in Y_+\backslash\{0\}$, what can be established under the very general condition (ii) of Theorem 3.4. Next we need to be able to prove that $\psi_0 \le r\phi_1$ for some $r > 0$. In concrete situations, we may take ψ_0 with compact support and then the above inequality is a consequence of the standard strong maximum principle. We deduce

$$\begin{aligned} \|\widetilde{S}(t) f\| &\le M\|f\| + Kr \sup_{0\le\tau\le t} \langle |\widetilde{S}(\tau) f|, \phi_1\rangle \\ &\le M\|f\| + Kr \sup_{0\le\tau\le t} \langle \widetilde{S}(\tau) |f|, \phi_1\rangle \\ &= M\|f\| + Kr\langle |f|, \phi_1\rangle, \end{aligned}$$

so that $(\widetilde{S}_t)$ is bounded. The hypothesis **(1)** is in fact a bit more demanding, but also quite natural. Assume that $S_{\mathcal{L}}$ enjoys the splitting structure introduced in section 3.1 and section 3.3, so that

$$\widetilde{S} = V + K, \tag{5.19}$$

with

$$V := \widetilde{S}_{\mathcal{B}} + \cdots + (\widetilde{S}_{\mathcal{B}}\mathcal{A})^{*(N-1)} * \bar{S}_{\mathcal{B}}, \quad K := (\widetilde{S}_{\mathcal{B}}\mathcal{A})^{(*N)} * \widetilde{S}, \quad \widetilde{S}_{\mathcal{B}}(t) := e^{-\lambda_1 t} S_{\mathcal{B}}(t).$$

In some applications, we typically have

$$\|V(t) f_0\| \le \Theta(t)\|f_0\|, \quad \|(\widetilde{S}_{\mathcal{B}}\mathcal{A})^{(*N)}\|_{\mathscr{B}(\mathcal{X},\mathcal{X}_1)} \le \Theta$$

with $\Theta \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $\mathcal{X}_1 \subset \mathcal{X}$ compact. In that situation, we deduce **(1)**.

Proof of Theorem 5.21. We fix $f \in X$ and without loss of generality, we may assume that $\langle f, \phi_1\rangle = 0$. We observe that

$$\langle |\widetilde{S}_t f|, \phi_1\rangle = \langle |\widetilde{S}_{t-s}\widetilde{S}_s f|, \phi_1\rangle \le \langle \widetilde{S}_{t-s}|\widetilde{S}_s f|, \phi_1\rangle = \langle |\widetilde{S}_s f|, \phi_1\rangle, \tag{5.20}$$

for any $t \ge s$. We deduce that $(\widetilde{S}_t)$ is a dynamical system with compact trajectoires in $\mathcal{X}_1$ and $\mathcal{H}(f) := \langle |f|, \phi_1\rangle$ is a Lyapunov functional. As a consequence, from the La Salle invariance principle, we have

$$\inf_{g\in\omega_{\mathcal{H}}} \langle |\widetilde{S}_t f - g|, \phi_1\rangle \to 0 \ \text{ as } \ t \to \infty, \tag{5.21}$$

with

$$\omega_{\mathcal{H}} := \{g \in X; \ \langle g, \phi_1\rangle = 0, \ \forall\, t \in \mathbb{R}, \ \mathcal{H}(\widetilde{S}_t g) = \inf_{s>0} \mathcal{H}(\widetilde{S}_s f)\}. \tag{5.22}$$

We next characterize $\omega_{\mathcal{H}}$. Picking up $g \in \omega_{\mathcal{H}}$, we observe that

$$\langle |\widetilde{S}_t g|, \phi_1\rangle = \langle |g|, \phi_1\rangle = \langle |g|, \widetilde{S}^*_t \phi_1\rangle = \langle \widetilde{S}_t |g|, \phi_1\rangle, \quad \forall\, t \ge 0,$$

so that

$$\langle \widetilde{S}_t |g| - |\widetilde{S}_t g|, \phi_1\rangle = 0, \quad \forall\, t \ge 0.$$

In particular, using that $|\widetilde{S}_t g| \le \widetilde{S}_t |g|$, we have

$$\widetilde{S}_t |g| = |\widetilde{S}_t g|, \quad \forall\, t \ge 0. \tag{5.23}$$

Because of the reverse positivity condition for semigroups (5.18), there exist $T > 0$ and $u_T \in \mathbb{S}^1$ such that

$$\widetilde{S}_T g = u_T \widetilde{S}_T |g|.$$

As a consequence, by definition of the set $\omega_{\mathcal{H}}$, we have

$$0 = \langle g, \phi_1 \rangle = \langle \widetilde{S}_T g, \phi_1 \rangle = u_T \langle \widetilde{S}_T |g|, \phi_1 \rangle = u_T \langle |g|, \phi_1 \rangle.$$

Because $u_T \neq 0$, we conclude that $g = 0$. In other words, we have established that $\omega_{\mathcal{H}} = \{0\}$ and together with (5.21), we obtain (5.17). □

We present a more concrete situation where the previous result can be invoked. Although the hypotheses are somehow restrictive, it is yet useful in many applications and its proof is very simple.

Corollary 5.22. *Consider a strongly continuous and positive semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C2)** *of existence, uniqueness and strict positivity of the first eigentriplet (λ_1, f_1, ϕ_1). Assume further that the reverse Kato's inequality condition (as defined in Definition 5.12) holds true for the (large) class*

$$\mathcal{C} := \{f \in D(\mathcal{L});\ \mathcal{L}|f| = \Re e(\mathrm{sign} f)\mathcal{L} f\},$$

that $X \subset L^1_{\rm loc}(E, \mathscr{E}, \mu)$ and that the space $\mathcal{X}^k$ defined in (4.26) *satisfies $\mathcal{X}^k \subset L^1_{\rm loc}$ with strongly compact embedding for some $k \geq 1$. Then the ergodicity property* **(E2)** *holds in the sense of strong topology of $L^1_{\phi_1}$.*

Proof of Corollary 5.22. Because of Step 3 in the proof Theorem 4.23, we see that condition (1) in Theorem 5.21 holds with $\mathcal{X}_1 := \mathcal{X}^k$. On the other hand, because of Remark 5.14 and the reverse Kato's inequality condition in $\mathcal{C}$, we see that condition (2) also holds, so that we may apply Theorem 5.21 and conclude. □

We present now a variant of the previous result which provides a convergence for various topologies, and relies on the (very general) assumption that the boundary spectrum is trivial rather than on the reverse positivity condition.

Theorem 5.23. *Consider a positive semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C3)** *on the existence, uniqueness and strict positivity of the first eigentriplet problem (λ_1, f_1, ϕ_1) and triviality of the boundary point spectrum. Setting $\widetilde{S}_t := e^{-\lambda_1 t} S_t$, we assume that we are in one of the following situations:*

(1) S is strongly continuous and the trajectories $(\widetilde{S}_t f)_{t \geq 0}$ are relatively compact for all $f \in X$, and we denote by $\mathscr{T}$ the strong topology of X;
(2) $X = Y'$, Y separable, and the trajectories $(\widetilde{S}_t f)_{t \geq 0}$ are bounded for all $f \in X$, and we denote by $\mathscr{T}$ the weak $*\,\sigma(Y', Y)$ topology;
(3) $X \subset L^1_{\rm loc}(E, \mathscr{E}, \mu)$, and we denote by $\mathscr{T}$ the weak topology of $L^1_{\phi_1}$;
(4) $X \subset L^1_{\rm loc}$, S is strongly continuous, and for some $k \geq 1$ the space $\mathcal{X}^k$ defined in (4.26) *satisfies $\mathcal{X}^k \subset L^1_{\rm loc}$ with strongly compact embedding, and we denote by $\mathscr{T}$ the strong topology of $L^1_{\phi_1}$.*

Then the ergodicity property **(E2)** *holds in the sense of the topology $\mathscr{T}$.*

Remark 5.24. *The case (4) of Theorem 5.23 enjoys some strong similarities with the main consequences of the General Relative Entropy technique developed in* [269]*, see in particular* [269, Thm. 3.2]*,* [269, Thm. 4.3] *and* [269, Thm. 5.2]*. In particular, the aperiodicity condition that the boundary point spectrum is trivial may be compared with the fact that the first eigenvector f_1 is the unique (normalized and nonnegative) vector $f \in X$ with vanishing dissipation of entropy $\mathcal{D}(f) = 0$ as defined in* [269] *or more generally that* $\mathrm{Span}(f_1)$ *is the unique invariant space on which the functional $\mathcal{D}$ vanishes. The present formulation is more abstract and probably more general. The drawback is the condition $\mathcal{X}^k \subset L^1_{\rm loc}$ with strongly compact embedding which can be avoided in* [269]*, by using some weak compactness argument and the lower semicontinuity property of $\mathcal{D}$. That is explained by the fact that our proof uses rather the La Salle invariance principle (similarly as in the proof of* [153, Thm. 3.2]*) instead of a entropy dissipation argument.*

In the case when the boundary point spectrum is not trivial but a discrete set, the same method of proof as for Theorem 5.22 allows us to accurately describe the periodic long time behaviour of the semigroup.

Theorem 5.25. *Consider a positive semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C2)** *on the existence and uniqueness of the first eigentriplet problem (λ_1, f_1, ϕ_1), and satisfies the complex Kato's inequality (5.7). Suppose furthermore that X and Y both enjoy the structure conditions* **(X2)** *and* **(X3)***, that λ_1 is an isolated eigenvalue and that the boundary spectrum is not trivial, i.e. $\Sigma_P^+ \neq \{\lambda_1\}$. Setting $\widetilde{S}_t := e^{-\lambda_1 t} S_t$, we assume that we are in one of the situations (1), (2), (3) or (4) listed in statement of Theorem 5.23. Then $\Sigma_P^+ = \{\lambda_1 + ik\alpha,\ k \in \mathbb{Z}\}$ for some $\alpha > 0$, there exists a sequence $(g_k, \psi_k)_{k\in\mathbb{Z}}$ such that $\mathcal{L}g_k = (\lambda_1 + ik\alpha) g_k$, $\mathcal{L}^*\psi_k = (\lambda_1 + ik\alpha)\psi_k$ and $\langle g_k, \psi_k\rangle = 1$, and for all $f \in X$, in the sense of the topology $\mathscr{T}$, the projection*

$$\Pi f = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n} \sum_{k=-\ell}^{\ell} \langle f, \psi_k\rangle g_k$$

is well defined and

$$\widetilde{S}_t f - \widetilde{S}_t \Pi f \to 0 \qquad \text{as } t \to +\infty.$$

Remark 5.26. *In Theorem 5.25, the assumptions that λ_1 is isolated and $\Sigma_P^+ \neq \{\lambda_1\}$ might seem difficult to check in practice. We indicate here some ways to verify them.*
(i) The condition that λ_1 is an isolated eigenvalue is for instance guaranteed under the assumptions of Theorem 5.6 or Theorem 6.5.
(ii) The condition that Σ_P^+ is not restricted to $\{\lambda_1\}$ can be guaranteed by verifying that **(E2)** *does not hold. Indeed, if $\Sigma_P^+ = \{\lambda_1\}$, then Theorem 5.23 imposes* **(E2)** *to hold.*

The result in Theorem 5.25 can be compared for instance to [41, Thm. 14.19], although our hypotheses are slightly more general. Our proof is also more direct than in [41] and it additionally provides an explicit expression of the projection on the boundary eigenspace $\overline{\mathrm{Span}}(g_k)_{k\in\mathbb{Z}}$. The proof of Theorems 5.23 and 5.25 relies on the theory of almost periodic functions which dates back to H. Bohr. There is a large literature on the subject and we refer for instance to the book of Corduneanu [113] for a comprehensive introduction. There are several equivalent definitions of almost periodic functions and we will use the following one. A function $f \in C_b(\mathbb{R}, X)$, *i.e.* a bounded continuous function from $\mathbb{R}$ to X, is said to be almost periodic if the set $\{f(\cdot + \tau),\ \tau \in \mathbb{R}\}$ is relatively compact in $C_b(\mathbb{R}, X)$. The set of almost periodic functions is a sub-algebra of $C_b(\mathbb{R}, X)$, and also the closure of the space of periodic functions in $C_b(\mathbb{R}, X)$. We start with the proof of Theorem 5.23 and Theorem 5.25 in the case when S satisfies the condition *(1)*. Then we deduce the cases *(2)*, *(3)* and *(4)* from the case *(1)*.

Proof of Theorems 5.23 and 5.25 in the case (1). Step 1. Let $f \in X$. Since the trajectory $(\widetilde{S}_t f)_{t\geq 0}$ is relatively compact, we infer from [201, Thm. 8] (with $U(\tau, t) = S_t$ and thus no periodicity condition on U) the existence of an almost periodic eternal solution g of the rescaled semigroup $\widetilde{S}$, *i.e.* a function $g : \mathbb{R} \to X$ such that $g(t + \tau) = \widetilde{S}_\tau g(t)$ for all $t \in \mathbb{R}$ and $\tau \geq 0$, such that

$$\lim_{t\to+\infty} \|\widetilde{S}_t f - g(t)\| = 0.$$

The end of the proof consists in characterizing the function g in the situations of Theorems 5.23 and 5.25. For $\lambda \in \mathbb{R}$, we define the Bohr transformation of the almost-periodic function g by

$$c_\lambda(g) = \lim_{T\to+\infty} \frac{1}{T} \int_0^T e^{-i\lambda t} g(t)\, dt,$$

which is known to exists, see [113, Thm. 3.4], since $e^{-i\lambda t} g(t)$ is also almost periodic. Since $e^{-i\lambda t} g(t)$ is besides an eternal solution of the semigroup $e^{-i\lambda t}\widetilde{S}_t$ with infinitesimal generator $\mathcal{L}_\lambda = \mathcal{L} - \lambda_1 - i\lambda$, we have that

$$\mathcal{L}_\lambda \int_0^T e^{-i\lambda t} g(t)\, dt = g(T) - g(0).$$

Dividing by T the above expression, passing to the limit $T \to +\infty$ and using that $\mathcal{L}_\lambda$ is a closed operator, we get

$$\mathcal{L}_\lambda c_\lambda(g) = 0.$$

In other words, we have established

$$\mathcal{L} c_\lambda(g) = (\lambda_1 + i\lambda) c_\lambda(g)$$

and $\lambda_1 + i\lambda$ is an eigenvalue of $\mathcal{L}$ if $c_\lambda(g) \neq 0$.

Step 2. We deduce that if the boundary spectrum is trivial, as in Theorem 5.23, then necessarily $c_\lambda(g) = 0$ for all $\lambda \neq 0$. By the uniqueness theorem, see for instance [113, Thm. 4.7], we get that g is constant. Due to the conservation law $\langle \widetilde{S}_t f, \phi_1\rangle = \langle f, \phi_1\rangle$ and the simplicity of the eigenvalue 0, we get that $g = \langle f, \phi_1\rangle f_1$ and the result of the case *(1)* in Theorem 5.23 is proved.

Step 3. In the case of Theorem 5.25, the boundary spectrum is not trivial and we know from Theorem 5.5 that $\Sigma_P^+(\widetilde{\mathcal{L}})$ is an additive subgroup of $i\mathbb{R}$, made of algebraically simple eigenvalues. Due to the assumption that λ_1 is isolated, this subgroup must be discrete and $\Sigma_P^+(\mathcal{L})$ is thus given by $\{\lambda_1 + i\alpha k, k \in \mathbb{Z}\}$ for some $\alpha > 0$. As a consequence, any λ such that $c_\lambda(g) \neq 0$ is necessarily of the form $\lambda = \alpha k$ for some $k \in \mathbb{Z}$. By the uniqueness theorem, g is then a α-periodic function which is given, due to Fejér's theorem, by

$$g(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n} \sum_{k=-\ell}^{\ell} c_{\alpha k}(g) e^{i\alpha k t}.$$

Consider (g_k, ψ_k) two positive direct and dual eigenvectors of $\mathcal{L}$ associated to the eigenvalue $i\alpha k$ such that $\langle g_k, \psi_k\rangle = 1$. From the conservation laws $\langle \widetilde{S}_t f, \psi_k\rangle = \langle f, \psi_k\rangle e^{i\alpha k t}$ and the algebraic simplicity of the eigenvalues $i\alpha k$, we get that $c_{\alpha k}(g) = \langle f, \phi_k\rangle g_k$, and the result is proved. □

Proof of Theorems 5.23 and 5.25 in the case (2). Since Y is separable, we can find a sequence $(\varphi_n)_{n\geq 1} \subset Y$ which satisfies $\|\varphi_n\| = 1$ and $\mathrm{span}(\varphi_n)$ is dense in Y. We can then define on X the norm $\|\cdot\|_*$ by setting

$$\|f\|_* = \sum_{n=1}^{\infty} 2^{-n} |\langle f, \varphi_n\rangle|. \tag{5.24}$$

On bounded subsets of X, the topology of this norm is the same as the weak-* topology, or more explicitly it is worth emphasizing

$$f_n \rightharpoonup f \ \ast\sigma(Y', Y) \ \Leftrightarrow\ (\sup \|f_n\| < \infty \quad \text{and} \quad \|f_n - f\|_* \to 0).$$

Since by assumption the trajectory $(\widetilde{S}_t f)$ is bounded, it is weakly-* relatively compact, and so relatively compact in $(X, \|\cdot\|_*)$. It is also clear that the semigroup S is continuous for the weak norm $\|\cdot\|_*$. The normed space $(X, \|\cdot\|_*)$ is not a Banach space, but the proof of Theorem 5.25 actually only requires, for applying [201, Thm. 8], that the closed balls of X are complete metric spaces, which is the case for the distance induced by $\|\cdot\|_*$. Applying the case *(1)* of Theorems 5.23 and 5.25 then yields the result. □

Proof of Theorems 5.23 and 5.25 in the case (3). We consider $f \in X$ and, repeating the proof of Step 2 in Theorem 4.23, we get that $(S_t f)_{t\geq 0}$ belongs to a weak compact set $\mathcal{G}$ of $L^1_{\phi_1}$. We define the norm $\|\cdot\|_*$ by (5.24) for a sequence $(\varphi_n)_{n\geq 1} \subset C_c(E)$ which satisfies $\|\varphi_n\|_{L^\infty} = 1$ and $\mathrm{span}(\varphi_n)$ is dense in $C_0(E)$. This norm induces a metric on $\mathcal{G}$ which is topologically equivalent to the weak convergence on $L^1_{\phi_1}$. The trajectory $(\widetilde{S}_t f)$ is then relatively compact in $(\mathcal{G}, \|\cdot\|_*)$ and the semigroup S is continuous for the weak norm $\|\cdot\|_*$. We conclude as in the proof of the case *(2)*. □

Proof of Theorems 5.23 and 5.25 in the case (4). From the step 3 of the proof of Theorem 4.23, we know that for any $f \in \mathcal{X}^k$ the trajectory $(\widetilde{S}_t f)$ is compact for the strong topology of $L^1_{\phi_1}$. We may then conclude similarly as in the case *(1)*, using that $\mathcal{X}^k$ is dense in X for the norm of $L^1_{\phi_1}$. □

5.6. **A word about spectral analysis argument.** The aim of this section is to recall some more or less classical results which makes possible to slightly improve the conclusions of the results presented in the previous section by additionally assume some spectral gap at the level of the operator or the semigroup. More precisely, we are interested by some accurate versions of a partial, but principal *spectral mapping theorem* which asserts that

$$\Sigma(e^{t\mathcal{L}}) \cap B^c(0, e^{\kappa t}) = e^{t\Sigma(\mathcal{L})\cap\Delta_\kappa}, \qquad \forall\, t \geq 0, \tag{5.25}$$

for some $\kappa < \lambda_1$, and even more precisely, we wish to establish the following geometric (or exponential) asymptotic stability

($\mathbf{E3}_1$) there exist some constants $\kappa < \lambda_1$ and $C \ge 1$ such that for any $f \in X$, there holds

$$\|\widetilde{S}(t)f - \langle f, \phi_1\rangle f_1\| \le \Theta(t)\|f - \langle f, \phi_1\rangle f_1\|, \quad \forall\, t \ge 0,\ \forall\, f \in X, \tag{5.26}$$

with the decay rate function $\Theta(t) := C\, e^{(\kappa-\lambda_1)t}$.

In order to discuss the several results we present, we recall the splitting framework

$$S = V + W * S, \qquad \|V(t)\|_{\mathscr{B}(X)} + \|W(t)\|_{\mathscr{B}(X)} \lesssim e^{\kappa t}, \tag{5.27}$$

for the same $\kappa \in \mathbb{R}$ as above. We start by recalling the spectral mapping theorem for the point spectrum, and its proof, which is instructive.

Lemma 5.27 (Spectral mapping theorem for point spectrum)**.** *For a semigroup $(S_t)_{t\ge0}$ with infinitesimal generator $\mathcal{L}$ we have*

$$\Sigma_P(S_t) \setminus \{0\} = e^{t\Sigma_P(\mathcal{L})}, \qquad \forall\, t \ge 0.$$

Proof of Lemma 5.27. The inclusion $e^{t\Sigma_P(\mathcal{L})} \subset \Sigma_P(S_t) \setminus \{0\}$ is clear. Now let $\xi \in \Sigma_P(S_t) \setminus \{0\}$, that is $\xi \in \mathbb{C} \setminus \{0\}$ such that $S_t f = \xi f$ for some $f \in X_{\mathbb{C}} \setminus \{0\}$, and let $\lambda \in \mathbb{C}$ such that $\xi = e^{\lambda t}$ and $\phi \in X'$ such that $\langle \phi, f\rangle \ne 0$. For any $k \in \mathbb{Z}$ we have $\xi = e^{\lambda t + 2ik\pi}$ and so

$$0 = e^{-(\lambda + \frac{2ik\pi}{t})t} S_t f - f = \Big(\mathcal{L} - \lambda - \frac{2ik\pi}{t}\Big) \int_0^t e^{-(\lambda+\frac{2ik\pi}{t})s} S_s f\, ds.$$

If the last integral is non-zero for some $k \in \mathbb{Z}$, we deduce that $\lambda + \frac{2ik\pi}{t}$ is an eigenvalue of $\mathcal{L}$ and the result is proved. Assume by contradiction that $\int_0^t e^{-(\lambda+\frac{2ik\pi}{t})s} S_s f\, ds = 0$ for all $k \in \mathbb{Z}$. This means that the continuous and periodic complex-valued function $s \mapsto e^{-\lambda s}\langle \phi, S_s f\rangle$ has all its Fourier coefficients equal to zero, which is not possible since this function is not equally zero (its value at $s = 0$ is not zero). □

We next present a very classical result about the exponential stability of f_1 which is based on the quasi-compact semigroup framework of Voigt [355] (see also [15, B-IV-2] and [152, Sec. V.3]) and which is a more accurate version of Lemma 2.7 and Theorem 5.7.

Theorem 5.28. *Let $(S_t)_{t\ge0}$ be a positive irreducible semigroup on a Banach lattice X satisfying the hypotheses of Lemma 2.7 and Theorem 5.7, in particular* **(H2)** *holds for a constant $\kappa_0 \in \mathbb{R}$ and there exists $T > 0$ such that the splitting*

$$S_T = V_T + K_T, \tag{5.28}$$

holds with $\|V_T\|_{\mathscr{B}(X)} \le e^{\kappa T}$, $\kappa < \kappa_0$, and $K_T \in \mathscr{K}(X)$. Then there exists a unique solution (λ_1, f_1, ϕ_1) to the eigentriplet and the exponential stability **($\mathbf{E3}_1$)** *holds (without constructive estimate).*

Remark 5.29. *In the splitting framework* (5.27) *the critical hypothesis $K_T \in \mathscr{K}(X)$ may be obtained by assuming that*

$$\|W(t)\|_{\mathscr{B}(X,\mathcal{X}_1)} \lesssim e^{\kappa t}, \quad \forall\, t \ge 0, \quad \mathcal{X}_1 \subset X \text{ compact.}$$

In fact, in many applications, we are also able to establish $\mathcal{X}_1 \subset D(\mathcal{L}^\beta)$, for some $\beta > 0$, without too much more work.

Theorem 5.28 is in fact nothing but [41, Thm. 14.18] (see also [360, Sec. 2], [152, Thm. V.3.7] or [15, C-IV, Thm. 2.1 & Rk. 2.2]). We give however a short proof of Theorem 5.28 since it is simpler and more direct than the ones we usual find in the literature and in particular does not refer to subtil results about the spectrum and its essential part.

Proof of Theorem 5.28. First step. From Lemma 2.7, we already know that **(H1)**, **(H2)** and **(H3)** hold. Together with the irreducibility which is nothing but **(H4)** from Lemma 4.8, we may apply Theorem 4.13 and conclude to the existence, uniqueness and strict positivity result about the eigentriplet solution (λ_1, f_1, ϕ_1).

Second step. We claim that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C},\ \Re e(z) \ge \kappa_0\}$ is also made of a finite number of isolated eigenvalues with finite geometric multiplicity. We indeed set $\beta_0 := e^{\kappa_0 T}$. Since for any $\lambda \in B^c_{\beta_0} := \{z \in \mathbb{C},\ |z| \ge \beta_0\}$ the operator $\lambda - V_T$ is invertible, we see that $\lambda \in B^c_{\beta_0}$ is in the spectrum of S_T if and only if 0 is in the spectrum of $I - (\lambda - V_T)^{-1}K_T$, or in the spectrum of $I - K_T(\lambda - V_T)^{-1}$. Indeed, solving $(\lambda - S_T)f = g$ is equivalent to, on the one hand,

$$(I - (\lambda - V_T)^{-1}K_T)f = (\lambda - V_T)^{-1}g,$$

and in the other hand,

$$(I - K_T(\lambda - V_T)^{-1})(\lambda - V_T)f = g.$$

So if $\lambda \in \Sigma(S_T) \cap B^c_{\beta_0}$ then $1 \in \Sigma((\lambda - V_T)^{-1}K_T)$. Since $(\lambda - V_T)^{-1}K_T$ is a compact operator, the classical Fredholm alternative (see for instance [71, Thm. 6.6]) asserts that its spectrum is made of eigenvalues with finite geometric multiplicity, and then so does for $\Sigma(S_T) \cap B^c_{\beta_0}$. We can also prove, by adapting the proof of [71, Lem. 6.2], that these eigenvalues are isolated, and thus $\Sigma(S_T) \cap B^c_{\beta_0}$ is made of a finite number of isolated eigenvalues with finite geometric multiplicity. Since $e^{T\Sigma(\mathcal{L})} \subset \Sigma(S_T)$, we deduce that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C},\ \Re e(z) \geq \kappa_0\}$ is also made of a finite number of isolated eigenvalues with finite geometric multiplicity.

Third step. We prove the existence of a spectral gap and we conclude.
Since $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C},\ \Re e(z) \geq \kappa_0\}$ is finite, λ_1 is simple, and the boundary spectrum of $\mathcal{L}$ is a group, we deduce the existence of $\varepsilon > 0$ such that $\Sigma(\mathcal{L}) \cap \{z \in \mathbb{C},\ \Re e(z) \geq \lambda_1 - \varepsilon\} = \{\lambda_1\}$. The spectral mapping theorem in Lemma 5.27 then ensures that $\Sigma(S_T) \cap \{z \in \mathbb{C},\ |z| \geq e^{(\lambda_1 - \varepsilon)T}\} = \{e^{\lambda_1 T}\}$ and that $e^{\lambda_1 T}$ is simple with eigenspace spanned by f_1. The restriction $S_T^\perp$ of S_T to the invariant subspace $X_\perp := \{f \in X,\ \langle \phi_1, f\rangle = 0\}$ thus has a spectral radius smaller than $e^{(\lambda_1 - \varepsilon)T}$. The spectral radius formula (see [334, Thm. 10.13] for instance) then ensures that

$$\lim_{n\to\infty} \|S_{nT}^\perp\|^{1/n} = r(S_T^\perp) \leq e^{(\lambda_1 - \varepsilon)T}.$$

This guarantees, for any $\eta \in (0, \varepsilon)$, the existence of a constant $C_\eta > 0$ such that for all $f \in X_\perp$ and all $t \geq 0$

$$\|e^{-\lambda_1 t} S_t f\| \leq C_\eta e^{-\eta t}\|f\|,$$

and the proof is complete. □

Let us now present a variant of another classical result known as the Gearhart-Prüss Theorem in [175, 326], see also the contributions of Herbst [211] and Greiner [15, A-III.7] as well as the more constructive proof [152, Thm. V.1.11] and recently [206] based on techniques developed in or related to [364, 61].

Theorem 5.30. *Consider a positive semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ satisfies the conclusions* **(C2)** *about the existence, positivity and uniqueness of the first eigentriplet* (λ_1, f_1, ϕ_1). *We assume furthermore that X is an Hilbert space and that there exist $\kappa < \lambda_1$ and $R > 0$ such that*

(i) $\sup_{z\in\Delta_\kappa\setminus B_R} \|\mathcal{R}_\mathcal{L}(z)\|_{\mathscr{B}(X)} < \infty$*;*

(ii) $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L}) \cap B_R$.

Then the exponential stability **(E3$_1$)** *holds (without constructive estimate).*

Proof of Theorem 5.30. The spectral information **(C2)** and (ii) together imply **(C3)** (because of Theorem 5.5) and that there exists $\kappa^* \in (\kappa, \lambda_1)$, such that $\Sigma(\mathcal{L}) \cap \Delta_{\kappa^*} = \{\lambda_1\}$. The operator $\mathcal{L}$ on $X_0 := (\mathrm{vect}\{f_1\})^\perp$ thus satisfies $\sup_{z\in\Delta_{\kappa^*}} \|\mathcal{R}_\mathcal{L}(z)\|_{\mathscr{B}(X_0)} < \infty$, and we conclude thanks to [152, Thm. V.1.11]. The lack of constructively here only comes from the fact that our assumptions do not provide any information on the spectral gap $\lambda_1 - \kappa > 0$. □

Remark 5.31. *Except of the Hilbert space framework, the assumptions made in Theorem 5.30 are slightly weaker than those of Theorem 5.28, and are indeed established during the proof of Theorem 5.28: such an information at the level of the resolvent is a bit easier to establish than a similar estimate at the level of the semigroup. In the splitting framework* (5.27) *and its resolvent counterpart* (2.22)*, we typically only have to show*

$$\sup_{\kappa\leq\Re ez\leq\kappa_1} \|\mathcal{V}(z)\|_{\mathscr{B}(X)} < \infty, \quad \lim_{r\to\infty} \sup_{\kappa\leq\Re ez\leq\kappa_1,\,|\Im mz|\geq r} \|\mathcal{W}(z)\|_{\mathscr{B}(X)} = 0, \tag{5.29}$$

for some $\kappa < \lambda_1$, and $\mathcal{W}(z) \in \mathscr{K}(X)$ for any $z \in \Delta_\kappa$. That last claim is classical (see for instance [190]*) and we only briefly sketch the proof. On the one hand, from the first and the last estimates,* *we deduce that* $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L})$ *thanks to Theorem 5.6. As in the proof of Theorem 5.6 and with the usual notations, we also have*

$$(I - \mathcal{W}(z))\mathcal{R}_\mathcal{L}(z) = \mathcal{V}(z), \quad \forall z \in \Delta_\kappa,$$

where $I - \mathcal{W}(z)$ *is invertible and* $\|(I - \mathcal{W}(z))^{-1}\|_{\mathcal{B}(X)} \le 2$ *for any* $z \in \mathbb{C}$ *such that* $\kappa \le \Re e z \le \kappa_1$, $|\Im m z| \ge R$ *and* R *is large enough. We immediately deduce that the condition (i) in Theorem 5.30 holds.*

We end this section by a more recent result which is similar to the Gearhart-Prüss Theorem but is not restricted to an Hilbert space.

Theorem 5.32. *Consider a positive semigroup* S *on a Banach lattice* X *such that its generator* $\mathcal{L}$ *satisfies the conclusions* **(C2)** *about the existence, positivity and uniqueness of the first eigentriplet* (λ_1, f_1, ϕ_1). *We further assume that* $\mathcal{L} = \mathcal{A} + \mathcal{B}$ *with* $0 \le \mathcal{A} \in \mathscr{B}(X)$, $S_\mathcal{B} \ge 0$ *and the associated operators* V *and* W *defined by* (3.14) *satisfy* (5.27) *for some* $\kappa < \lambda_1$ *and that the resolvent counterpart* $\mathcal{W}$ *defined by* (2.22) *satisfies* (5.29) *and more precisely*

$$\sup_{\kappa \le \Re e z \le \kappa_1} \|\langle z \rangle^\alpha \mathcal{W}(z)\|_{\mathscr{B}(X)} < \infty,$$

with $\alpha > 1$. *Then the exponential stability* $(\mathbf{E3}_1)$ *holds (without constructive estimate).*

The proof of Theorem 5.32 is a mere adaptation of [278, Thm. 3.1] (see also [273]) and it is thus skipped. The needed estimates are a bit stronger than those of Remark 5.31, but in the applications, they are not really more demanding. They also hold at the level of the resolvent instead of what is assumed in the statement of Theorem 5.28.

We conclude by emphasizing again on the fact that all the above results are not constructive. We propose in the next part an alternative approach which is constructive.

6. Quantitative stability

In this section we establish some quantitative stability results in the spirit of the Doblin, Harris, Meyn-Tweedie theory for Markov semigroup.

6.1. **About quantified positivity conditions.** We briefly discuss some positivity conditions related to the strong maximum principle and barriers techniques. The issue is about how quantify the strong maximum principle

$$f \in X_+ \backslash \{0\}, \ (\kappa_1 - \mathcal{L}) f \ge 0 \ \text{ imply } \ f > 0 \text{ or } f \gg 0$$

or the related strong positivity of the associated semigroup. A possible way can be achieved with the help of a barrier functions family $\mathcal{G} \subset X_+$ and a second weaker (semi)norm $[\cdot]$ used for normalization. Let us then introduce the two conditions

$$\forall R > 0, \ \exists g_i \in \mathcal{G}, \ \forall f \in X_+, \ [f] = 1, \ \|f\| \le R, \tag{6.1}$$

we have
(i) $S_T f \ge g_1$ (for some $T > 0$)
or
(ii) $f \ge g_2$ if $(\kappa_1 - \mathcal{L}) f \ge 0$.

Point (ii) is a quantified version of the strong maximum principle when $\mathcal{G} \subset X_{++}$ and it is always a consequence of the positivity condition (i). Assume indeed that (i) holds (for some $T > 0$) and that f satisfies the requirements (6.1) and $(\kappa_1 - \mathcal{L}) f \ge 0$. We then write

$$\frac{d}{dt}(e^{(\mathcal{L}-\kappa_1)t} f) = e^{(\mathcal{L}-\kappa_1)t}(\mathcal{L} - \kappa_1) f \le 0,$$

so that

$$f \ge e^{(\mathcal{L}-\kappa_1)T} f = e^{-\kappa_1 T} S_T f \ge e^{-\kappa_1 T} g_2 =: g_1,$$

with g_2 given by condition (i). The reciprocal implication is not clear, see however Lemma 4.8-**(3)**.

Let us now make a list of possible quantified positivity conditions of Doblin-Harris type for a linear (and continuous) operator $A : X \to X$:

(P1′) $\exists\, g_0 \in X_+\backslash\{0\}$, $\exists\, \psi_0 \in X_+\backslash\{0\}$, $\forall\, f \in X_+$, $Af \geq g_0 \langle f, \psi_0 \rangle$;
(P2′) $\exists\, g_0 \in X_+\backslash\{0\}$, $\exists\, \psi_0 \in X'_{++}$, $\forall\, f \in X_+$, $Af \geq g_0 \langle f, \psi_0 \rangle$;
(P3′) $\exists\, g_0 \in X_{++}$, $\exists\, \psi_0 \in X'_+\backslash\{0\}$, $\forall\, f \in X_+$, $Af \geq g_0 \langle f, \psi_0 \rangle$;
(P4′) $\exists\, g_0 \in X_{++}$, $\exists\, \psi_0 \in X'_{++}$, $\forall\, f \in X_+$, $Af \geq g_0 \langle f, \psi_0 \rangle$.

We summarize some elementary relations between these conditions and those listed in Section 4.2.

Lemma 6.1. *We have (P2′) ⇒ (P2) ⇒ (P1), (P3′) ⇒ (P3) ⇒ (P1), (P4′) ⇒ ((P4), (P3′), (P2′)) as well as (P4) ⇒ ((P3), (P2)).*

We also have: A satisfies (P2′) iff A^* satisfies (P3′); A satisfies (P3′) iff A^* satisfies (P2′); A satisfies (P4′) iff A^* satisfies (P4′).

We finally have: A satisfies (P2′) implies $\exists\, g_0 \in X_+\backslash\{0\}$, $\exists\, \kappa > 0$, $Ag_0 \geq \kappa g_0$

Proof of Lemma 6.1. We assume $Af \geq g_0 \langle f, \psi_0 \rangle$ for any $f \in X_+$ and some $g_0 \in X_+$, $\psi_0 \in X'_+$. For any $\phi \in X'\backslash\{0\}$ and $f \in X_+$, we have

$$\langle A^*\phi, f \rangle = \langle \phi, Af \rangle \geq \langle \phi, g_0 \langle f, \psi_0 \rangle \rangle,$$

which implies $A^*\phi \geq \psi_0 \langle \phi, g_0 \rangle$. We thus deduce that A satisfies (P2′) (resp. (P3′), (P4)) implies that A^* satisfies (P3′) (resp. (P2′), (P4)). The other implications can be established in a similar or even simpler way. □

We conclude this introductory section by emphasizing on the fact (as already mentioned above) that $S_{\mathcal{L}}$ satisfies (Pi') implies $\mathcal{R}_{\mathcal{L}}(\lambda)$ satisfies (Pi') for any $\lambda \geq \lambda_1$ and $i = 1, \dots, 4$.

6.2. **Asymptotic stability under Doblin condition.** We start with a simple situation. We assume the Doblin condition, namely

$$\exists\, T > 0,\ \exists\, \psi_0 \gg 0,\ \exists\, g_0 > 0, \quad \forall\, f \geq 0,\ S_T f \geq g_0 \langle \psi_0, f \rangle, \tag{6.2}$$

together with the companion positivity condition

$$\exists\, r_0 > 0, \quad \langle \phi_1, g_0 \rangle \geq r_0, \tag{6.3}$$

as well as the strong additional boundedness assumption

$$\exists\, R_0 > 0, \quad \phi_1 \leq R_0 \psi_0. \tag{6.4}$$

When $\psi_0 := 1 \in X' \subset L^\infty$, the condition in (6.4) is automatically satisfied with $R_0 := \|\phi_1\| = 1$. Let us first emphasize that (6.3) is a natural condition when $S^*_{\mathcal{L}}$ enjoys a splitting structure similar to (5.19). More precisely, when

$$\|\widetilde{S}^*(t)\phi\| \leq \Theta(t)\|\phi\| + \int_0^t \Theta(t-s)[\widetilde{S}^*(s)\phi]_{g_0}\, ds,$$

with $\Theta \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, we deduce that

$$1 = \|\phi_1\| = \|\widetilde{S}^*(t)\phi_1\| \leq \Theta(t) + \int_0^t \Theta(t-s)[\phi_1]_{g_0}\, ds, \quad \forall\, t > 0.$$

Passing to the limit $t \to \infty$, we get (6.3) with $r_0 := \|\Theta\|^{-1}_{L^1}$. Also (6.4) can be deduced from a splitting structure condition on the dual problem. More precisely, we assume that $D(\mathcal{L}^\infty) \subset L^1_{\text{loc}}$ and the splitting property $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} \in \mathscr{B}(X)$, $\mathcal{R}_{\mathcal{B}}(\lambda) \in \mathscr{B}(X) \cap \mathscr{B}(X_+)$ for any $\lambda \geq \kappa$, with $\kappa < \kappa_0 \leq \lambda_1$, and the additional regularity condition

$$(\mathcal{R}_{\mathcal{B}^*}(\lambda)\mathcal{A}^*)^N : L^1_{g_0} \to L^\infty_{\psi_0^{-1}}, \quad \forall\, \lambda > \kappa. \tag{6.5}$$

Since the dual eigenvector ϕ_1 satisfies

$$(\lambda_1 - \mathcal{B}^*)\phi_1 = \mathcal{A}^*\phi_1, \quad \lambda_1 > \kappa,$$

and then $\phi_1 = (\mathcal{R}_{\mathcal{B}^*}(\lambda_1)\mathcal{A}^*)^N\phi_1$, we may use estimate (6.5) and we get that (6.3)-(6.4) holds with the normalization condition $r_0 := 1$ and $R_0 := \|(\mathcal{R}_{\mathcal{B}^*}(\lambda)\mathcal{A}^*)^N\|_{\mathscr{B}(L^1_{g_0}, L^\infty_{\psi_0^{-1}})}$.

We are then able to formulate a first quantified stability result.

Theorem 6.2. *Consider a semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusion* **(C1)** *on the existence of the first eigentriplet* (λ_1, f_1, ϕ_1). *We assume furthermore the Doblin condition* (6.2)–(6.4)-(6.3). *Then the exponential stability* **(E3$_1$)** *in the norm* $[\cdot]_{\psi_0}$ *holds true, with constructive constants.*

The proof closely follows the usual contraction argument in the Doblin result, see for instance [268], [168, Thm. 11] or [81, Thm. 2.1]. We do not explicitly assume the irreducibility of the semigroup, but the Doblin condition (6.2)–(6.4)-(6.3) is in many aspects a strong positivity condition. In particular, our result implies the uniqueness of the first eigentriplet (λ_1, f_1, ϕ_1) and the triviality of the boundary spectrum.

Proof of Theorem 6.2. The two conditions (6.2) and (6.4) together imply the modified Doblin condition

$$\exists\, T > 0,\ \exists\, g_1 > 0, \quad \forall\, f \ge 0,\ S_T f \ge g_1 \langle \phi_1, f\rangle,$$

with $g_1 := g_0/R_0$. Take f such that $\langle \phi_1, f\rangle = 0$, so that $\langle \phi_1, f_\pm\rangle = r := \langle \phi_1, |f|\rangle/2 \ge 0$ and thus

$$S_T f_\pm \ge g_1 \langle \phi_1, f_\pm\rangle = r g_1.$$

We write

$$|S_T f| \le |S_T f_+ - r g_1| + |S_T f_- - r g_1| = S_T|f| - 2 r g_1.$$

We deduce

$$\langle \phi_1, |S_T f|\rangle \le \langle S_T^* \phi_1, |f|\rangle - 2r\langle \phi_1, g_1\rangle = \Big(e^{\lambda_1 T} - \langle \phi_1, g_1\rangle\Big)\langle \phi_1, |f|\rangle.$$

In other words, setting $\widetilde{S}_t := e^{-\lambda_1 t} S_t$, we have

$$[\widetilde{S}_T f]_{\phi_1} \le \gamma [f]_{\phi_1},$$

with $\gamma < 1$ which depends explicitly of r_0, R_0, T and the estimates on λ_1. We then classically deduce the exponential convergence in the $[\cdot]_{\phi_1}$ norm. Now, the dual condition associated to the Doblin hypothesis (6.2) is

$$\forall\, \psi \in X'_+, \quad S_T^* \psi \ge \psi_0 \langle \psi, g_0\rangle.$$

In particular, the first dual eigenvector ϕ_1 satisfies

$$\phi_1 = e^{-\lambda_1 T} S_T^* \phi_1 \ge e^{-\lambda_1 T} \psi_0 \langle \phi_1, g_0\rangle = e^{-\lambda_1 T} r_0 \psi_0. \tag{6.6}$$

Together with condition in (6.4), we see that $[\cdot]_{\phi_1}$ and $[\cdot]_{\psi_0}$ are equivalent norm, and we immediately obtain the exponential convergence in the $[\cdot]_{\psi_0}$ norm (with constructive constants). ☐

6.3. **Asymptotic stability under Harris condition.** The Doblin condition (6.2)–(6.4)-(6.3) is too much demanding for many applications. In this section, we make the following somehow more general Harris type condition complemented with a Lyapunov condition. More precisely, we assume that there exists $T > 0$ such that $\widetilde{S}_T := S_T e^{-\lambda_1 T}$ first satisfies the Lyapunov condition

$$\|\widetilde{S}_T f\| \le \gamma_L \|f\| + K[f]_{\phi_1}, \tag{6.7}$$

with $\gamma_L \in (0,1)$, $K \ge 0$. We next assume that $\widetilde{S}_T$ satisfies the Harris condition

$$\begin{cases} \exists\, A > K/(1-\gamma_L),\ \exists\, g_A > 0 \text{ such that} \\ \forall\, f \ge 0,\ \|f\| \le A[f]_{\phi_1} \text{ there holds } S_T f \ge g_A [f]_{\phi_1}. \end{cases} \tag{6.8}$$

We finally replace the positivity condition (6.3) by

$$\exists\, r_A > 0, \quad \langle \phi_1, g_A\rangle \ge r_A. \tag{6.9}$$

As we have seen several times, condition (6.7) is some kind of regularity hypothesis which is natural under a splitting structure on the semigroup $S_{\mathcal{L}}$. We emphasize that conditions (6.7)-(6.8)-(6.9) slightly generalize the usual set of hypothesizes for the Harris theorem, see for instance [81, Sect. 3]. We also point out that there is a connection between the condition (6.8) and the notion of *partial integral* or *partial kernel* operators, see for instance [178, Cor. 5.3]. The long term convergence of semigroups that contain a partially integral operator was studied in particular in [317, 177, 180].

Theorem 6.3. *Consider a semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C1)** *on the existence of the first eigentriplet (λ_1, f_1, ϕ_1). We assume furthermore the Harris condition* (6.8) *together with the Lyapunov condition* (6.7) *and the positivity condition* (6.9)*. Then the exponential stability* **(E3$_1$)** *in the norm of X holds true, with constructive constants.*

Of course, in order that Theorem 6.3 really gives a constructive convergence result, we have to establish (6.8), (6.7) and (6.9) in a constructive way.

Proof of Theorem 6.3. On the one hand, we have

$$[\widetilde{S}_T f]_{\phi_1} \le \langle \widetilde{S}_T |f|, \phi_1\rangle = \langle |f|, \widetilde{S}_T^* \phi_1\rangle = [f]_{\phi_1}. \tag{6.10}$$

On the other hand, we wish to establish the coupling property

$$[\widetilde{S}_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1} \quad \text{if} \quad \|f\| \le A'[f]_{\phi_1} \text{ and } \langle f, \phi_1\rangle = 0, \tag{6.11}$$

for some $\gamma_H \in (0,1)$ and with $A' := A/2$. We thus consider $f \in X$, such that $\langle f, \phi_1\rangle = 0$ and $\|f\| \le A'[f]_{\phi_1}$, so that

$$\|f_\pm\| \le \|f\| \le A'[f]_{\phi_1} = A[f_\pm]_{\phi_1}.$$

Using the Harris condition (6.8), we deduce

$$\widetilde{S}_T f_\pm \ge \vartheta g_A, \quad \vartheta := \tfrac{1}{2} e^{-\lambda_1 T} [f]_{\phi_1}.$$

Similarly as in the proof of Theorem 6.2, we next compute

$$|\widetilde{S}_T f| \le |\widetilde{S}_T f_+ - \vartheta g_A| + |\widetilde{S}_T f_- - \vartheta g_A| \le \widetilde{S}_T |f| - 2\vartheta g_A$$

and then

$$\begin{aligned} [\widetilde{S}_T f]_{\phi_1} &\le \langle \widetilde{S}_T |f| - 2\vartheta g_A, \phi_1\rangle \\ &= \langle |f|, \widetilde{S}_T^* \phi_1\rangle - 2\vartheta \langle g_A, \phi_1\rangle \\ &= \big(1 - e^{-\lambda_1 T}\langle g_A, \phi_1\rangle\big)[f]_{\phi_1}, \end{aligned}$$

which in turn implies (6.11) with $\gamma_H := 1 - e^{-\kappa_0 T} r_A$.

Now, the two estimates (6.10) and (6.11) together give

$$[\widetilde{S}_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1} + \frac{1-\gamma_H}{A'} \|f\|. \tag{6.12}$$

From (6.12) and the Lyapunov condition (6.7), we deduce that

$$U^{n+1} = M U^n$$

with

$$U^n := \begin{pmatrix} \|\widetilde{S}_T^n f\| \\ [\widetilde{S}_T^n f]_{\phi_1} \end{pmatrix} = \begin{pmatrix} \|\widetilde{S}_{nT} f\| \\ [\widetilde{S}_{nT} f]_{\phi_1} \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} \gamma_L & K \\ \frac{1-\gamma_H}{A} & \gamma_H \end{pmatrix}.$$

The eigenvalues of M are

$$\mu_\pm := \frac{1}{2}\big(T \pm \sqrt{T^2 - 4D}\big),$$

with

$$T := \mathrm{tr} M = \gamma_L + \gamma_H, \quad D := \det M = \gamma_L \gamma_H - (1-\gamma_H)\frac{K}{A'}.$$

We observe that

$$\gamma_L \gamma_H > D > \gamma_L \gamma_H - (1-\gamma_H)(1-\gamma_L) = T - 1,$$

so that

$$(\gamma_H - \gamma_L)^2 = T^2 - 4\gamma_L\gamma_H < T^2 - 4D < T^2 - 4(T-1) = (T-2)^2$$

and finally

$$\alpha := \max(|\mu_+|, |\mu_-|) < \max(\gamma_H, \gamma_L, |T-1|, 1) = 1.$$

We conclude that $\|M^n\| \lesssim \alpha^n$, from what we immediately conclude. □

Remark 6.4. *It is useful to emphasize that the existence of f_1 is not required in the proof of Theorem 6.3 for proving that $\|M^n\| \lesssim \alpha^n$, and this estimate can actually be used to derive the existence of f_1. In ordet to prove that last claim, we first observe that Theorem 6.3 ensures that $(\|\widetilde{S}_{nT} f_0\|)_n$ is a Cauchy sequence for any $f_0 \in X$. Indeed, for any $p \in \mathbb{N}$, $f = f_0 - \widetilde{S}_{pT} f_0$ verifies*

$$\langle f, \phi_1 \rangle = \langle f_0, \phi_1 \rangle - \langle f_0, \widetilde{S}^*_{pT} \phi_1 \rangle = \langle f_0, \phi_1 \rangle - \langle f_0, \phi_1 \rangle = 0,$$

and we then have

$$\|\widetilde{S}_{nT} f_0 - \widetilde{S}_{(n+p)T} f_0\| + [\widetilde{S}_{nT} f_0 - \widetilde{S}_{(n+p)T} f_0]_{\phi_1} \lesssim \alpha^n \big(\|f_0 - \widetilde{S}_{pT} f_0\| + [f_0 - \widetilde{S}_{pT} f_0]_{\phi_1}\big).$$

Choosing $f_0 \in X_+$ such that $[f_0]_{\phi_1} = 1$, we deduce that $(\widetilde{S}_{nT} f)$ converges to a fixed point f_1 of $\widetilde{S}_T$, which is not zero because

$$[f_1]_{\phi_1} = \lim [\widetilde{S}_{nT} f_0]_{\phi_1} = [f_0]_{\phi_1} = 1,$$

and f_1 is the unique fixed point with normalization $[f_1]_{\phi_1} = 1$. Besides, $f_1 \in X_+$ because of the positivity of S and f_0. This ensures that

$$[\widetilde{S}_t f_1]_{\phi_1} = \langle \widetilde{S}_t f_1, \phi_1 \rangle = \langle f_1, \widetilde{S}^*_t \phi_1 \rangle = \langle f_1, \phi_1 \rangle = 1,$$

for any $t > 0$. Since on the other hand

$$\widetilde{S}_T \widetilde{S}_t f_1 = \widetilde{S}_{t+T} f_1 = \widetilde{S}_t \widetilde{S}_T f_1 = \widetilde{S}_t f_1,$$

we deduce from the uniqueness of the fixed point that $\widetilde{S}_t f_1 = f_1$, which yields that $f_1 \in D(\mathcal{L})$ and $\mathcal{L} f_1 = \lambda_1 f_1$.

6.4. **Quantified isolation of the first eigenvalue.** In terms of the geometry of the spectrum, an immediate consequence of Theorem 6.3 is that the conditions (6.8), (6.7) and (6.9) ensure the existence of a spectral gap, namely the existence of $\varepsilon > 0$ such that

$$\Sigma(\mathcal{L}) \cap \Delta_{\lambda_1 - \varepsilon} = \{\lambda_1\}.$$

We still assume that the Lyapunov condition (6.7) holds for some $T > 0$, $\gamma_L \in (0,1)$ and $K \geq 0$, but we relax (6.8) into the time-averaged condition

$$\text{(6.13)} \quad \begin{cases} \exists\, A > K/(1-\gamma_L),\ \exists\, g_A > 0 \text{ such that} \\ \forall\, f \geq 0,\ \|f\| \leq A[f]_{\phi_1} \text{ there holds } \displaystyle\int_0^T S_t f \, dt \geq g_A [f]_{\phi_1}. \end{cases}$$

It is worth emphasizing that (6.13) does not imply anymore the existence of a spectral gap, and there can be a non-trivial boundary spectrum, see Section 9.2 for an example. However, it is strong enough for guaranteeing that λ_1 is isolated from the rest of the spectrum, in the sense that

$$\text{(6.14)} \quad \Sigma(\mathcal{L}) \cap B(\lambda_1, \varepsilon) = \{\lambda_1\},$$

for some $\varepsilon > 0$. In particular, if not trivial, the boundary spectrum must be discrete from Theorem 5.5 (under the additional assumptions listed in the statement of this last result).

Theorem 6.5. *Consider a semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C1)** *on the existence of the first eigentriplet (λ_1, f_1, ϕ_1). We assume furthermore the time-averaged Harris condition* (6.13) *together with the Lyapunov condition* (6.7) *and the positivity condition* (6.9)*. Then* (6.14) *holds true for some constructive constant $\varepsilon > 0$.*

Proof. First, we readily deduce from (6.13) and the inversion formula (2.13) that

$$\text{(6.15)} \quad \begin{cases} \exists\, A > K/(1-\gamma_L),\ \exists\, g_A > 0 \text{ such that} \\ \forall\, f \geq 0,\ \|f\| \leq A[f]_{\phi_1} \text{ there holds } \widetilde{\mathcal{R}}(\lambda) f \geq \widetilde{g}_A [f]_{\phi_1},\ \forall\, \lambda > \lambda_1, \end{cases}$$

where $\widetilde{\mathcal{R}}(\lambda) := (\lambda - \lambda_1)\mathcal{R}_{\mathcal{L}}(\lambda)$ and $\widetilde{g}_A := (\lambda - \lambda_1) e^{-\lambda T} g_A$. It is worth emphasizing that $\widetilde{\mathcal{R}}(\lambda) f_1 = f_1$ and $1 \in \Sigma(\widetilde{\mathcal{R}}(\lambda) \subset \overline{B(0,1)}$. Next, we claim that the Lyapunov condition (6.7) ensures the existence of $\lambda > \lambda_1$ such that

$$\text{(6.16)} \quad \|\widetilde{\mathcal{R}}(\lambda) f\| \leq \gamma'_L \|f\| + K' [f]_{\phi_1}$$

for all $f \in X$ and some $\gamma'_L < 1$ and $K' > 0$. Indeed, by iteration of (6.7), we have

$$\|\widetilde{S}_{nT} f\| \leq \gamma_L^n \|f\| + \frac{K}{1-\gamma_L} [f]_{\phi_1},$$

for all integer n, from which we deduce

$$\|\widetilde{S}_t f\| \le C\gamma_L^{\lfloor t/T \rfloor} \|f\| + \frac{CK}{1-\gamma_L}[f]_{\phi_1},$$

for all $t \ge 0$ and where $C = \sup_{0\le t\le T} \|\widetilde{S}_t\|$. We finally infer from the inversion formula (2.13) that

$$\|\widetilde{\mathcal{R}}(\lambda) f\| \le \frac{C_1(\lambda-\lambda_1)}{\lambda-\lambda_1+\log\frac{1}{\gamma_L}}\|f\| + \frac{C_2}{1-\gamma_L}[f]_{\phi_1},$$

for all $\lambda > \lambda_1$ and some $C_1, C_2 > 0$. Then we only need to choose λ close enough to λ_1 so that $\frac{C_1(\lambda-\lambda_1)}{\lambda-\lambda_1+\log\frac{1}{\gamma_L}} < 1$ and we obtain (6.16).

We have proved that $\widetilde{\mathcal{R}}(\lambda)$ satisfies (6.16) and (6.15). Together with the positivity condition (6.9), we can thus repeat the proof of Theorem 6.3 for the operator $\widetilde{\mathcal{R}}$ instead of $\widetilde{S}$ and we obtain the existence of constructive constants $\alpha \in (0,1)$ and $C \ge 1$ such that

$$\|\widetilde{\mathcal{R}}(\lambda)^n f\| \le C\alpha^n \|f\|, \quad \forall\, n \ge 1,$$

for any $f \in X$, $\langle f, \phi_1\rangle = 0$. By the spectral radius formula, we deduce

$$\Sigma\big(\widetilde{\mathcal{R}}(\lambda)\big) \cap \{z \in \mathbb{C},\ |z| > \alpha\} = \{1\}.$$

The spectral mapping theorem for the resolvent, which ensures that

$$\Sigma\big(\widetilde{\mathcal{R}}(\lambda)\big) \setminus \{0\} = \frac{\lambda-\lambda_1}{\lambda-\Sigma(\mathcal{L})},$$

then yields (6.14) with $\varepsilon = (\alpha^{-1}-1)(\lambda-\lambda_1)$. □

6.5. **The weak dissipativity case.** In this section, we consider a weak dissipative semigroup (S_t) as considered in Section 3.3 and in a sense we make precise now. We consider four Banach lattices $X_3 \subset X_2 \subset X_1 \subset X_0 = X$. We first make the same kind of Harris type condition as in the previous section, namely

Hypothesis (H) (Doblin-Harris) condition (6.8) holds for the same time $T > 0$ and for both norms $\|\cdot\| = \|\cdot\|_{X_0}$ and $\|\cdot\| = \|\cdot\|_{X_2}$ as well as the companion positivity condition (6.9) holds.

Instead of the strong Lyapunov condition (6.7), we assume

Hypothesis (L) (weak Lyapunov) there exist a constant $K \ge 0$ such that

$$\begin{aligned} \|\widetilde{S}f\|_1 + \|\widetilde{S}f\|_0 &\le \|f\|_1 + K[f]_{\phi_1}, \quad \forall\, f \in X_1,\\ \|\widetilde{S}f\|_3 + \|\widetilde{S}f\|_2 &\le \|f\|_3 + K[f]_{\phi_1}, \quad \forall\, f \in X_3, \end{aligned}$$

with $\widetilde{S} = S_T e^{-\lambda_1 T}$.

Hypothesis (I) (interpolation) there exists an increasing function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, $\lambda \mapsto \xi_\lambda$, such that

$$\lambda\|f\|_1 \le \|f\|_0 + \xi_\lambda \|f\|_3,\ \forall\, \lambda > 0, \quad \xi_\lambda/\lambda \to 0 \ \text{ as } \lambda \to 0.$$

Theorem 6.6. *Consider a semigroup S on a Banach lattice X such that its generator $\mathcal{L}$ enjoys the conclusions* **(C1)** *on the existence of the first eigentriplet (λ_1, f_1, ϕ_1). We assume furthermore the three above conditions of weak confinement* **(L)***, Doblin-Harris strong irreducibility* **(H)** *and interpolation* **(I)***. Then, there exist some constructive decay rate functions Θ and $\widetilde{\Theta}$ such that*

$$\|S^n f\|_{X_1} \lesssim \Theta(n)\|f\|_{X_3}, \quad \forall\, n \ge 1, \tag{6.17}$$

and

$$\|S^n f\| \lesssim \widetilde{\Theta}(n)\|f\|_{X_3}, \quad \forall\, n \ge 1, \tag{6.18}$$

for any $f \in X_3$, $\langle f, \phi_1\rangle = 0$. More precisely, the decay rate functions Θ and $\widetilde{\Theta}$ are defined by

$$\Theta(t) := \inf_\lambda \Theta_{\zeta\lambda}(t), \quad \widetilde{\Theta}(t) := t^{-1}\Theta([t/2]), \tag{6.19}$$

for a constructive constant $\zeta \in (0,1)$, the infimum being taken over all the decreasing function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$, $t \mapsto \lambda_t$, and Θ_λ is defined by

$$\Theta(t) :\simeq \inf_{\lambda>0} \big(e^{-\lambda t} + \frac{\xi_\lambda}{\lambda}\big). \tag{6.20}$$

The proof follows closely the proof of [81, Thm. 4.8]. We start with the following key argument of non expansive mapping result on a well chosen norm.

Proposition 6.7. *Consider a positive semigroup (S_t) which satisfies both above conditions of weak confinement* **(L)** *and Doblin-Harris strong irreducibility* **(H)**. *There exist some equivalent norms $|||\cdot|||_1$ to $\|\cdot\|_1$ and $|||\cdot|||_3$ to $\|\cdot\|_3$ such that $\widetilde{S}_t$ is a non expansive mapping for the two new norms $|||\cdot|||_1$ and $|||\cdot|||_3$. More precisely, there exists $\alpha > 0$ such that*

$$|||\widetilde{S}f|||_1 + \alpha\|\widetilde{S}f\|_0 \quad \le \quad |||f|||_1, \quad \forall\, f \in X_1,\ \langle f, \phi_1\rangle = 0, \tag{6.21}$$

$$|||\widetilde{S}f|||_3 + \alpha\|\widetilde{S}f\|_2 \quad \le \quad |||f|||_3, \quad \forall\, f \in X_3,\ \langle f, \phi_1\rangle = 0. \tag{6.22}$$

Proof of Proposition 6.7. We define

$$|||f|||_1 := [f]_{\phi_1} + \delta\|f\|_0 + \beta\|f\|_1, \tag{6.23}$$

with $\beta > \delta > 0$ conveniently chosen. We take $\beta := (1-\gamma_H)/K$, $\delta := (1-\gamma_H)/A$. We define $|||\cdot|||_3$ in the same way. In what follows, we then only establish (6.21), the proof of (6.22) being exactly the same.

We fix $f \in X_1$, $\langle f, \phi_1\rangle = 0$, and we recall

$$[\widetilde{S}f]_{\phi_1} \le [f]_{\phi_1}. \tag{6.24}$$

We also recall that from (6.11), for any $A > 0$, there exists $\gamma_H = \gamma_H(A) \in (0,1)$ such that the following coupling property holds

$$[\widetilde{S}f]_{\phi_1} \le \gamma_H [f]_{\phi_1} \quad \text{if} \quad \|f\|_0 \le A[f]_{\phi_1}. \tag{6.25}$$

We fix $A > K$ and we observe that the following alternative holds

$$\|f\|_0 \le A[f]_{\phi_1} \tag{6.26}$$

or

$$\|f\|_0 > A[f]_{\phi_1}. \tag{6.27}$$

Case 1. Under condition (6.26), we use (6.25) and the first estimate in **(L)**, and we deduce

$$\begin{aligned} |||\widetilde{S}f|||_1 &= [\widetilde{S}f]_{\phi_1} + \delta\|\widetilde{S}f\|_0 + \beta\|\widetilde{S}f\|_1 \\ &\le \gamma_H[f]_{\phi_1} + \beta\|f\|_1 + \beta K[f]_{\phi_1} - (\beta-\delta)\|\widetilde{S}f\|_0. \end{aligned}$$

From our choice of $\beta > 0$ we have $\gamma_H + \beta K = 1$, and we conclude that (6.21) holds with $\alpha := \beta - \delta > 0$.

Case 2. Under condition (6.27), the first Lyapunov condition in **(L)** implies

$$\|\widetilde{S}f\|_1 + \|\widetilde{S}f\|_0 \le \|f\|_1 + \frac{K}{A}\|f\|_0.$$

Together with the non expansivity estimate (6.24), we get

$$[\widetilde{S}f]_{\phi_1} + \beta\|\widetilde{S}f\|_1 + \beta\|\widetilde{S}f\|_0 \le [f]_{\phi_1} + \beta\|f\|_1 + \delta\|f\|_0,$$

and we conclude to (6.21) again. □

The subgeometric convergence result is a straightforward consequence of Proposition 6.7 and an interpolation argument.

Proposition 6.8. *Assume that S satisfies the hypotheses of Theorem 6.6. Then* (6.17) *and* (6.18) *hold true with the same decay rate functions Θ and $\widetilde{\Theta}$ given by* (6.19) *(up to a modification of the constant ζ).*

Proof of Proposition 6.8. We recall that we have already proven (6.21) and (6.22). From (6.21) and the interpolation condition **(I)**, we deduce

$$|||\widetilde{S}f|||_1 + \lambda\alpha\|\widetilde{S}f\|_1 \le |||f|||_1 + \xi_\lambda\alpha\|\widetilde{S}f\|_3.$$

We observe next that from the very definition of the $|||\cdot|||_1$ norm

$$|||\widetilde{S}f|||_1 + \frac{\alpha}{\lambda}\|\widetilde{S}f\|_1 \ge Z_\lambda|||\widetilde{S}f|||_1, \quad Z_\lambda = 1+\kappa\lambda \in (1,2],$$

for some $\kappa > 0$ and any $\lambda \in (0,\lambda_0)$, $\lambda_0 > 0$, and that from the very definition of the $|||\cdot|||_3$ norm

$$\alpha\xi_\lambda\|\widetilde{S}f\|_3 \le B\xi_\lambda|||\widetilde{S}f|||_3,$$

for some $B > 0$. The three above estimates together imply

$$Z_\lambda|||\widetilde{S}f|||_1 \le |||f|||_1 + B\xi_\lambda|||\widetilde{S}f|||_3.$$

Using the second estimate (6.22) and repeating the same proof, we have

$$Z_{\lambda_{n+1}}|||\widetilde{S}^{n+1}f|||_1 \le |||\widetilde{S}^n f|||_1 + B\xi_{\lambda_{n+1}}|||f|||_3,$$

for any $n \ge 0$ and for any $\lambda_{n+1} > 0$. The discrete Grönwall lemma implies

$$|||\widetilde{S}^n f|||_1 \le A_n|||f|||_1 + \sum_{k=1}^n A_{k,n}\xi_{\lambda_k}B|||f|||_3, \quad \forall\, n \ge 0, \tag{6.28}$$

where we have defined

$$A_n := \prod_{k=1}^n a_k, \quad A_{k,n} = A_n/A_k = \prod_{i=k+1}^n a_i, \quad a_i := Z_{\lambda_i}^{-1}.$$

Observing that

$$A_{k,n} \lesssim e^{-\kappa\sum_{i=k}^n \lambda_i} \lesssim e^{\kappa(\Lambda(k)-\Lambda(n))}, \quad \text{with} \quad \Lambda(t) := \int_0^t \lambda_s\, ds,$$

and $\lambda_s := \lambda_i$ if $s \in (i-1,i]$, we immediately conclude that the first estimate (6.17) holds true. We come back to the first inequality in (6.21) that we iterate and sum up in order to obtain

$$|||\widetilde{S}^n f|||_1 + \alpha\sum_{k=[n/2]+1}^n \|\widetilde{S}^k f\|_0 \le |||\widetilde{S}^{[n/2]}f|||_1,$$

for any $n \ge 1$. Together with the non expansion inequality

$$[\widetilde{S}^n f]_{\phi_1} \le [\widetilde{S}^k f]_{\phi_1} \lesssim \|\widetilde{S}^k f\|_0, \quad \forall\, n \ge k,$$

and the first estimate (6.17), we deduce

$$\big(n-[n/2]-1\big)\alpha[\widetilde{S}^n f]_{\phi_1} \lesssim \Theta([n/2])|||f|||_3,$$

which is nothing but (6.18). □

7. Parabolic equations

In this part, we consider a general elliptic operator in divergence form

$$\mathcal{L}f := \partial_i(a_{ij}\partial_j f) + b_i\partial_i f + \partial_i(\beta_i f) + cf, \quad f \in H_0^1(\Omega), \tag{7.1}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain (i.e. an open and connected set) or $\Omega = \mathbb{R}^d$, and we always assume $d \ge 3$ (in order to simplify the discussions when using the Sobolev inequality). We also always assume at least a boundedness and ellipticity condition on the (a_{ij}) matrix, namely

$$a_{ij} \in L^\infty(\Omega), \quad \exists \nu > 0,\ \forall \xi \in \mathbb{R}^d,\ a_{ij}\xi_i\xi_j \ge \nu|\xi|^2, \tag{7.2}$$

and some conditions on the coefficients b_i, β_j and c which will be described below.

We aim to establish the existence of (λ_1, f_1, ϕ_1) solution to the first eigentriplet problem

$$\lambda_1 \in \mathbb{R}, \quad 0 < f_1 \in H_0^1,\ \mathcal{L}f_1 = \lambda_1 f_1, \quad 0 < \phi_1 \in H_0^1,\ \mathcal{L}^*\phi_1 = \lambda_1\phi_1, \tag{7.3}$$

and the existence of some (constructive) rate function Θ such that the rescaled semigroup $\widetilde{S}$ associated to the generator $\widetilde{\mathcal{L}} = \mathcal{L} - \lambda_1$ satisfies

$$\|\widetilde{S}(t)f - \langle f, \phi_1\rangle f_1\|_{H_0} \le \Theta(t)\|f - \langle f, \phi_1\rangle f_1\|_H, \tag{7.4}$$

for any $t \ge 0$ and any $f \in H$, with $H \subset H_0 \subset L^2$.

7.1. **Diffusion with rough coefficients in a bounded domain.** In this section, we consider the general elliptic operator in divergence form (7.1) in the case of a bounded and smooth enough domain $\Omega \subset \mathbb{R}^d$ with general elliptic condition on a_{ij} as formulated above. We further assume that

$$b_i, \beta_j \in L^r(\Omega), \quad c \in L^{r/2}(\Omega), \quad r > d. \tag{7.5}$$

In that situation, the first eigentriplet problem (7.3) has been considered by Chicco in [106, 107] and revisited in a slightly less general framework (all the coefficients belong to L^∞) in [252], where the conclusions **(C2)** are established. We explain with all details the existence proof by following more or less the arguments presented in [252] stressing on the constructive way for obtaining the estimates, and next we present a proof of the geometric part and the stability part by taking advantage of the abstract material developed in the previous sections. It is worth emphasizing that our proof of the uniqueness of the first eigenfunction significantly differs from the one presented in [252] which is based on a dissipativity argument, probably related to the reverse Kato's inequality condition. The framework considered here is the usual generalized solutions or weak solutions framework which goes back at least to Stampacchia [339, 340], but it is reminiscent of previous contributions by Friedrichs [165, 166], Gårding [173], De Giorgi [127], Nash [297], Morrey [289], Moser [290, 291, 292], Ladyzhenskaya, Solonnikov, Ural'ceva [244, 242], Oleinik, Kruzhkov [303] and many others. Lot of the functional arguments are picked up from the book of Gilbarg and Trudinger, and more specifically from [179, Chapter 8], and also in recent notes by Kavian [230] and Vasseur [349]. It is worth emphasizing that the present analysis does not apply directly to elliptic operators in non divergence form, although this framework is considered in [252]. We expect that all the results developed below can be generalized to a non divergence form framework, for example the one developed in [47], but we do not follow this line of research in the present work.

The proof of (7.3) and (7.4) are straightforward consequences of the abstract results developed in the previous sections once we have been able to check that the corresponding hypotheses are fulfilled. In the sequel, we will then show how these hypotheses are met in the present context.

Condition (H1). We recall that a weak (or variational) solution to the elliptic equation

$$\mathcal{L}f = g \in H^{-1}(\Omega), \quad f \in H^1_0(\Omega),$$

is a function $f \in H^1_0(\Omega)$ such that

$$D_{\mathcal{L}}(f, w) = \langle g, w\rangle, \quad \forall\, w \in H^1_0(\Omega), \tag{7.6}$$

where the (negative) Dirichlet form $D_{\mathcal{L}}$ is defined by

$$D_{\mathcal{L}}(f, w) := -\int_\Omega (a_{ij}\partial_j f + \beta_i f)\partial_i w + \int_\Omega (b_i \partial_i f w + cfw),$$

for any $f, w \in H^1_0(\Omega)$. Most of the time, we will simply write

$$\langle \mathcal{L}f, w\rangle = \langle g, w\rangle, \quad \forall\, w \in H^1_0(\Omega), \tag{7.7}$$

instead of (7.6). For the reader convenience, we repeat here some estimates picked up in [340]. For $\lambda \in \mathbb{R}$ and $f \in H^1_0(\Omega)$, we start with

$$\begin{aligned} \langle(\lambda - \mathcal{L})f, f\rangle &= \int_\Omega a_{ij}\partial_i f \partial_j f + \int_\Omega (\beta_i - b_i)\partial_i f f + \int_\Omega (\lambda - c) f^2 \\ &\ge \|f\sqrt{c_-}\|^2_{L^2} + \nu\|\nabla f\|^2_{L^2} - \||\beta - b|f\|_{L^2}\|\nabla f\|_{L^2} - \|\sqrt{c_+}f\|^2_{L^2} + \lambda\|f\|^2_{L^2} \\ &\ge \|f\sqrt{c_-}\|^2_{L^2} + \frac{\nu}{2}\|\nabla f\|^2_{L^2} - \frac{1}{2\nu}\||\beta - b|f\|^2_{L^2} - \|\sqrt{c_+}f\|^2_{L^2} + \lambda\|f\|^2_{L^2}, \end{aligned}$$

using the Cauchy-Schwarz inequality and the Young inequality, and next

$$\begin{aligned}\langle(\lambda-\mathcal{L})f,f\rangle &\geq \|f\sqrt{c_-}\|^2_{L^2}+\frac{\nu}{4}\|\nabla f\|^2_{L^2}+(\lambda-\frac{M}{2\nu}-M^{1/2})\|f\|^2_{L^2}\\ &\quad+\frac{\nu}{4}C_\Omega\|f\|^2_{L^{2^*}}-\frac{1}{2\nu}\||\beta-b|\mathbf{1}_{|\beta-b|\geq M}f\|^2_{L^2}-\|\sqrt{c_+}\mathbf{1}_{c_+\geq M}f\|^2_{L^2}\\ &\geq \|f\sqrt{c_-}\|^2_{L^2}+\frac{\nu}{4}\|\nabla f\|^2_{L^2}+(\lambda-\frac{M}{2\nu}-M^{1/2})\|f\|^2_{L^2}\\ &\quad+\big(\frac{\nu}{4}C_\Omega-\frac{1}{2\nu}\||\beta-b|\mathbf{1}_{|\beta-b|\geq M}\|^2_{L^d}-\|c_+\mathbf{1}_{c_+\geq M}\|_{L^{d/2}}\big)\|f\|^2_{L^{2^*}},\end{aligned}$$

using the Sobolev inequality (with associated constant C_Ω) and the Holder inequality. Choosing $M>0$ large enough in such a way that the last term is positive, and next $\kappa_1>0$ large enough, we deduce for instance that

$$\langle(\lambda-\mathcal{L})f,f\rangle\geq\|f\sqrt{c_-}\|^2_{L^2}+\frac{\nu}{4}\|\nabla f\|^2_{L^2}+\|f\|^2_{L^2},\quad\forall\,\lambda\geq\kappa_1. \tag{7.8}$$

Thanks to the Lax-Milgram theorem and the above coercivity estimate, we deduce that $\lambda-\mathcal{L}$ is invertible, and more precisely the mapping $(\lambda-\mathcal{L})^{-1}:H^{-1}\to H^1_0(\Omega)$ is well defined. We also claim that $\lambda-\mathcal{L}$ enjoys a weak principle maximum, and more precisely

$$f\in H^1_0(\Omega),\quad(\lambda-\mathcal{L})f\geq 0\quad\text{imply}\quad f\geq 0. \tag{7.9}$$

Indeed, for such a function $f\in H^1_0(\Omega)$, we take $w=f_-\in H^1_0(\Omega)$, as a test function, and elementary Sobolev space calculus together with the previous estimate yields

$$\begin{aligned}0 &\leq \langle(\lambda-\mathcal{L})f,f_-\rangle=-\langle(\lambda-\mathcal{L})f_-,f_-\rangle\\ &\leq -\|f_-\sqrt{c_-}\|^2_{L^2}-\frac{\nu}{4}\|\nabla f_-\|^2_{L^2}-\|f_-\|^2_{L^2}\leq 0,\end{aligned}$$

so that $f_-=0$ and $f\geq 0$. We thus deduce $(\lambda-\mathcal{L})^{-1}:L^2_+\to L^2_+$, and from J.-L. Lions theory on parabolic equation (see for instance [251, Chapter 3]), we next deduce that $\mathcal{L}$ is the generator in L^2 of a positive semigroup $S_\mathcal{L}$, so that **(H1)** holds. It is worth emphasizing at this point that the semigroup S built thanks to Lions's theory is defined by $S(t)f_0=f$ for any $f_0\in L^2$, where $f\in\mathcal{E}:=C([0,\infty);L^2)\cap L^2_{\rm loc}([0,\infty);H^1_0)\cap H^1_{\rm loc}([0,\infty);H^{-1})$ is the unique (variational) solution to the equation

$$(f(T),g(T))_{L^2}-(f_0,g(0))_{L^2}=\int_0^T\{\langle\partial_t g,f\rangle_{H^{-1},H^1_0}+D_\mathcal{L}(f,g)\}ds, \tag{7.10}$$

for any $T>0$ and $g\in\mathcal{E}$. Choosing $g=f$ in the above equation, we classically compute

$$\frac12\|f(t)\|^2_{L^2}-\frac12\|f_0\|^2_{L^2}-\int_0^t D_\mathcal{L}(f,f)ds=0,\quad\forall\,t>0,$$

which together with (7.8) implies

$$\frac1t\int_0^t\frac{\nu}{4}\|\nabla f\|^2_{L^2}ds\leq-\big(\frac{f(t)-f_0}{t},\frac{f(t)+f_0}{2}\big)_{L^2}+\frac{\kappa_1}{t}\int_0^t\|f\|^2_{L^2}ds,\quad\forall\,t>0.$$

When $f_0\in D(\mathcal{L})$, the RHS is bounded and there thus exists a sequence $t_n\to 0$ such that $\|\nabla f(t_n)\|_{L^2}$ is bounded. That implies $f_0\in H^1_0(\Omega)$ and thus $D(\mathcal{L})\subset H^1_0(\Omega)$. Similarly, we may consider the dual Dirichlet form $D^*(f,g):=D_\mathcal{L}(g,f)$ and build an associated positive semigroup S^* through Lions's theory described above. More precisely $S^*(t)g_0=g$ for any $t\geq 0$ and $g_0\in L^2$, where $g\in\mathcal{E}$ is the unique (variational) solution to the equation

$$(g(t),f(t))_{L^2}-(g_0,f(0))_{L^2}=\int_0^t\{\langle\partial_t f,g\rangle_{H^{-1},H^1_0}+D^*(g,f)\}ds,$$

for any $t>0$ and $f\in\mathcal{E}$. Now, we fix $T>0$, $g_T\in L^2$ and we set $g(t):=S^*(T-t)g_T$, so that g is a solution to the backward evolution equation

$$-\partial_t g=\mathcal{L}^*g,\quad g(T)=g_T,$$

with

$$\mathcal{L}^*g:=\partial_j(a_{ij}\partial_i g)-\partial_i(b_i g)-\beta_i\partial_i g+cg.$$

The variational formulation of this last problem is

$$(g_T, f(T))_{L^2} - (g(0), f(0))_{L^2} = \int_0^T \{\langle \partial_t f, g\rangle_{H^{-1},H^1_0} - D^*(g,f)\} ds, \tag{7.11}$$

for any $f \in \mathcal{E}$. Summing up (7.10) and (7.11) with $f(t) := S(t) f_0$ for $f_0 \in L^2$ and $g(t) := S^*(T-t) g_T$ for $g_T \in L^2$, we deduce

$$(S(T) f_0, g_T)_{L^2} = (S^*(T) g_T, f_0)_{L^2}.$$

In other words, we have established that $S^* = (S_{\mathcal{L}})^*$ and thus that $\mathcal{L}^*$ is the generator of the semigroup S^*.

Condition (H2). Let us consider a ball B_R, $R > 0$, such that $B_{4R} \subset \Omega$ and next the solution

$$f_0 \in H^1_0(\Omega), \quad (\kappa_1 - \mathcal{L}) f_0 = \mathbf{1}_{B_R}, \tag{7.12}$$

which exists from the above discussion. We next recall some classical results. On the one hand, from [339, Sec. 3 & Sec. 4] or [179, Thm. 8.15] (see also the original papers [127, 297, 290]), the following global L^∞ De Gorgi-Nash-Moser type estimate

$$\|f_+\|_{L^\infty(\Omega)} \lesssim \|f_+\|_{L^2(\Omega)} + \|g\|_{L^{r/2}(\Omega)} \tag{7.13}$$

holds for any subsolution

$$f \in H^1_0(\Omega), \quad (\lambda - \mathcal{L}) f \le g \in L^{r/2}(\Omega).$$

The local estimate variant [179, Thm. 8.18] (or *weak Harnack inequality*)

$$\|f\|_{L^p(B_{2R})} \lesssim \inf_{B_R} f + \|g\|_{L^{r/2}(\Omega)}, \quad \forall p \in [1, 2^*/2), \tag{7.14}$$

also holds for a nonnegative supersolution

$$f \in H^1(\Omega), \quad f \ge 0 \text{ on } B_{4R} \subset \Omega, \quad (\lambda - \mathcal{L}) f \ge g \in L^{r/2}(\Omega),$$

from what one deduces that a strong maximum principle [179, Thm. 8.19] holds. More precisely, under the additional one side pointwise bound

$$c + \mathrm{div}\beta \le c_0 \quad \text{or} \quad c - \mathrm{div} b \le c_0, \tag{7.15}$$

for some $c_0 \in \mathbb{R}$, we have that, for any $f \in H^1_0(\Omega)$,

$$\mathcal{L} f \le 0 \text{ in } \Omega, \ f \ge 0 \text{ in } \Omega \quad \text{imply} \quad f \equiv 0 \text{ or } f > 0 \text{ a.e. in } \Omega. \tag{7.16}$$

When indeed $f \not\equiv 0$, we may choose $B_{4R} \subset \Omega$ such that $\|f\|_{L^1(B_{2R})} > 0$ and thus $\inf_{B_R} f > 0$ from (7.14) (with $g = 0$) and because constants are supersolutions thanks to the first condition in (7.15). In the case only the second condition holds in (7.15), the same argument implies that $\mathcal{L}^*$ satisfies the strong strong maximum principle and thus also $\mathcal{L}$ thanks to Lemma 4.9. We conclude that f is positive by a connexity argument. An alternative and less demanding proof is presented in [106, Cor. 1] where (7.16) is established without the additionnal assumption (7.15).
On the other hand, the following Hölder regularity estimate [339, Théorème 7.1] and [179, Thm. 8.29] (see also the original papers [127, 297, 290]) of De Gorgi-Nash-Moser type

$$\|f\|_{C^\alpha(\Omega)} \le C \, \|(\lambda - \mathcal{L}) f\|_{L^\infty(\Omega)} \tag{7.17}$$

holds true for some $\alpha = \alpha(a_{ij}) \in (0,1)$ and $C > 0$. These last two pieces of information together and the fact that $f_0 \not\equiv 0$ imply that there exists a constant $\theta > 0$ such that $f_0 \ge \theta \mathbf{1}_{B_R}$, and thus

$$\mathcal{L} f_0 \ge (\kappa_1 - \theta^{-1}) f_0.$$

That is condition **(i)** in Lemma 2.4, so that condition **(H2)** holds thanks to Lemma 2.4. Presented in that way, the above estimate is not really constructive, but the constant $\theta := \inf_{B_R} (\kappa_1 - \mathcal{L})^{-1} \mathbf{1}_{B_R}$ can also be considered as a geometric quantity associated to geometric properties of the operator and the domain.

First constructive argument for (H2). In the case when $\mathcal{L}$ is self-adjoint, that corresponds to the case $a_{ij} = a_{ji}$ and $b_i + \beta_i = 0$, we classically know (that has been recalled in Section 2.3, see (2.35)) that

$$\lambda_1 = \inf_{f \in X_+ \backslash \{0\}} \frac{\langle \mathcal{L} f, f\rangle}{\|f\|^2} = \inf_{f \in H^1_0, \, \|f\|_{L^2} = 1} \int_{\mathcal{O}} \{a \nabla f \cdot \nabla f + c f^2\},$$

from what and the Sobolev imbedding, we get

$$\lambda_1 \geq \inf_{f\in H^1_0,\, \|f\|_{L^2}=1} \big\{ (\nu C_\Omega - \|c_- \mathbf{1}_{c_-\geq M}\|_{L^{d/2}}) \|f\|^2_{L^{2^*}} - M \big\} \geq -M,$$

by choosing M large enough. That gives an explicit lower bound on λ_1.

Second constructive argument for (H2). We give another constructive argument without assuming any self-adjointness property. We rather assume

$$(\partial_i b_i - c)_+ \in M^1(\Omega), \quad b_i + \beta_i - \partial_j a_{ij} \in M^1(\Omega). \tag{7.18}$$

We fix $h_0 \in C^2_c(\Omega)$ such that $c_0 \mathbf{1}_{B_\rho} \leq h_0 \leq c_0 \mathbf{1}_{B_{3\rho/2}}$ with $B_{8\rho} \subset \Omega$ and $\|h_0\|_{L^2} = 1$. We next define f_0 as the (positive) solution to

$$f_0 \in H^1_0(\Omega), \quad (\kappa_1 - \mathcal{L}) f_0 = h_0, \tag{7.19}$$

so that $f_0 \in C^\alpha(\Omega)$ from (7.13) and (7.17), and similarly

$$\widetilde{f}_0 \in H^1_0(B_{2\rho}), \quad (\kappa_1 - \mathcal{L}) \widetilde{f}_0 = h_0, \tag{7.20}$$

so that $\widetilde{f}_0 \in C^\alpha(B_{2\rho})$ from (7.13) and (7.17). We observe that $0 \leq \widetilde{f}_0 \leq f_0$ thanks to the weak maximum principle. We then compute

$$1 = \|h_0\|^2_{L^2} = \int_{B_{2\rho}} h_0 (\kappa_1 - \mathcal{L}) \widetilde{f}_0 = \int_{B_{2\rho}} \widetilde{f}_0 (\kappa_1 - \mathcal{L}^*) h_0 \leq \|\widetilde{f}_0\|_{L^\infty} \|(\kappa_1 - \mathcal{L}^*) h_0\|_{M^1},$$

where the last term is finite because of the additional hypothesis (7.18). We conclude to a first constructive lower bound $\|\widetilde{f}_0\|_{L^\infty(B_{2\rho})} \geq c_1 > 0$. Because of the Holder continuity, we also have $\|\widetilde{f}_0\|_{L^1(B_{2\rho})} \geq c_2$ with constructive constant $c_2 = c_2(c_1, \alpha, d) > 0$. Thanks to (7.14) (with $g = 0$), we obtain

$$\begin{aligned} f_0 &\geq \mathbf{1}_{B_{3\rho/2}} \inf_{B_{3\rho/2}} f_0 \geq \mathbf{1}_{B_{3\rho/2}} C_{wH} \|f_0\|_{L^1(B_{3\rho/2})} \\ &\geq \mathbf{1}_{B_{3\rho/2}} C_{wH} \|\widetilde{f}_0\|_{L^1(B_{3\rho/2})} \geq C_{wH} c_2 c_0^{-1} h_0. \end{aligned}$$

Because all the inequalities are constructive and proceeding as above, we deduce that condition **(ii)** in Lemma 2.4 holds and thus also **(H2)** with constructive constant $\kappa_0 := \kappa_1 - C^{-1}_{wH} c_2^{-1} c_0$. Finally, because of $(\kappa_1 - \mathcal{L}) f_0 = 0$ on $\Omega \backslash B_{3\rho/2}$, we may apply the Harnack inequality [179, Cor. 8.21], and we classically deduce there exist constructive constants $C > 0$ and $C_\varrho > 0$ for any $\varrho > 0$ such that

$$C_\varrho \mathbf{1}_{\omega_\varrho} \leq f_0 \leq C, \tag{7.21}$$

with $\omega_\varrho := \{x \in \Omega;\, \delta(x) > \varrho\}$ and $\delta(x) := d(x, \partial\Omega)$ is the distance to the boundary function.

We can also get a constructive argument for **(H2)** by asking that condition **(i)** in Lemma 2.4 holds. We may for instance verify that the dual counterpart of the above constructive argument holds when $(c + \partial_i \beta_i)_- \in M^1$ and $b_i + \beta_i + \partial_j a_{ji} \in M^1$. More precisely, we establish in a similar way as above that the solution to the problem

$$\phi_0 \in H^1_0(\Omega), \quad (\kappa_1 - \mathcal{L}^*) \phi_0 = h_0, \tag{7.22}$$

satisfies

$$\kappa_0 \phi_0 \leq \mathcal{L}^* \phi_0 \leq \kappa_1 \phi_0, \tag{7.23}$$

for some constructive constants $\kappa_0 \leq \kappa_1$. Similarly as above again, there exist constructive constants $C > 0$ and $C_\varrho > 0$ for any $\varrho > 0$ such that

$$C_\varrho \mathbf{1}_{\omega_\varrho} \leq \phi_0 \leq C. \tag{7.24}$$

Third constructive argument for (H2). We write

$$\mathcal{L} f = a_{ij} \partial^2_{ij} f + \tilde{b}_i \partial_i f + \tilde{c} f, \tag{7.25}$$

with $\tilde{b}_i := b_i + \partial_j a_{ji} + \beta_i$ and $\tilde{c} := c + \partial_i \beta_i$. We further assume $\tilde{b}_i, \tilde{c} \in L^\infty$. In that case, we may also obtain an explicit lower bound on λ_1 by proceeding in the following way. We define $f_0(x) := \chi(|x|)$ with $\chi \in C^1_c(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$, $\mathbf{1}_{[0,1/3]} \leq \chi \leq \mathbf{1}_{[0,1]}$, $\chi' \leq 0$ on $[0,1]$, $\chi(s) := n^2(1-s)^2/2$ on $[\iota_n, 1]$,

$\iota_n := 1 - 1/(2n)$, for some $n \ge 1$ to be chosen. As a consequence, $\chi'' = n^2$ on $[\iota_n, 1]$, $|\chi'| \le n$ on $[\iota_n, 1]$ and $\chi \ge 1/2$ on $[0, \iota_n]$. Denoting $s := |x|$, we compute

$$\mathcal{L} f_0 = a_{ij} \big\{ \chi''(s) \hat x_i \hat x_j + \chi'(s) \frac{\delta_{ij} - \hat x_i \hat x_j}{s} \big\} + \tilde b(x) \cdot \hat x \chi'(s) + \tilde c(x) \chi(s).$$

For n large enough, we get

$$\begin{aligned} \mathcal{L} f_0 &\ge n^2 \nu - n2A - nB - C \ge 0 \ \text{ on } \ B_1 \backslash B_{\iota_n}, \\ \mathcal{L} f_0 &\ge -A \big\{ \|\chi''\|_{L^\infty} + \|\chi'(s)/s\|_{L^\infty} \big\} - B \|\chi'\|_{L^\infty} - C \ge \kappa_0 \chi \ \text{ on } \ B_{\iota_n}, \end{aligned}$$

with $A := \|a\|_{L^\infty(B_1)}$, $B := \|\tilde b\|_{L^\infty(B_1)}$, $C := \|\tilde c\|_{L^\infty(B_1)}$ and $\kappa_0 \in \mathbb{R}_-$. As a conclusion, we have again established condition **(ii)** in Lemma 2.4, so that condition **(H2)** holds.

Fourth constructive argument for (H2). We present a last situation when we are able to prove a quantitative version of condition **(H2)**. We assume that $a \in C^0(\bar\Omega)$, $\mathrm{div}\beta \in L^{r/2}$, as well as $\tilde b_i \in L^r$ and $\tilde c \in L^{r/2}$ in the definition of (7.25). We define h_0 and f_0 as in the second constructive argument for **(H2)**, so that (7.18) holds. Choosing $p \in (1, 2)$ defined by $1/p := 1/r + 1/2 > 2/r + 1/2^*$, we observe that

$$\begin{aligned} \|\kappa_1 f_0 - \tilde b_i \partial_i f_0 - \tilde c f_0 - h_0\|_{L^p} &\lesssim \kappa_1 \|f_0\|_{L^2} + \|\tilde b_i\|_{L^r} \|\partial_i f_0\|_{L^2} + \|\tilde c\|_{L^{r/2}} \|f_0\|_{L^{2^*}} + \|h_0\|_{L^2} \\ &\lesssim \|h_0\|_{L^2}, \end{aligned}$$

from equation (7.19) and the coercivity estimate (7.8). From the Calderon-Zygmond regularity theory [85] or [179, Thm. 9.14], we also know that

$$\|f_0\|_{W^{2,p}(\Omega)} \lesssim \|a_{ij} \partial^2_{ij} f_0\|_{L^p(\Omega)}. \tag{7.26}$$

Writing $a_{ij} \partial^2_{ij} f_0 = \kappa_1 f_0 - \tilde b_i \partial_i f_0 - \tilde c f_0 - h_0$ and using the two above estimates, we deduce

$$\|f_0\|_{W^{2,1}(\Omega)} \lesssim \|h_0\|_{L^2(\Omega)}. \tag{7.27}$$

On the other hand, from (7.7) and the Poincaré inequality, we have

$$1 = \|h_0\|^2_{L^2} = \langle (\kappa_1 - \mathcal{L}) f_0, h_0 \rangle \lesssim \|\nabla f_0\|_{L^2} \|\nabla h_0\|_{L^2}.$$

Together with the estimate (7.27) and the Gagliardo-Niremberg inequality

$$\|\nabla f\|_{L^2} \lesssim \|D^2 f\|^{1/2}_{L^1} \|f\|^{1/2}_{L^\infty},$$

we obtain a lower bound $\|f_0\|_{L^\infty} \ge C_0 > 0$. We then conclude as in the second constructive argument for **(H2)**.

Condition (H3). Because of Rellich-Kondrachov theorem on the compact embedding $H^1_0 \subset L^2$, the mapping $(\lambda - \mathcal{L})^{-1} : L^2 \to L^2$ is compact for any $\lambda \ge \kappa_1$. As a consequence, introducing the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} := \kappa_1 - \kappa_\mathcal{B}$, $\kappa_\mathcal{B} \in \mathbb{R}$ arbitrary, the operator $\mathcal{R}_\mathcal{B}(\lambda) = (\lambda + \kappa_1 - \kappa_\mathcal{B} - \mathcal{L})^{-1}$ is bounded uniformly on $\lambda \ge \kappa_\mathcal{B}$ and it is compact for any $\lambda \ge \kappa_\mathcal{B}$. We deduce from Lemma 2.8-(2) that **(H3)** holds for both the primal and the dual problems.

We may thus apply Theorem 2.21 and deduce the existence of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem

$$\lambda_1 \in \mathbb{R}, \quad 0 \le f_1 \in H^1_0, \ \mathcal{L} f_1 = \lambda_1 f_1, \quad 0 \le \phi_1 \in H^1_0, \ \mathcal{L}^* \phi_1 = \lambda_1 \phi_1, \tag{7.28}$$

where both equations must be understood in the variational sense as a consequence of the discussion at the end of the proof of condition **(H1)**.

Condition (H4). The strong maximum principle holds as already mentioned in the paragraph dedicated to condition **(H2)**. As a consequence and thanks to Theorem 4.13, we know that the first eigentriplet problem (7.3) has a unique solution (λ_1, f_1, ϕ_1) which satisfies $f_1 > 0$, $\phi_1 > 0$, $N(\mathcal{L} - \lambda_1)^k = \mathrm{Span}(f_1)$ and $N(\mathcal{L}^* - \lambda_1)^k = \mathrm{Span}(\phi_1)$ for any $k \ge 1$.

Condition (H5). Consider $f \in D(\mathcal{L}^\infty)$ such that $0 < |f| \in D(\mathcal{L}^\infty)$ and

$$\mathcal{L}|f| = \Re e(\mathrm{sign} f) \mathcal{L} f,$$

so that multiplying both term of the equation by $|f|$ and integrating, we have

$$\Re e \langle \mathcal{L} f, \bar f \rangle = \langle \mathcal{L}|f|, |f| \rangle.$$

We next compute

$$\Re e\langle \mathcal{L}f, \bar f\rangle = -\int_\Omega a_{kj}\Re e(\partial_j f \partial_k \bar f) + \int_\Omega (b_k - \beta_k)\Re e(\bar f \partial_k f) + \int_\Omega c|f|^2,$$

and

$$\langle \mathcal{L}|f|, |f|\rangle = -\int_\Omega a_{kj}\partial_j |f| \partial_k |f| + \int_\Omega (b_k - \beta_k)\Re e(\bar f \partial_k f) + \int_\Omega c|f|^2,$$

where in the last equality, we have used that $\partial_k |f| = \frac{1}{|f|}\Re e(\bar f \partial_k f)$. From the three above equations, we deduce

$$\int_\Omega a_{kj}[\partial_j |f| \partial_k |f| - \Re e(\partial_j f \partial_k \bar f)] = 0.$$

Introducing the real and complex part decomposition $f = u + iv$, and similarly as in [231, Proof of Theorem 5.1], we next compute

$$\begin{aligned}
&\partial_j |f| \partial_k |f| - \Re e(\partial_j f \partial_k \bar f) \\
&= \frac{1}{|f|^2}\big[uv(\partial_k u \partial_j v + \partial_k v \partial_j u) - u^2 \partial_j v \partial_k v - v^2 \partial_j u \partial_k u\big] \\
&= \frac{1}{|f|^2}(u\partial_j v - v\partial_j u)(u\partial_k v - v\partial_k u),
\end{aligned}$$

so that from the ellipticity condition on a, we have $u\partial_k v - v\partial_k u = 0$ a.e. on Ω. On the other hand, from De Girogi-Nash-Moser regularity estimates (7.13) and (7.17), f has Hölder regularity. In particular both functions u and v are continuous. Because $|f| \not\equiv 0$, one of the two function is not identically vanishing, say for instance $v \not\equiv 0$. There exists some points $x_0 \in \Omega$ such that $v(x_0) \neq 0$, say for instance $v(x_0) > 0$. Denoting by ω the connected component of the set $\{x \in \Omega;\ v(x) > 0\}$ containing x_0, we have $\nabla(u/v) = 0$ on ω. Hence $u = \alpha\, v$ on ω for some $\alpha \in \mathbb{R}$, which implies that there exists $\sigma \in \mathbb{S}^1$ such that $f = \sigma |f|$ on ω. If $\omega \neq \Omega$, we would have $|f| = 0$ on $\partial\omega \cap \Omega \neq \emptyset$, which would be a contradiction with the fact that $|f| > 0$. We conclude that $\omega = \Omega$ and thus that $f = \sigma|f|$, which is nothing but the reverse Kato's inequality condition **(H5)**.

At this stage, we may use Theorem 5.16, in order to get the conclusion **(C3)** on the triviality of the boundary punctual spectrum.

In order to go one step further and establish the asymptotic stability of f_1, we may use the two following approaches which are consequences respectively of Lemma 7.1 and Lemma 7.2.

Lemma 7.1. *For any $R > 0$, the set*

$$\mathcal{K} := \{f \in D(\mathcal{L});\ [f] \le R,\ [\mathcal{L}f] \le R\}$$

is strongly compact in $L^1_{\rm loc}(\Omega)$, where $[g] := \|g\|_{L^1_{\phi_1}}$.

Proof of Lemma 7.1. Consider $f \in \mathcal{K}$ so that $f \in H^1_0(\Omega)$ and

$$\partial_i(a_{ij}\partial_j f) + b_i \partial_i f + \partial_i(\beta_i f) + cf = g \in L^2(\Omega).$$

From the renormalization theory of elliptic equations and the GRE trick (see for instance [269] and the references therein) for any renormalizing function $H \in C^2(\mathbb{R})$, there holds

$$\begin{aligned}
H''(u) f_1 \phi_1 a \nabla u \cdot \nabla u \quad &= \quad \mathrm{div}(a\phi_1 \nabla(H(u) f_1)) - \mathrm{div}(f_1 H(u) a \nabla \phi_1) \\
&\qquad + \mathrm{div}((b + \beta) H(u) f_1 \phi_1) + g H'(u) f_1 \phi_1,
\end{aligned}$$

with $u := f/f_1$. Considering $H \in W^{2,\infty}$ the even (and convex) function such that $H(0) = 0$ and $H'' := \mathbf{1}_{[n,n+1]}$, so that in particular $|H'(s)| \le 1$, and integrating the previous equation, we deduce

$$\nu \int_{|u| \in [n,n+1]} |\nabla u|^2 f_1 \phi_1 \le \int |g| f_1 \phi_1 \le \|f_1\|_{L^\infty} R.$$

We proceed along the line of the proof of [62, Thm. 1]. For a fixed $\omega \subset\subset \Omega$, we define $B_n := \{x \in \omega;\ |u(x)| \in [n, n+1]\}$. Using that $f_1 > 0$ and $\phi_1 > 0$, there exists a constructive constant $C_{\omega,R} > 0$ such that

$$\int_{B_n} |\nabla u|^2 \le C_\omega^2, \quad \forall\, n \ge 0.$$

From the Cauchy-Schwarz inequality, we have

$$\int_{B_n} |\nabla u| \le C_\omega \mathrm{meas}(B_n)^{1/2}, \quad \forall\, n \ge 0. \tag{7.29}$$

On the other hand, denoting by $1^* := d/(d-1)$ the Sobolev exponent, we have

$$\int_{B_n} |\nabla u| \le C_{\omega,R} \Big(n^{-1^*} \int_{B_n} |u|^{1^*} \Big)^{1/2}.$$

Summing up and using the Cauchy-Schwarz inequality again, we have

$$\begin{aligned} \sum_{n\ge1} \int_{B_n} |\nabla u| &\le C_{\omega,R} \Big(\sum_{n\ge1} n^{-1^*} \Big)^{1/2} \Big(\sum_{n\ge1} \int_{B_n} |u|^{1^*} \Big)^{1/2} \\ &\le C_{\omega,R} \Big(\sum_{n\ge1} n^{-1^*} \Big)^{1/2} \|u\|_{L^{1^*}}^{1^*/2}. \end{aligned}$$

Together with (7.29) for $n = 0$, we deduce

$$\|\nabla u\|_{L^1(\omega)} \le C'_{\omega,R} (1 + \|\nabla u\|_{L^1(\omega)}^{1^*/2}).$$

Because $1^*/2 \le 3/4 < 1$ (recall that $d \ge 3$), we can kill the last term, and we obtain the estimate

$$\|\nabla (f/f_1)\|_{L^1(\omega)} \le C'', \quad \forall\, f \in \mathcal{K},$$

for some constant $C'' := C''_{\omega,R} > 0$. We classically conclude thanks to the Rellich-Kondrachov theorem. ☐

From the above lemma and Theorem 5.23, we deduce that $\widetilde{S}(t) f \to \langle f, \phi_1 \rangle f_1$ in the $L^1_{\phi_1}$ norm sense as $t \to \infty$ for any $f \in L^2(\Omega)$. The alternative approach is based on the following result.

Lemma 7.2. *Setting $\kappa := \kappa_0 - 1$, there exist $A, \alpha, R > 0$ such that*

(i) $\sup_{z\in\Delta_\kappa} \langle y \rangle^\alpha \|\mathcal{R}_\mathcal{B}(z)\|_{\mathscr{B}(L^2;H^1_0)} + \sup_{z\in\Delta_\kappa\setminus B_R} \|\mathcal{R}_\mathcal{L}(z)\|_{\mathscr{B}(L^2;H^1_0)} < \infty$,

(ii) $\Sigma(\mathcal{L}) \cap \Delta_\kappa \subset \Sigma_d(\mathcal{L}) \cap B_R$,

where $\mathcal{B} := \mathcal{L} - A$ and $z = x + iy$, $x, y \in \mathbb{R}$.

Proof of Lemma 7.2. Let us consider an a priori solution to the stationary problem

$$f \in H^1_0, \quad z = x + iy \in \Delta_\kappa, \quad (\mathcal{L} + z) f = g \in L^2.$$

This one satisfies

$$\Big| - \int (a\nabla f + \beta f) \cdot \nabla \bar f + \int b \cdot \nabla f \bar f + (c+z)|f|^2 \Big| = \Big| \int g \bar f \Big|.$$

Using the elliptic condition, the Cauchy-Schwarz inequality and triangular inequalities, we get

$$\begin{aligned} \Big| \int g\bar f \Big| &\ge \Big| \int a\nabla f \nabla \bar f + ((c+x)_+ + iy)|f|^2 \Big| - \Big| \int b\cdot\nabla f \bar f - \beta f \cdot \nabla \bar f + (c+x)_- |f|^2 \Big| \\ &\ge \frac{\nu}{2} \|\nabla f\|^2_{L^2} + \big(\frac{|y|}{2} - x_- \big) \|f\|^2_{L^2} - \|(|b| + |\beta|) f\|_{L^2} \|\nabla f\|_{L^2} - \|\sqrt{c_-} f\|^2_{L^2}. \end{aligned}$$

Using next similar arguments and those introduced in the paragraph dedicated to condition **(H1)** and with similar definition for the constant $M := M(b, \beta, c) > 0$, we deduce

$$\Big| \int g \bar f \Big| \ge \big(\frac{|y|}{2} - x_- - M \big) \|f\|^2_{L^2} + \frac{\nu}{4} \|\nabla f\|^2_{L^2}.$$

Defining the sectorial set

$$\mathcal{S} := \big\{ z = x + iy \in \mathbb{C};\ |y| > 2x_- + M \big\},$$

we have established the a priori estimates

$$\begin{aligned} \|f\|_{L^2} &\le \big(\frac{|y|}{2} - x_- - M \big)^{-1/2} \|g\|_{L^2}, \\ \|\nabla f\|_{L^2} &\le 2\nu^{-1/2} \big(\frac{|y|}{2} - x_- - M \big)^{-1/4} \|g\|_{L^2}, \end{aligned}$$

for any $z \in \mathcal{S}$. We classically and immediately deduce that $\rho(\mathcal{L}) \supset \mathcal{S}$ and the resolvent estimate $\|\mathcal{R}_\mathcal{L}(z)\|_{\mathscr{B}(L^2,H^1_0)} \lesssim \big(\frac{|y|}{2} - x_- - M\big)^{-1/2} + \big(\frac{|y|}{2} - x_- - M\big)^{-1/4}$ for any $z \in \mathcal{S}$, and in particular the estimate **(i)** holds true.

On the other hand, because $\mathcal{L}$ has compact resolvent as established just above or during the proof of **(H3)** and using the Fredholm alternative, we have $\Sigma(\mathcal{L}) = \Sigma_d(\mathcal{L})$ and $\Sigma(\mathcal{L}) \cap \Delta_\kappa$ is finite for any $\kappa \in \mathbb{R}$, what is nothing but the property **(ii)**. ☐

From the above lemma and Theorem 5.30 or Theorem 5.32, we deduce that $\widetilde{S}(t)f \to \langle f, \phi_1\rangle f_1$ in the L^2 norm sense as $t \to \infty$ for any $f \in L^2(\Omega)$ with exponential rate.

We may summarize our analysis in the following result.

Theorem 7.3. *Consider the elliptic operator* (7.1) *in a bounded domain and assume that the coefficients satisfy* (7.2), (7.5) *and* (7.18)*. Then the conclusions* **(C3)** *holds as well as* **(E2)** *in* $L^1_{\phi_1}$ *norm and* **(E3$_1$)** *in* L^2 *with non constructive rate.*

It is however worth emphasizing again that the above approach is definitively not constructive. We propose now an alternative approach which is constructive.

Quantitative estimate of stability.
Using the Doblin-Harris type approach presented in Section 6, we are able to establish a rate of convergence to the principal dynamic, at least in a regular framework. We thus make some regularity assumptions on Ω and additional regularity assumptions on the coefficients.
- For the domain, we assume that there exists a constant $r_\Omega > 0$ such that for any $x \in \Omega$ there is $y \in \Omega$ such that $x \in B(y, r_\Omega) \subset \Omega$, in particular, for any $x \in \partial\Omega$ there is $y \in \Omega$ such that $x \in \partial B(y, r_\Omega)$, $B(y, r_\Omega) \subset \Omega$. We also assume that Ω is $C^{1,1}$.
- For the coefficients, we assume $a_{ij} \in C(\bar\Omega)$, $\tilde b_i, \tilde c \in L^\infty(\Omega)$, where $\tilde b_i$ and $\tilde c$ are defined in (7.25).

Theorem 7.4. *Consider the elliptic operator* (7.1) *in a bounded domain and assume that the assumptions of Theorem 7.3 hold together with the above additional regularity assumptions on the coefficients and the boundary. Then the conclusion* **(E3$_1$)** *holds with constructive exponential rate.*

The proof of Theorem 7.4 follows from Theorem 6.3. We split the proof into several steps.

- Step 1. Regularity estimates. Thanks to De Giorgi-Nash-Moser regularity technique for parabolic equations developed for instance in [243] (in Russian), [346, Thm. 1.3, Thm. 2.2] as well as more recently in [230, Lem. 2.7] and [191, Thm. 1.1], there exists $\alpha = \alpha(a_{ij}) \in (0,1)$ and for any $T_1 > T_0 > 0$ and any $\varrho \in (0,1)$, there exist constructive constants $C_i = C_i(\|f\|_{L^\infty_t L^2_x}, T, \tau, r)$ such that any solution $f \in L^\infty(0,\infty; L^2(\Omega))$ to the parabolic equation $\partial_t f = \mathcal{L} f$ satisfies

$$\|f\|_{L^\infty([T_0,T_1]\times\Omega)} \le C_1, \quad \|f\|_{C^\alpha([T_0,T_1]\times\omega_\varrho)} \le C_2, \tag{7.30}$$

with $\omega_r := \{x \in \Omega;\ d(x,\partial\Omega) > r\}$. More precisely, in order to establish the second estimate in (7.30) with constructive constant, one may observe that the proof of [191, Prop. 2.4] may be repeated in order to get that solutions to the parabolic equation considered in the present framework fall into De Giorgi classes as defined in [191, Definition 2.3], and thus [191, Thm. 1.1] applies.

On the other hand, in this context and because of the regularity assumptions, we may establish a more accurate regularity estimate. More precisely, by gathering the Sobolev inequality and the Calderon-Zygmond estimate (7.26), we obtain the classical constructive regularity estimate

$$\|u\|_{C^{0,1}(\Omega)} \lesssim \|u\|_{W^{2,d+1}(\Omega)} \lesssim \|(\kappa_1 - \mathcal{L})u\|_{L^{d+1}(\Omega)}, \tag{7.31}$$

see for instance Theorem 7.10, Theorem 7.25 and Lemma 9.17 in [179]. Iterating the same kind of arguments, we get

$$\|u\|_{C^{0,1}(\Omega)} \le C\|(\kappa_1 - \mathcal{L})^k u\|_{L^2(\Omega)}, \tag{7.32}$$

with constructive constants C and k.

- Step 2. Harnack estimate. We claim that for any $T > t_0 > 0$ and $\varrho > 0$, there exist a constant $C_H > 0$ such that, for any $f_0 \in L^2$, the associated solution $f := S_\mathcal{L} f_0$ satisfies

$$\sup_{\omega_\varrho} f_{t_0} \le C_H \inf_{\omega_\varrho} f_T. \tag{7.33}$$

The proof mainly follows form Aronson-Serrin [25] (see also [292, 222, 224, 240, 223, 345, 346, 241] for similar results). First, we know from [25, Thm. 3] that

$$\max_{Q^*(\rho)} f \le C \min_{Q(\rho)} f, \tag{7.34}$$

for any $\rho > 0$, $t > 0$ such that $Q^*(3\rho) \subset (0,\infty) \times \Omega$, where $Q(\rho) := [t-\rho^2, t] \times \mathcal{C}(\rho)$, $Q^*(\rho) := [t - 8\rho^2, t - 7\rho^2] \times \mathcal{C}(\rho)$ and $\mathcal{C}(\rho)$ is a cube with length ρ. To avoid technical issues we assume that w_ϱ is convex. In other case, the geometrical condition given above implies that there is $N \in \mathbb{Z}_+$ such that any two points $x, y \in \Omega$ can be connected by a polygonal path of at most N segments, and we can argue as follows for any segment. We define $D := \sup_{a,b\in\Omega} d(a, b)$ the diameter of Ω and we choose $r' < \varrho/7$ such that

$$7(\lfloor \frac{D}{2r'} \rfloor + 1)(r')^2 < T - t_0.$$

For any $x, y \in \omega_\varrho$, we also define $N_c = \lfloor \frac{|x-y|}{r'} \rfloor$. Since ω_ϱ is convex, $r' < \varrho/7$, we have that the family of cubes $\{\mathcal{C}(x_i, 2r')\}_{i=0,N_c}$ of center x_i and length $2r'$ for $x_i = x + \frac{(x-y)i}{N_c}$ satisfy that $C(x_i, 6r') \subset \Omega$ and $\mathcal{C}(x_i, 2r') \cap C(x_{i+1}, 2r') \neq \emptyset$ for any $i = 0, \dots, N_c$. As a consequence, we can apply Aronson-Serrin estimate (7.34) for each cube to obtain

$$\max_{C(x_i,2r')} f_{t_i} \le C_{2r'} \min_{C(x_i,2r')} f_{t_{i+1}},$$

with $t_i = t_0 + 7i(2r')^2$. Taking $y_i \in C(x_i, 2r') \cap C(x_{i+1}, 2r')$, we deduce

$$\max_{C(x_i,2r')} f_{t_i} \le C_{2r'} \min_{C(x_i,2r')} f_{t_{i+1}} \le C_{2r'} f_{t_{i+1}}(y_i) \le C_{2r'} \max_{C(x_{i+1},2r')} f_{t_{i+1}} \le C^2_{2r'} \min_{C(x_{i+1},2r')} f_{t_{i+2}}.$$

By induction, we obtain

$$f_{t_0}(x) \le \max_{C(x_1,2r')} f_{t_1} \le C^{N_c}_{2r'} \min_{C(x_{N_c},2r')} f_{t_{N_c}} \le C^{N_c}_{2r'} f_{t_{N_c}}(y),$$

with $t_{N_c} = t_0 + 7N_c r'^2 \le T$. Note that in any case the constant $C_{2r'}$ is the same since it only depends on the length $2r'$ and the coefficient of the equation. We have thus established (7.33) with $C_H := C_{2r'}^{\lfloor \frac{D}{2r'} \rfloor + 1}$.

On the other hand, we state an improved version of the already mentioned stationary Harnack inequality. Because of the interior ball condition the Hopf Lemma (see for instance the proof of [179, Lem. 3.4]) claims that for any $\varrho \in (0, r_\Omega/2]$ there exists a constructive constant $\alpha > 0$ such that if $u \in W^{2,p}(\Omega)$, $p > d$, is such that

$$u \ge \mathbf{1}_{\omega_\varrho}, \quad (\kappa_1 - \mathcal{L}^*)u \ge 0,$$

then u satisfies

$$u \ge \chi(x) := e^{-\alpha(2\varrho - \delta(x))^2} - e^{-\alpha(2\varrho)^2} \text{ on } \omega^c_\varrho. \tag{7.35}$$

Let us give two applications of the above sharp regularity and positivity estimates. First, recalling (7.24) and using (7.31) and (7.35), we deduce that there exist two constructive constants $c_i \in (0,\infty)$ such that

$$c_0 \delta \le \phi_0 \le c_1 \delta \text{ on } \Omega. \tag{7.36}$$

Consider now $f_1 \in H^1_0(\Omega)$ the positive first eigenfunction with normalization $\|f_1\|_{L^2} = 1$. Using the estimate of regularity (7.32) on the iterated equation $(\kappa_1 - \mathcal{L})^k f_1 = (\kappa_1 - \lambda_1)^k f_1$, we have

$$\|f_1\|_{L^\infty(\Omega)} \le \|f_1\|_{C^{0,1}(\Omega)} \le C_1,$$

for some constructive constant $C_1 \in (0,\infty)$. Next using the elementary inequality

$$1 = \int_\Omega f_1^2 \le \|f_1\|_{L^\infty} \|f_1\|_{L^1} \le C_1 \|f_1\|_{L^1},$$

we deduce

$$\begin{aligned} |\Omega| \sup_{\omega_\varrho} f_1 &\ge \int_{\omega_\varrho} f_1 = \|f_1\|_{L^1} - \int_{\omega^c_\varrho} f_1 \\ &\ge 1/C_1 - C_1 |\omega^c_\varrho| \ge 1/(2C_1), \end{aligned}$$

by choosing $\varrho \in (0, r_\Omega/2)$ small enough. Then, from the Harnack inequality [179, Cor. 8.21], we deduce

$$\inf_{\omega_\varrho} f_1 \ge C_H \sup_{\omega_\varrho} f_1 \ge C_H (2C_1|\Omega|)^{-1}.$$

Finally, from the above Hopf lemma and the above Lipschitz continuity, we have established

$$c_0 \delta \le f_1 \le c_1 \delta \quad \text{on} \quad \Omega, \tag{7.37}$$

for two constructive constants $c_i \in (0,\infty)$. The same arguments on the normalized and positive first dual eigenfunction ϕ_1 lead to the same estimate

$$c_0 \delta \le \phi_1 \le c_1 \delta \quad \text{on} \quad \Omega. \tag{7.38}$$

In particular, for any such $\varrho \in (0, r_\Omega/2)$, we have

$$\langle \phi_1, \mathbf{1}_{\omega_\varrho} \rangle \ge r_\varrho, \tag{7.39}$$

with constructive constant r_ϱ, what is nothing but condition (6.9) in the Harris theorem that we will use below.

- Step 3. Splitting of $\mathcal{L}$. We introduce the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, with $\mathcal{A} f = \mathscr{M} \mathbf{1}_{\omega_\varrho} f$, $\mathscr{M} \ge 0$ large enough and $\varrho > 0$ small enough that we fix just below. Using (7.8), we observe that

$$\begin{aligned} (\mathcal{B} f, f)_{L^2} &= (\mathcal{L} f, f)_{L^2} - \mathscr{M} \|f\|_{L^2} + \mathscr{M} \int_{\omega_\varrho^c} f^2 \\ &\le -\frac{\nu}{4} \|\nabla f\|_{L^2}^2 + (\kappa_1 - \mathscr{M}) \|f\|_{L^2}^2 + \mathscr{M} |\omega_\varrho^c|^{4/d} \|f\|_{L^{2^*}}^2 \le \kappa_0 \|f\|_{L^2}^2, \end{aligned}$$

by choosing first $\mathscr{M} \ge \kappa_1 - \kappa_0$ and next $\varrho > 0$ small enough in order to be able to throw away the last term using the negative first term and the Sobolev inequality. We deduce

$$S_\mathcal{B}(t) : L^2 \to L^2 \quad \text{with bound} \quad \mathcal{O}(e^{\kappa_0 t}). \tag{7.40}$$

On the other hand, denoting $f_t := S_\mathcal{L}(t) f$ for $f \in L^2(\Omega)$ and recalling that ϕ_0 defined by (7.22) satisfies (7.23), we have

$$\frac{d}{dt} \int |f_t| \phi_0 \le \int \mathcal{L} |f_t| \phi_0 \le \int |f_t| \mathcal{L}^* \phi_0 \le \kappa_1 \int |f_t| \phi_0,$$

so that

$$\int |f_t| \phi_0 \le e^{\kappa_1 t} \int |f_0| \phi_0. \tag{7.41}$$

Arguing in the same way for $S_\mathcal{B}$ and using (7.36), we have established

$$S_\mathcal{L}(t), S_\mathcal{B}(t) : L^1_\delta \to L^1_\delta \quad \text{with bound} \quad \mathcal{O}(e^{\kappa_1 t}). \tag{7.42}$$

For a solution to the evolution equation $\partial_t f = \mathcal{C} f$, $\mathcal{C} = \mathcal{L}$ or $\mathcal{C} = \mathcal{B}$, we also classically compute

$$\begin{aligned} \frac{d}{dt} \int f^2 \phi_0 &= 2 \int (\mathcal{C} f) f \phi_0 \\ &= -2 \int (\nabla f \cdot a \nabla f) \phi_0 + \int f^2 \mathcal{C}^* \phi_0. \end{aligned}$$

Thanks to (7.23) again, we have

$$\frac{d}{dt} \int f^2 \phi_0 \le -2\nu \int |\nabla f|^2 \phi_0 + \kappa_1 \int f^2 \phi_0, \tag{7.43}$$

from what we deduce

$$S_\mathcal{L}(t), S_\mathcal{B}(t) : L^2(\delta) \to L^2(\delta) \quad \text{with bound} \quad \mathcal{O}(e^{\kappa_1 t/2}).$$

In the sequel, we will need the following version of Nash inequality.

Lemma 7.5 (weighted Nash inequality)**.** *There exists a constructive constant C_N such that*

$$\|f\|_{L^2(\delta)} \le C_N \|\nabla f\|_{L^2(\delta)}^{\frac{d+1}{d+2}} \|f\|_{L^1_\delta}^{\frac{1}{d+2}}, \quad \forall f \in H^1(\delta). \tag{7.44}$$

Proof of Lemma 7.5. For $\varepsilon > 0$, we define

$$f_\varepsilon(x) := \frac{1}{\delta_\varepsilon(x)} \int_{B(x,\varepsilon)} f(y)\, \delta(y) dy, \quad \delta_\varepsilon(x) = \delta(B(x,\varepsilon)) := \int_{B(x,\varepsilon)} \delta(y) dy,$$

and $B(x,\varepsilon) := \{y \in \Omega;\ |x-y| < \varepsilon\}$. It is worth emphasizing that

$$\varepsilon^{d+1} \lesssim \delta_\varepsilon(x) \lesssim \varepsilon^d, \quad \forall\, \varepsilon > 0. \tag{7.45}$$

For $f \in H^1(\delta)$, we compute

$$\begin{aligned}
\|f - f_\varepsilon\|^2_{L^2(\delta)} &= \int_\Omega \Big| \frac{1}{\delta_\varepsilon(x)} \int_{B(x,\varepsilon)} (f(y) - f(x))\, \delta(y) dy \Big|^2 \delta(x) dx \\
&\le \int_\Omega \int_\Omega \mathbf{1}_{|y-x|\le\varepsilon} |f(y) - f(x)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy \\
&\le \varepsilon^2 \int_0^{1/2} \int_\Omega \int_\Omega |\nabla f((1-t)x + ty)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy dt \\
&\quad + \varepsilon^2 \int_{1/2}^1 \int_\Omega \int_\Omega |\nabla f((1-t)x + ty)|^2 \frac{\delta(y)}{\delta_\varepsilon(x)} \delta(x) dx dy dt \\
&\lesssim \varepsilon^2 \int_0^{1/2} \int_\Omega \int_\Omega |\nabla f(z)|^2 \frac{\delta(y)}{\varepsilon^{d+1}} dy 2\delta(z) \frac{dz}{(1-t)^d} dt \\
&\quad + \varepsilon^2 \int_{1/2}^1 \int_\Omega \int_\Omega |\nabla f(z)|^2 \frac{2\delta(z)}{\varepsilon^{d+1}} \delta(x) \frac{dz}{t^d} dt dx
\end{aligned}$$

where for the last inequality we have used the first inequality in (7.45), the fact that $\delta(x) \le 2\delta(z)$ when $0 < t < 1/2$ and the fact that $\delta(y) \le 2\delta(z)$ when $1/2 < t < 1$. Using the second inequality in (7.45), we straightforwardly obtain

$$\|f - f_\varepsilon\|^2_{L^2(\delta)} \le C_1\, \varepsilon\, \|\nabla f\|^2_{L^2(\delta)}, \quad \forall\, \varepsilon > 0,$$

for a constant $C_1 > 0$. On the other hand, we also observe that

$$\|f_\varepsilon\|_{L^\infty} \le \frac{C_2}{\varepsilon^{d+1}} \|f\|_{L^1_\delta}.$$

Writing now

$$f^2 = f(f - f_\varepsilon) + f f_\varepsilon$$

and using the above two estimates, we deduce

$$\begin{aligned}
\|f\|^2_{L^2_\delta} &\le \|f\|_{L^2_\delta} \|f - f_\varepsilon\|_{L^2_\delta} + \|f\|_{L^1_\delta} \|f_\varepsilon\|_{L^\infty}. \\
&\le \|f\|_{L^2_\delta} C_1\, \varepsilon^{1/2} \|\nabla f\|_{L^2_\delta} + C_2\, \varepsilon^{-d-1} \|f\|^2_{L^1_\delta} \\
&\le \frac{1}{2} \|f\|^2_{L^2_\delta} + \frac{C_1}{2} \varepsilon \|\nabla f\|^2_{L^2_\delta} + C_2\, \varepsilon^{-d-1} \|f\|^2_{L^1_\delta},
\end{aligned}$$

and we obtain the weighted Nash inequality (7.44) by choosing $\varepsilon := (\|f\|^2_{L^1_\delta} / \|\nabla f\|^2_{L^2_\delta})^{1/(d+2)}$. □

Defining

$$u := \int |f_t| \phi_0 dx e^{-2\kappa t}, \quad v := \int f_t^2 \phi_0 dx e^{-2\kappa t},$$

with $\kappa := \kappa_{1+}$, coming back to (7.43) and using (7.36), the Nash inequality (7.44) and the estimate (7.41), we get

$$\begin{aligned}
v'(t) &\le -2\nu c_0 \int |\nabla f_t|^2 \delta e^{-2\kappa t} \\
&\le -2\nu c_0 C_N^{-2\frac{d+2}{d+1}} \frac{\Big(\|f_t\|^2_{L^2(\delta)} e^{-2\kappa t} \Big)^{\frac{d+2}{d+1}}}{\Big(\|f_t\|^2_{L^1(\delta)} e^{-2\kappa t} \Big)^{\frac{1}{d+1}}} \\
&\le -C \frac{v(t)^{1+\alpha}}{u(0)^{2\alpha}},
\end{aligned}$$

with $C := 2\nu C_N^{-2\frac{d+2}{d+1}} c_0^{1+2\frac{d+2}{d+1}} c_1^{-\frac{2}{d+1}}$ and $\alpha := 1/(d+1)$. Integrating in time, we deduce

$$v(t) \quad \leq \quad \frac{\alpha^{1/\alpha}}{C^{1/\alpha}} \frac{u(0)^2}{t^{1/\alpha}}, \quad \forall\, t > 0.$$

We have thus established that there exist constructive constants $K > 0$ and $\kappa \geq 0$ such that

$$\|S_{\mathcal{C}}(t) f\|_{L^2(\phi_0)} \leq K \frac{e^{\kappa t}}{t^{(d+1)/2}} \|f\|_{L^1(\phi_0)}, \quad \forall\, f \in L^1(\phi_0). \tag{7.46}$$

From that last result, the estimates (7.36) and the properties of $\mathcal{A}$, we deduce that for $N \geq 1$ large enough

$$(S_{\mathcal{B}}\mathcal{A})^{(*N)} : L^1(\delta) \to L^2(\delta) \ \text{ with bound } \ \mathcal{O}(e^{\kappa t}). \tag{7.47}$$

We refer to [190, Prop. 3.9], [276, Prop. 2.5] and [231, Lem. 2.4] for details.

- Step 4. Lyapunov condition. We may next write

$$\widetilde{S}_{\mathcal{L}} = V + W * \widetilde{S}_{\mathcal{L}},$$

with

$$V := \widetilde{S}_{\mathcal{B}} + \cdots + (\widetilde{S}_{\mathcal{B}}\mathcal{A})^{(*(N-1))}, \quad V := (\widetilde{S}_{\mathcal{B}}\mathcal{A})^{(*N)}.$$

On the one hand, using that $\mathcal{A} : L^2 \to L^2$ is bounded and (7.40), we deduce that

$$V : L^2 \to L^2, \ \text{ with bound } \ \mathcal{O}(e^{\kappa t}),$$

for any $\kappa \in (\kappa_0 - \kappa_1, 0)$. On the other hand, using that $\mathcal{A} : L^2_\delta \to L^2$ is bounded as well as (7.42) for $S_{\mathcal{L}}$, (7.47), (7.38), (7.36) and (7.40), we deduce that

$$W * \widetilde{S}_{\mathcal{L}} : L^1_{\phi_1} \to L^2, \ \text{ with bound } \ \mathcal{O}(e^{\kappa' t}),$$

for any $\kappa' > \kappa_1 - \kappa_0$. We may thus fix $t = T$ large enough such that the following Lyapunov inequality holds

$$\|\widetilde{S}_{\mathcal{L}}(T) f\|_{L^2} \leq \frac{1}{2}\|f\|_{L^2} + M_T \|f\|_{L^1_{\phi_1}}, \tag{7.48}$$

which is nothing but (6.7) in the hypothesis of the Harris theorem.

- Step 5. Harris condition Let $A > 0$ and consider $0 \leq f_0 \in L^2$ such that $\|f_0\|_2 \leq A\langle f_0, \phi_0\rangle$. We set $\widetilde{f}_t := e^{-\lambda_1 t} S_{\mathcal{L}}(t) f_0$. From the first inequality in (7.23), we have

$$\frac{d}{dt}\langle \widetilde{f}_t, \phi_0\rangle = \langle \widetilde{f}_t, (\mathcal{L}^* - \lambda_1)\phi_0\rangle \geq -(\lambda_1 - \kappa_0)\langle \widetilde{f}_t, \phi_0\rangle,$$

and then, thanks to Gronwall lemma again, we obtain,

$$\langle \widetilde{f}_t, \phi_0\rangle \geq e^{-(\lambda_1 - \kappa_0)t}\langle f_0, \phi_0\rangle.$$

This estimate, together with the previous step, shows that

$$\begin{aligned}
\int_{\omega_\varrho} \widetilde{f}_{t_0}(x)\phi_0 dx &= \int_{\Omega} \widetilde{f}_{t_0}(x)\phi_0 dx - \int_{\omega_\varrho^c} \widetilde{f}_{t_0}(x)\phi_0 dx \\
&\geq e^{-(\lambda_1 - \kappa_0)t_0}\langle f_0, \phi_0\rangle - \|\widetilde{f}_{t_0}\|_2 \|\phi_0\|_\infty |\omega_\varrho^c|^{1/2} \\
&\geq e^{-(\kappa_1 - \kappa_0)t_0}\langle f_0, \phi_0\rangle - e^{(\kappa_1 - \kappa_0)t_0}\|f_0\|_2 \|\phi_0\|_\infty |\omega_\varrho^c|^{1/2} \\
&\geq \left(e^{-(\kappa_1 - \kappa_0)t_0} - A e^{(\kappa_1 - \kappa_0)t_0}\|\phi_0\|_\infty |\omega_\varrho^c|^{1/2}\right)\langle f_0, \phi_0\rangle.
\end{aligned}$$

Choosing $\varrho > 0$ small enough, we get

$$\int_{\omega_\varrho} \widetilde{f}_{t_0}(x)\phi_0 dx \geq \gamma\langle f_0, \phi_0\rangle, \quad \gamma := \frac{1}{2} e^{-(\lambda_1 - \kappa_0)t_0}.$$

As a consequence, there is $x^f_{t_0} \in \omega_\varrho$ such that

$$\widetilde{f}_{t_0}(x^f_{t_0}) \geq \frac{1}{|\omega_\varrho|}\int_{\omega_\varrho} \widetilde{f}_{t_0}(x) dx \geq \frac{1}{|\Omega| c_1 \varrho}\int_{\omega_\varrho} \widetilde{f}_{t_0}(x)\phi_0 dx \geq \frac{\gamma}{|\Omega| c_1 \varrho}\langle f_0, \phi_0\rangle.$$

On the other hand, from the Harnack inequality (7.33) established in Step 2, we know that for any $T > t_0$, there exits C_H such that

$$\widetilde{f}_{t_0}(x^f_{t_0}) \le \sup_{\omega_\varrho} \widetilde{f}_{t_0} \le C_H \inf_{\omega_\varrho} \widetilde{f}_T.$$

The two last estimates together with (7.38) and (7.36) imply the Harris type estimate

$$\widetilde{f}_T = \widetilde{S}(T) f_0 \ge g_A \langle f_0, \phi_1\rangle, \tag{7.49}$$

with $g_A := \frac{c_0 \gamma}{C_H |\Omega| c_1^2 \varrho} \mathbf{1}_{w_\varrho}$, which is nothing but (6.8) in Harris theorem.

- Step 6. Conclusion. Because of the constructive estimates (7.39), (7.48) and (7.49), we may apply the Harris type Theorem 6.3, and we conclude to the exponential stability **(E3$_1$)** in the norm of $L^2(\Omega)$ with constructive constants.

7.2. **Diffusion in the whole space with strong potential confinement.** We consider in this section the elliptic operator

$$\mathcal{L} f := \Delta f + b \cdot \nabla f + c f, \quad f \in H^1(\mathbb{R}^d), \tag{7.50}$$

with $b \in L^\infty_{\rm loc}(\mathbb{R}^d)$, $c \in L^2_{\rm loc}(\mathbb{R}^d)$ and a confinement condition that we impose through the properties of the *potential function* c, which is roughly speaking $c \to -\infty$ as $|x| \to \infty$. More precisely, we assume

$$\sigma_{i+} \in L^{d/2}, \quad \text{meas}\{\sigma_i \ge K\} < \infty, \ \forall K < 0, \tag{7.51}$$

with either $\sigma_1 := c + |b|^2/\kappa$ for some constant $\kappa \in (0,4)$ or either $\sigma_2 := c + \text{div} b/2$. When we assume that

$$c \sim -|x|^\gamma \quad \text{and} \quad b \sim x|x|^{\beta-1} \quad \text{as} \quad |x| \to \infty,$$

the condition (7.51) for σ_1 is reached when $\gamma > \max(0, 2\beta)$ or $\gamma = 2\beta > 0$ and some conditions on the constants involved in the behavior of the coefficients. In that context, the condition (7.51) for σ_2 is more general since it is reached when $\gamma > \max(0, \beta - 1)$ or $\gamma = \beta - 1 > 0$ and some conditions on the constants involved in the behavior of the coefficients.
A similar framework is considered in [252] and for the reader convenance we just briefly check that it falls in the framework developed before by slightly modifying the arguments presented in the previous section. The integrability conditions on b and c may be probably weaken. For the sake of clarity we do not follow this line of research but rather focus on the new arguments which are necessary in order to deal with the unbounded domain $\Omega = \mathbb{R}^d$.

Condition (H1). The definition of the operator is still made through the formula (7.7). Under assumption (7.51) on σ_1, denoting $\theta_1 := 1 - \kappa/4$ and proceeding exactly as in the previous section during the proof of (7.8), for any $f \in H^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} \langle (\lambda - \mathcal{L}) f, f\rangle &= \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} f\, b \cdot \nabla f + \int_{\mathbb{R}^d} (\lambda - c) f^2 \\ &\ge \theta_1 \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} (\lambda - \sigma_1) f^2, \end{aligned}$$

by using successively the Cauchy-Schwarz inequality and the Young inequality. On the other hand, under assumption (7.51) on σ_2, denoting $\theta_2 := 1$, for any $f \in H^1(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we write

$$\langle (\lambda - \mathcal{L}) f, f\rangle = \theta_2 \int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} (\lambda - \sigma_2) f^2,$$

by performing one integration by part in the previous equation. In both cases, for and any $M > 0$, proceeding again as in the previous section during the proof of (7.8), and denoting from now on $\sigma = \sigma_i$, $\theta = \theta_i$ we have

$$\langle (\lambda - \mathcal{L}) f, f\rangle \ \ge \ \frac{\theta}{2} \|\nabla f\|^2_{L^2} + \|\sqrt{\sigma_-} f\|^2_{L^2} + (\lambda - M)\|f\|^2_{L^2} + (\frac{\theta C_S}{2} - \|\sigma \mathbf{1}_{\sigma \ge M}\|_{L^{d/2}}) \|f\|^2_{L^{2^*}},$$

by using the Sobolev inequality (with associated constant C_S) and the Holder inequality. Taking $M > 0$ large enough, and next $\kappa_1 > 0$ large enough, we finally obtain

$$\langle (\lambda - \mathcal{L}) f, f\rangle \ge \frac{\theta}{2} \|\nabla f\|^2_{L^2} + \|\sqrt{\sigma_-} f\|^2_{L^2} + \|f\|^2_{L^2}, \quad \forall \lambda \ge \kappa_1. \tag{7.52}$$

With the same arguments as in the previous section, we conclude that $\mathcal{L}$ is the generator in L^2 of a positive semigroup $S_{\mathcal{L}}$, so that **(H1)** holds.

Condition (H2). We may for instance use the third constructive argument (which is local) presented in section 7.1 and we establish

$$\exists\, f_0 \in H^1_0\backslash\{0\},\ f_0 \ge 0,\ \exists\, \kappa_0 \in \mathbb{R}, \quad \mathcal{L} f_0 \ge \kappa_0 f_0.$$

That is condition **(ii)** in Lemma 2.4, so that condition **(H2)** holds.

Condition (H3). We introduce again the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} := \kappa_1 - \kappa_0 + 1$, so that from (7.52), the operator $\lambda - \mathcal{B} = (\lambda - \kappa_0 + 1) + (\kappa_1 - \mathcal{L})$ is invertible for any $\lambda \ge \kappa_{\mathcal{B}} := \kappa_0 - 1$. We claim that the operator $(\lambda - \mathcal{B})^{-1}$ is compact for any $\lambda \ge \kappa_{\mathcal{B}}$. For that purpose, let us consider a sequence (f_n) such that $(\lambda - \mathcal{B}) f_n$ is bounded in L^2 and we have to prove that (f_n) is relatively strongly compact. When condition (7.51) holds and because of the estimate (7.52) and the very definition of $\mathcal{B}$, we have

$$\frac{\theta}{2}\|\nabla f_n\|^2_{L^2} + \|\sqrt{\sigma_-} f_n\|^2_{L^2} + \|f_n\|^2_{L^2} \le C, \tag{7.53}$$

for some constant $C \in \mathbb{R}_+$. Because of the Rellich-Kondrachov theorem, we just have to show that

$$\lim_{R\to\infty} \sup_n \int_{B^c_R} f_n^2 = 0.$$

But that last convergence may be established using the assumption (7.51) in the following way. We write

$$\begin{aligned}\int_{B^c_R} f_n^2 &= \int_{B^c_R \cap \{\sigma \ge K\}} f_n^2 + \int_{B^c_R \cap \{\sigma < K\}} f_n^2 \\ &\le \|f_n\|^{\frac{d-2}{d}}_{L^{2^*}} [\mathrm{meas}(B^c_R \cap \{\sigma \ge K\})]^{\frac{2}{d}} + \frac{1}{|K|}\int \sigma_- f_n^2,\end{aligned}$$

for any $K < 0$, by using the Holder inequality. Using next the Sobolev inequality, the estimate (7.53) and the assumption (7.51), we deduce

$$\limsup_{R\to\infty} \int_{B^c_R} f_n^2 \lesssim \limsup_{R\to\infty} \inf_{K<0} \Big\{ [\mathrm{meas}(B^c_R \cap \{\sigma \ge K\})]^{\frac{2}{d}} + \frac{1}{|K|}\Big\} = 0,$$

and the claim is proved. As a consequence, we may apply Lemma 2.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

Condition (H4). As in [231, Prop. 5.4], we establish the strong maximum principle by exhibiting a barrier function and using Lemma 4.11. An alternative argument should be to adapt the proof based on the Harnack inequality as presented in the previous section. Let us then consider $f \in D(\mathcal{L}^k) \cap X_+\backslash\{0\}$ such that $(\lambda - \mathcal{L}) f \ge 0$ with k large enough ($k > d/2$ must be suitable) and $\lambda \ge \lambda_1$ large enough but fixed ($\lambda \ge \kappa_1$ is suitable). Using a very classically bootstrap argument based on iterated application of the Calderon-Zygmond elliptic regularity theorem and the Morrey estimate, we have $f \in C(\mathbb{R}^d)$. By assumption, there thus exist $x_0 \in \mathbb{R}^d$, and two constants $\tau, r > 0$ such that $f \ge \tau$ on $B(x_0, r)$ and we take choose $x_0 = 0$ in order to simplify the notations. We next fix $R > r$ and we observe that the function

$$g(x) := \tau^*(g_0(|x|) - g_0(R)), \quad g_0(s) := \exp(\sigma r^2/2 - \sigma s^2/2),$$

satisfies

$$\begin{aligned}(\tau^*)^{-1}(\lambda - \mathcal{L}) g &= (\lambda - c)(g_0 - g_0(R)) + (d\sigma - \sigma b\cdot x - \sigma^2 r^2)\, g_0 \\ &\le [2(|\lambda| + \|c\|_{L^\infty(B_R)}) + \sigma(d + \|b\cdot x\|_{L^\infty(B_R)}) - \sigma^2 r^2]\, g_0 \le 0\end{aligned}$$

on $\mathcal{O} := B(0,R)\backslash B(0,r)$ for $\sigma > 0$ large enough. We next fix τ^* such that $g = \tau$ on $\partial B(x_0, r)$. We also observe that from (7.52), $\lambda - \mathcal{L}$ is coercive on $\mathcal{O}$, in the sense that

$$\forall\, h \in H^1_0(\mathcal{O}) \quad ((\lambda - \mathcal{L})h, h)_{L^2(\mathcal{O})} \ge \|h\|_{L^2(\mathcal{O})}.$$

In particular, $\lambda - \mathcal{L}$ satisfies the weak maximum principle as explained in the proof of (7.9). Arguing as in the proof of Lemma 4.11, we deduce that $f \ge g > 0$ on $\mathcal{O}$, what we also see directly by observing that $h := (g - f)_+ \in H^1_0(\mathcal{O})$, $(\lambda - \mathcal{L})h \le 0$ and using that the weak maximum principle implies $h \le 0$, thus $h \equiv 0$ and finally $f \ge g$. Because $R > r$ can be chosen arbitrarily large, we conclude with $f > 0$ on $\mathbb{R}^d$.

Condition (H5). The reverse Kato's inequality condition is proved by using local arguments, so that it holds for the same reasons as in the previous section. Similarly, because the argument are local, the conclusion of Lemma 7.1 holds here.

As a consequence, using Theorem 2.21, Theorem 4.13, Theorem 5.16 and Theorem 5.23, we may summarize our analysis in the following result.

Theorem 7.6. *Consider the elliptic operator* (7.50) *in the whole space and assume that the coefficients satisfy* (7.51). *Then the conclusions* **(C3)** *holds as well as* **(E2)** *in* $L^1_{\phi_1}$.

We do not present an exponential constructive estimate, which we believe is possible to prove, but would require significantly more development.

7.3. **Diffusion in the whole space with weak potential confinement.** We consider in this section the same elliptic operator (7.50) with now a weak confinement condition assuming that c converges to a constant. With no loss of generality, we may assume $c \to 0$. More precisely, we consider the elliptic operator

$$\mathcal{L}f := \Delta f + b \cdot \nabla f + rcf, \tag{7.54}$$

with $c \in C_0(\mathbb{R}^d)$, $b \in C_0(\mathbb{R}^d)$ and $r \in \mathbb{R}_+$ a parameter. When not necessary in the discussion we will take $r = 1$. The associated first eigenvalue problem in such a situation has been studied in [252, 8th and 9th courses] to which we refer for more details. We define

$$\lambda_1 = \lambda_1(r) := \inf\{\kappa \in \mathbb{R};\ (\lambda - \mathcal{L})^{-1} \text{ well defined and positive for any } \lambda \ge \kappa\}.$$

Proceeding exactly as in the proof of **(H1)** in the preceding section, we see that the operator $\lambda - \mathcal{L}$ is invertible for any $\lambda > \|c_+\|_{L^\infty}$, and then its inverse is positive. Because the proof of **(H2)** in the preceding section also applies here, we deduce that the infimum λ_1 of the set $\mathcal{I}$ of real resolvent values is well defined with $\lambda_1 \in (\kappa_0, \kappa_1)$, for some constructive constants $\kappa_i \in \mathbb{R}$.

We split now the discussion into two cases.

Case 1. We start considering **the case** $b = 0$. In that case, $\mathcal{L}$ is self-adjoint so that λ_1 is also characterized by

$$\lambda_1 = \sup_{\|f\|_{L^2}=1} \mathcal{E}(f),$$

with

$$\mathcal{E}(f) := (\mathcal{L}f, f) = r\int cf^2 - \int |\nabla f|^2.$$

We make the following elementary observations :

• We claim that $\lambda_1 \ge 0$. Taking $f_n(x) := n^{-d/2}u(x/n)$ for some function $u \in H^1(\mathbb{R}^d)$, $\|u\|_{L^2} = 1$, we compute

$$\begin{aligned} -\mathcal{E}(f_n) &= \int |\nabla f_n|^2 - \int_{B_R} rcf_n^2 - \int_{B_R^c} rcf_n^2 \\ &\le \frac{1}{n^2}\int |\nabla u|^2 + \|rc\|_{L^\infty(B_R)} \int_{B_{R/n}} u^2 + \|rc\|_{L^\infty(B_R^c)}, \end{aligned}$$

for any $R > 0$, so that

$$-\lambda_1 \le \limsup(-\mathcal{E}(f_n)) \le 0.$$

• We claim that $\lambda_1 = 0$ when $c \le 0$. In that case, we have $\mathcal{E}(f) \le 0$ for any $f \in H^1(\mathbb{R}^d)$, and we deduce the reverse inequality $\lambda_1 \le 0$. In particular, as a function $\lambda_1 = \lambda_1(r)$ of $r \ge 0$, we have $\lambda_1(0) = 0$. We also claim that $\lambda_1(r) \to \infty$ as $r \to \infty$ when $c_+ \not\equiv 0$. We may indeed fix $f \in H^1(\mathbb{R}^d)$, $\|f\|_{L^2} = 1$, $\operatorname{supp} f \subset \operatorname{supp} c_+$, and we compute

$$\mathcal{E}(f) = r\int_{\mathbb{R}^d} c_+ f^2 - \int |\nabla f|^2 \to \infty, \quad \text{as} \quad r \to \infty.$$

• We finally observe that $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is convex since it is defined as the supremum of linear functions $r \to \mathcal{E}(f)$ for any fixed $f \in H^1(\mathbb{R}^d)$. As a consequence, we have the following alternative:

- $\lambda_1 \equiv 0$;
- $\exists r_0 \in [0,\infty)$ such that $\lambda_1(r) = 0$ for $r \le r_0$ and $\lambda_1(r) > 0$ for $r > r_0$.

Concerning the value of r_0, it may happen that $r_0 > 0$, and that is the case when $c \in L^{d/2}$ because of the Sobolev inequality, or that $r_0 = 0$, and that is the case for instance when $c \ge 0$, $c(x) = |x|^{-m}$ for $x \in B_R^c$, $m \in (0,2)$, $R > 0$. To prove that last claim, we may take the same sequence (f_n) as above, and we compute

$$\begin{aligned}\mathcal{E}(f_n) &\ge \int_{B_R^c} r|x|^{-m} f_n^2 - \int |\nabla f_n|^2 dx \\ &= \frac{r}{n^m}\int_{B_{R/n}^c} |x|^{-m} u^2 - \frac{1}{n^2}\int |\nabla u|^2 dx > 0,\end{aligned}$$

for n large enough (whatever is the value of $r > 0$).

About condition (H3). It is established in [252] that when $\lambda_1 = 0$, the condition **(H3)** is not satisfied and there does not exist a first eigenfunction $f_1 \in L^2(\mathbb{R}^d)$ to the operator $\mathcal{L}$ defined by (7.54). We refer to [252, 8th course] for a proof of that result. On the other hand, we claim that the condition **(H3)** is satisfied when $\lambda_1 > 0$. Consider indeed three sequences (λ_n) of $\mathbb{R}$, (f_n) of $H^1(\mathbb{R}^d)$ and (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L})f_n = \varepsilon_n$, $\varepsilon_n, f_n \ge 0$, $\|f_n\|_{L^2} = 1$, for any $n \ge 1$, $\lambda_n \to \lambda_1$ and $\varepsilon_n \to 0$ in L^2 as $n \to \infty$. We then have

$$\lambda_n - \mathcal{E}(f_n) = \langle(\lambda_n - \mathcal{L})f_n, f_n\rangle = \langle \varepsilon_n, f_n\rangle \to 0,$$

as $n \to \infty$. By definition of $\mathcal{E}$ and boundedness of c, we see that (f_n) is bounded in H^1. As a consequence, up to the extraction of a subsequence, we have $f_n \to f_1 \ge 0$ in $L^2_{\rm loc}$ and thus next $(\lambda_1 - \mathcal{L})f_1 = 0$ in the variational sense and

$$\int c f_n^2 \to \int c f_1^2, \quad \|\nabla f_1\|_{L^2} \le \liminf \|\nabla f_n\|_{L^2},$$

where we have used the dominated convergence theorem of Lebesgue and the fact that $c \to 0$ at infinity in order to get the first convergence. We finally deduce

$$\mathcal{E}(f_1) \ge \limsup \mathcal{E}(f_n) = \lambda_1 > 0,$$

so that $f_1 \not\equiv 0$, and the condition **(H3)** is verified.

As a conclusion, for a self-adjoint operator, condition **(H3)** is automatically fulfilled by its adjoint and the conditions **(H4)** and **(H5)** have been proved in a general situation, including the present framework. The same conclusions of existence, uniqueness and asymptotic stability of the first eigentriplet solution (λ_1, f_1, ϕ_1) as in section 7.2 hold true when $\lambda_1 > 0$.

Case 2. We consider the **general case** $b \in C_0(\mathbb{R}^d)$.

• We claim that $\lambda_1 \ge 0$. Adapting the second constructive argument in the proof of **(H2)** in Section 7.1, we consider $\chi \in C^1_c(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$ such that $\mathbf{1}_{[0,1/2]} \le \chi \le \mathbf{1}_{[0,1]}$, $\chi' \le 0$ on $[0,1]$, $\chi(s) := (1-s)^2/2$ on $[\eta,1]$ with $\eta \in (1/2,1)$ large enough in such a way that

$$\chi''(s) + (d-1)\chi'(s)/s \ge 1/2, \quad \forall s \in (\eta,1), \tag{7.55}$$

and define $f_0(x) := \chi(|x-x_0|/n)$ for $|x_0|$ large enough to be chosen later. We have supp $f_0 \subset B_n(x_0)$ for any $n \ge 1$ and we compute

$$\mathcal{L}f_0(x) = \frac{1}{n^2}\{\chi''(r/n) + \frac{d-1}{r/n}\chi'(r/n)\} + \frac{1}{n} b(x)\cdot \hat{y}\,\chi'(r/n) + c(x)\chi(r/n)$$

where $y = x - x_0$ and $r = |y|$. On $B_{n\eta}(x_0)$, we have

$$\mathcal{L}f_0(x) \ge -\frac{\|\chi''\|_\infty}{n^2} - \frac{d-1}{n^2}\Big\|\frac{\chi'(r)}{r}\Big\|_\infty - \frac{\|\chi'\|_\infty}{n}\sup_{B_n(x_0)}|b| - \|\chi\|_\infty \sup_{B_n(x_0)}|c|.$$

On $B_n(x_0) \setminus B_{\eta n}(x_0)$, thanks to (7.55), we have

$$\mathcal{L}f_0(x) \ge \frac{1}{2n^2} - \frac{\|\chi'\|_\infty}{n}\sup_{B_n(x_0)}|b| - \|\chi\|_\infty \sup_{B_n(x_0)}|c|.$$

Let now fix $\varepsilon > 0$ and choose first n large enough so that

$$-\frac{\|\chi''\|_\infty}{n^2} - \frac{d-1}{n^2}\Big\|\frac{\chi'(r)}{r}\Big\|_\infty \ge -\frac{\varepsilon}{2}\inf_{(0,\eta)}\chi.$$

Then, using that $b, c \in C_0(\mathbb{R}^d)$, we can take $|x_0|$ large enough so that

$$-\frac{\|\chi'\|_\infty}{n}\sup_{B_n(x_0)}|b| - \|\chi\|_\infty\sup_{B_n(x_0)}|c| \ge -\frac{\varepsilon}{2}\inf_{(0,\eta)}\chi$$

and

$$\frac{\|\chi'\|_\infty}{n}\sup_{B_n(x_0)}|b| + \sup_{B_n(x_0)}|c| \le \frac{1}{2n^2}.$$

Gathering the above inequalities, we obtain

$$\mathcal{L}f_0 \ge -\varepsilon f_0,$$

and the condition **(H2)** is verified with $\kappa_0 = -\varepsilon$. Because $\varepsilon > 0$ can be choose arbitrarily small, we conclude with $\lambda_1 \ge 0$.

• We claim that $\lambda_1 = 0$ when $\sigma_2 \le 0$. Indeed, we have already seen that

$$\langle \mathcal{L}f, f\rangle = -\int_{\mathbb{R}^d}|\nabla f|^2 + \int_{\mathbb{R}^d}\sigma_2 f^2,$$

from which we deduce that

$$\frac{d}{dt}\|S_t f\|^2 = 2\langle \mathcal{L}f, f\rangle \le 0.$$

This ensures that **(H1)** is verified with $\kappa_1 = 0$ and so $\lambda_1 \le 0$.

• We claim that $\lambda_1 > 0$ when $c_+ \not\equiv 0$ and $r > 0$ is large enough. For simplifying notations and up to translation and dilatation, we may reduce to the case $c \ge c_0 \mathbf{1}_{B(0,1)}$ with $c_0 > 0$. Adapting the second constructive argument in the proof of **(H2)** in Section 7.1, we consider $\chi \in C^1_c(\mathbb{R}_+) \cap W^{2,\infty}(\mathbb{R}_+)$, $\mathbf{1}_{[0,1/2]} \le \chi \le \mathbf{1}_{[0,1]}$, $\mathrm{supp}\chi = [0,1]$, $\chi''(1) = 1$, $\chi' \le 0$ on $[0,1]$ and we set $f_0(x) := \chi(|x|)$. We compute

$$\mathcal{L}f_0 = \chi''(|x|) + \chi'(|x|)((d-1)/|x| + b\cdot\hat{x}) + rc(x)\chi(|x|).$$

On the one hand, we fix $\eta \in (1/2, 1)$, $1-\eta$ small enough, in such a way that

$$\|\chi'\|_{L^\infty(\eta,1)}\big(2(d-1) + \|b\|_{L^\infty}\big) \le 1/4, \quad 1/2 \le \|\chi''\|_{L^\infty(\eta,1)},$$

and thus

$$\mathcal{L}f_0 \ge \frac14 \ge \frac14 f_0 \quad \text{on} \quad B(0,\eta)^c.$$

On the other hand, we fix $r > 0$, large enough, in such a way that

$$\|\chi''\|_{L^\infty} + \|\chi'\|_{L^\infty}\big(2(d-1) + \|b\|_{L^\infty}\big) \le \kappa(r) := \frac12 rc_0 \inf_{[0,\eta)}\chi,$$

and thus

$$\mathcal{L}f_0 \ge \kappa(r) \ge \kappa(r) f_0 \quad \text{on} \quad B(0,\eta).$$

As a conclusion, we have established that condition **(ii)** in the statement Lemma 2.4 holds with $\kappa_0 := \min(1/4, \kappa(r))$, and that ends the constructive proof of condition **(H2)** by using Lemma 2.4. That implies in particular the claim since then $\lambda_1 \ge \kappa_0 > 0$.

• We finally claim that **(H3)** holds when $\lambda_1 > 0$. To see that, we consider again three sequences (λ_n) of $\mathbb{R}$, (f_n) of $H^1(\mathbb{R}^d)$ and (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L})f_n = \varepsilon_n$, $\varepsilon_n, f_n \ge 0$, $\|f_n\|_{L^2} = 1$, for any $n \ge 1$, $\lambda_n \searrow \lambda_1$ and $\varepsilon_n \to 0$ in L^2 as $n \to \infty$. As a consequence, we have

$$\lambda_n + \int |\nabla f_n|^2 - \int f_n b\cdot\nabla f_n - \int c f_n^2 = ((\lambda_n - \mathcal{L})f_n, f_n) = \langle \varepsilon_n, f_n\rangle \to 0,$$

as $n \to \infty$. Using the boundedness of c, b and λ_n, we see that (f_n) is bounded in H^1. As a consequence, up to the extraction of a subsequence, we have $f_n \to f_1 \ge 0$ in L^2_{loc}. We assume by contradiction that $f_1 \equiv 0$. We deduce that

$$\int c f_n^2 \to 0, \quad \int f_n b\cdot\nabla f_n \to 0,$$

where we have used the dominated convergence theorem of Lebesgue and the fact that $b, c \to 0$ at infinity. We thus obtain

$$0 < \lambda_1 \le \lambda_n + \int |\nabla f_n|^2 = \int f_n b \cdot \nabla f_n + \int c f_n^2 + \langle \varepsilon_n, f_n \rangle \to 0,$$

and our contradiction. So that $f_1 \not\equiv 0$, and the condition **(H3)** is verified.

For the dual problem, from the above analysis, we know that there exist two sequences (ϕ_n) of $H^1(\mathbb{R}^d)$, (ε_n) of $L^2(\mathbb{R}^d)$ such that $(\lambda_n - \mathcal{L}^*)\phi_n = \varepsilon_n$, $\varepsilon_n, \phi_n \ge 0$ and $\|\phi_n\|_{L^2} = 1$, for any $n \ge 1$, and $\varepsilon_n \to 0$ in L^2 as $n \to \infty$. But we face the same situation as previously, since again

$$\lambda_n + \int |\nabla \phi_n|^2 - \int \phi_n b \cdot \nabla \phi_n - \int c \phi_n^2 = ((\lambda_n - \mathcal{L}^*)\phi_n, \phi_n) = (\varepsilon_n, \phi_n) \to 0,$$

and thus the same conclusion, namely $\phi_n \to \phi_1$, with $\phi_1 \in H^1(\mathbb{R}^d)$, $\phi_1 \ge 0$, $\phi_1 \not\equiv 0$.

Conclusion. The conditions **(H4)** and **(H5)** have been proved in a general situation, including the present framework. The same conclusions as in section 7.2 hold true when $r > 0$ is large enough (and thus $\lambda_1 > 0$).

7.4. **Diffusion in the whole space with drift confinement.** We now consider the elliptic operator

$$\mathcal{L} f := \Delta f + b \cdot \nabla f + c f,$$

with a drift confinement as it is the case for the Fokker-Planck operator. More precisely, and for the sake of simplicity, we assume here

$$b = \nabla U, \quad U(x) = \frac{1}{\gamma} \langle x \rangle^\gamma, \quad \gamma > 0. \tag{7.56}$$

When $\gamma = 2$ and $c = x$, that operator corresponds to the classical harmonic Fokker-Planck operator which is known to be related to the standard Poincaré inequality and to the standard log-Sobolev inequality, see [29, 26, 343] or more recently [27, 231] and the references therein. When $c = \mathrm{div} b$, the operator $\mathcal{L}$ is on divergence form and $\mathcal{L}^* 1 = 0$, so that $(0, 1) \in \mathbb{R} \times L^\infty(\mathbb{R}^d)$ is a solution to the dual first eigenvalue problem. Existence of stationary solution f_1 (which is also the first eigenfunction) and its stability have been widely studied. We refer for instance to [344, 332, 171, 28] as well as to [231, 274, 190] which techniques will be adapted here.

In the present situation, we impose that the contribution of c has lower influence at the infinity that the drift term b and we assume

$$c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d), \quad \exists\, C_0, R_0 > 0, \ \forall\, x \in B^c_{R_0}, \ |c(x)| = o(|x|^{2(\gamma-1)}). \tag{7.57}$$

We further assume that

$$c \ge \mathrm{div} b \quad \text{when} \quad \gamma \in (0, 1]. \tag{7.58}$$

The action of the drift term will be revealed through the choice of a convenient "confining space". More precisely, for a weight function $m : \mathbb{R}^d \to [1, \infty)$, we will work in a weighted Lebesgue space. Our analysis is based on the following elementary computation which can be readily adapted from [231, Lem. 2.1], [274, Lem. 3.8] and [190, Lem. 3.8].

Lemma 7.7. *For any $p \in [1, \infty)$, any weight function m and any smooth, rapidly decaying function f, we have*

$$\int (\mathcal{L} f)\, f |f|^{p-2} m^p = -(p-1) \int |\nabla f|^2 |f|^{p-2} m^p + \int |f|^p m^p \varphi_1, \tag{7.59}$$

with

$$\varphi_1 := (p-1) \frac{|\nabla m|^2}{m^2} + \frac{\Delta m}{m} + \left(c - \frac{1}{p} \mathrm{div}\, b \right) - b \cdot \frac{\nabla m}{m} \tag{7.60}$$

as well as

$$\int (\mathcal{L} f)\, f |f|^{p-2} m^p = -(p-1) \int |\nabla (f m)|^2 |f m|^{p-2} + \int |f|^p m^p \varphi_2, \tag{7.61}$$

with

$$\varphi_2 := 2(1-\frac{1}{p})\frac{|\nabla m|^2}{m^2} + (\frac{2}{p}-1)\frac{\Delta m}{m} + \left(c - \frac{1}{p}\mathrm{div}\, b\right) - b\cdot\frac{\nabla m}{m}. \tag{7.62}$$

In order to simplify the discussion, we restrict ourself to the exponent $p = 2$ and to the exponential weight function $m = e^{a\langle x\rangle^s}$, $s \in (0,\gamma]$, $a > 0$. We thus work in the Banach lattice $X := L^2_m$. We observe that

$$\begin{aligned}
\frac{\nabla m}{m} &= sax\langle x\rangle^{s-2} \sim sa|x|^{s-1},\\
\frac{\Delta m}{m} &= sad\langle x\rangle^{s-2} + s(s-2)a|x|^2\langle x\rangle^{s-4} + (sa)^2|x|^2\langle x\rangle^{2s-4} \sim (sa)^2|x|^{2s-2},\\
\mathrm{div} b &= d\langle x\rangle^{\gamma-2} + (\gamma-2)|x|^2\langle x\rangle^{\gamma-4} \sim (d+\gamma-2)|x|^{\gamma-2},\\
b\cdot\frac{\nabla m}{m} &= sax\langle x\rangle^{s-2}\cdot x\langle x\rangle^{\gamma-2} \sim sa|x|^{s+\gamma-2},
\end{aligned}$$

so that the contribution of $(c - \mathrm{div} b/2)$ is always negligible at infinity, and we get

$$\varphi_i \sim (sa)^2|x|^{2s-2} - sa|x|^{s+\gamma-2}. \tag{7.63}$$

We denote

$$\begin{aligned}
a' &:= sa > 0 \quad \text{if} \quad s \in (0,\gamma),\\
a' &:= a\gamma - 2(a\gamma)^2 > 0 \quad \text{if} \quad s = \gamma \text{ and } a \in (0, 1/(\sqrt{2}\gamma)).
\end{aligned}$$

We then face to three cases :

(i) $\gamma > 1$: taking $s \in ((2-\gamma)_+, \gamma)$, we have $\varphi_i \sim -a'|x|^{s+\gamma-2} \to -\infty$ with $s+\gamma-2 > 0$;
(ii) $\gamma = 1$: taking $s = \gamma$, $a < 1/(\sqrt{2}\gamma)$, we have $\varphi_i \to -a'$;
(iii) $\gamma \in (0,1)$: taking $s = \gamma$, $a < 1/(\sqrt{2}\gamma)$, we have $\varphi_i \sim -a'|x|^{2\gamma-2} \to 0$ with $2\gamma - 2 < 0$.

Condition (H1). In any of the above cases, we have from (7.59)

$$((\lambda - \mathcal{L})f, f) = \int |\nabla f|^2 m^2 + \int (\lambda - \varphi_1) f^2 m^2,$$

for $\lambda \in \mathbb{R}$, with $\inf(\lambda - \varphi_1) > 0$ for $\lambda \geq \kappa_1$ and $\kappa_1 > 0$ large enough. We deduce that $\lambda - \mathcal{L}$ is coercive for $\lambda \geq \kappa_1$. With the same arguments as in section 7.1, we conclude that $\mathcal{L}$ is the generator in L^2_m of a positive semigroup $S_\mathcal{L}$, so that **(H1)** holds.

Condition (H2). When $\gamma > 1$, the same arguments as in Section 7.2 imply that condition **(H2)** holds for some $\kappa_0 \in \mathbb{R}$. When $\gamma \in (0,1]$, we have $\mathcal{L}^*1 = c - \mathrm{div} b \geq 0$ from (7.58) and **(H2)** holds with $\kappa_0 = 0$.

Conditions (H4) and (H5). The strong maximum principle holds here because for instance we may apply the same barrier function argument as presented in Section 7.2. The reverse Kato's inequality condition is proved by using local arguments, so that it holds for the same reasons as in the previous section.

Condition (H3). We define the multiplication operator $\mathcal{A}$ and the elliptic operator $\mathcal{B}$ by

$$\mathcal{A} := M\chi_R, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

for $M, R > 0$ and $\chi_R(x) := \chi(x/R)$ with $\chi \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \leq \chi \leq \mathbf{1}_{B_2}$. We fix $\kappa_\mathcal{B} < \kappa_0$ in case (i), $\kappa_\mathcal{B} := -a'/4$ in case (ii) and $\kappa_\mathcal{B} := 0$ in case (iii), and we set $a'' := a'/2$. Choosing $M, R > 0$ large enough, from Lemma 7.7 and the discussion which follows, we deduce that

$$((\mathcal{B} - \alpha)f, f) \leq -\int |\nabla f|^2 m^2 - a'' \int f^2(\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c}|x|^{s+\gamma-2})m^2, \tag{7.64}$$

for any $\alpha \geq \kappa_\mathcal{B}$ and any nice function f. We classically deduce that $\alpha - \mathcal{B}$ is coercive and thus invertible. We discuss the three different cases.

- In the first case $\gamma > 1$, so that $s + \gamma - 2 > 0$, we see that the operator $\mathcal{R}_\mathcal{B}(\alpha)$ is compact from Rellich-Kondrachov theorem, so that also $\mathcal{W}(\alpha) := \mathcal{R}_\mathcal{B}(\alpha)\mathcal{A}$ for any $\alpha \geq \kappa_\mathcal{B}$. We may thus apply Lemma 2.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

- In the case $\gamma = 1$, so that $2\gamma - 2 \leq 0$, the operator $\mathcal{R}_\mathcal{B}(\alpha)$ is not compact anymore. However, for any sequence (f_n) which is bounded in L^2_m, we define the sequence (g_n) by $g_n := \mathcal{A}f_n$, and (g_n) is bounded in $L^2_{\tilde m}$ with $\tilde m := e^{\tilde a\langle x\rangle^\gamma}$, $\tilde a \in (a, 1/\sqrt{2\gamma})$. Using the dissipativity estimate (7.64) in

$L^2_{\tilde m}$, we see that $\mathcal{B}-\alpha$ is dissipative in $L^2_{\tilde m}$ for any $\alpha \ge \kappa_{\mathcal{B}}$, and more precisely the sequence (h_n) defined by $h_n := \mathcal{R}_{\mathcal{B}}(\alpha) g_n$ satisfies

$$\int |\nabla h_n|^2 m^2 + \tilde a'' \int h_n^2 (\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c} |x|^{2\gamma-2}) \tilde m^2 \le \int g_n^2 \tilde m^2.$$

Using that $|x|^{2\gamma-2}\tilde m^2/m^2 \to \infty$ as $|x| \to \infty$, that implies that (h_n) is relatively compact in L^2_m. More precisely, the above estimates show that $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A} : L^2_m \to H^1_m \cap L^2_{m^\sharp}$ with $m^\sharp := m^{1/2}\tilde m^{1/2}$ and in particular we have established that $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}$ is a compact operator in L^2_m uniformly on $\alpha \ge \kappa_{\mathcal{B}}$ because of the Rellich-Kondrachov theorem and the fact that $m = o(m^\sharp)$. Since $\mathcal{R}_{\mathcal{B}}(\alpha)$ is bounded in $\mathscr{B}(L^2_m)$ uniformly for any $\alpha > \kappa_{\mathcal{B}}$, the operator $\mathcal{L}$ satisfies the splitting structure **(HS1)** and, applying Lemma 2.8-(2), we deduce that **(H3)** holds for both the primal and the dual problems.

At this stage, when $\gamma \ge 1$, we obtain a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem (7.3) by using Theorem 2.21.

Condition (HS3). In the case $\gamma \in (0,1)$, the same as in the case $\gamma = 1$ holds except that $\mathcal{R}_{\mathcal{B}}(\alpha)$ is not uniformly bound in $\mathscr{B}(L^2_m)$ for $\alpha \ge \kappa_{\mathcal{B}}$ because we are in the critical case $\kappa_{\mathcal{B}} = \kappa_0$. We do not know how to adapt the stationary approach in that situation and we thus aim to use a dynamical approach through the use of Theorem 3.4 with the above splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $N := [d/4] + 1$. We set $X = X_1 := L^2_m$ and $X_0 := L^1$. The proof of condition **(HS3)** is an immediate consequence of the following estimate.

Proposition 7.8. *We define $\Theta_\zeta(t) := e^{-\zeta t^{\gamma/(2-\gamma)}}$. For $N := [d/4] + 1$, there hold*

(i) $S_{\mathcal{B}} \in L^\infty_t(\mathscr{B}(X_1))$;
(ii) $S_{\mathcal{B}}\mathcal{A}\Theta_\zeta^{-1} \in L^\infty_t(\mathscr{B}(X_i))$ *for $i = 0, 1$ and any $\zeta \in (0, \zeta^*)$;*
(iii) $(S_{\mathcal{B}}\mathcal{A})^{(*N)}\Theta_\zeta^{-1} \in L^\infty_t(\mathscr{B}(X_0, X_1))$ *for any $\zeta \in (0, \zeta^*/2)$.*

The proof of Proposition 7.8 is similar to the proofs of [231, Lem. 2.1], [231, Lem. 2.2], [231, Lem. 2.3] and [231, Lem. 2.4]. For the sake of completeness we however present the main lines of the proof. We start with a technical result that we will use during the proof of Proposition 7.8.

Lemma 7.9. *Consider two Banach spaces X_0, X_1 and a function $u : \mathbb{R}_+ \to \mathscr{B}(X_0) + \mathscr{B}(X_1)$ which satisfies*

(a) $u\Theta^{-1} \in L^\infty(0,\infty; \mathscr{B}(X_0) \cap \mathscr{B}(X_1))$;
(b) $u\wp \in L^\infty(0,\infty; \mathscr{B}(X_0, X_1))$;

for any exponentially decaying function $\Theta = \Theta_\zeta = e^{-\zeta t^\varsigma}$, $\forall \zeta \in (0,\zeta^*)$, and for the power function $\wp := t^{-\alpha}$, with $\zeta^* > 0$, $\varsigma \in (0,1]$ and $\alpha \ge 0$ fixed. Then

(c) *there exists N such that $u^{(*N)}\widetilde\Theta \in L^\infty(0,\infty; \mathscr{B}(X_0, X_1))$,*

for any $\widetilde\Theta = \Theta_{\tilde\zeta}$, $\tilde\zeta \in (0, \zeta^*/2)$.

Proof of Lemma 7.9. A similar argument is developed in [190, Lem. 2.17], [274, Lem. 2.4], [276, Prop. 2.5] and [231, Lem. 2.4].

Step 1. Consider two functions v and w which satisfy the estimate **(a)**. For $\mathcal{X} = X_0$ or $\mathcal{X} = X_1$, we compute

$$\begin{aligned} \|v * w(t)\|_{\mathcal{X}\to\mathcal{X}} &\le \int_0^t \|v(t-s)w(s)\|_{\mathcal{X}\to\mathcal{X}}\, ds \\ &\le \int_0^t C^v_{\mathcal{X}}\Theta(t-s)\, C^w_{\mathcal{X}}\Theta(s)\, ds \le C^v_{\mathcal{X}} C^w_{\mathcal{X}}\, t\, \Theta(t), \end{aligned}$$

with obvious notation and where we have used that $\Theta(t-s)\,\Theta(s) \le \Theta(t)$ for any $0 < s < t$. Since for any $\zeta' \in (0,\zeta)$, there exists a constant C such that $t\Theta_\zeta(t) \le C\,\Theta_{\zeta'}(t)$ for any $t \ge 0$, we see that the function $v * w$ satisfies the same estimate **(a)** for any $\Theta = \Theta_\zeta$, $\zeta \in (0,\zeta^*)$.

Step 2. Consider two functions v and w which satisfy the estimates **(a)** and **(b)** with $\alpha \ge 1$. We compute

$$\begin{aligned}\|v * w(t)\|_{X_0\to X_1} &\le \int_0^{t/2} \|v(t-s)w(s)\|_{X_0\to X_1}\,ds + \int_{t/2}^t \|v(t-s)w(s)\|_{X_0\to X_1}\,ds \\ &\le \int_0^{t/2} C^v_{01}(t-s)^{-\alpha}\,C^w_0\Theta(s)\,ds + \int_{t/2}^t C^v_1\Theta(t-s)\,C^w_{01}s^{-\alpha}\,ds \\ &= [C^v_1\,C^w_{01} + C^v_{01}\,C^w_0]\,\Theta(0)\,t^{-\alpha+1}\int_0^{1/2}(1-\tau)^{-\alpha}\,d\tau,\end{aligned}$$

with obvious notation and we have used that Θ is a decaying function. As a consequence, the function $v * w$ satisfies estimate **(b)** with an exponent $\alpha - 1$ instead of α.

Step 3. Consider two functions v and w which satisfy the estimates **(a)** and **(b)** with $\alpha \in [0,1)$. We compute

$$\begin{aligned}\|v * w(t)\|_{X_0\to X_1} &\le \int_0^{t/2} \|v(t-s)w(s)\|_{X_0\to X_1}\,ds + \int_{t/2}^t \|v(t-s)w(s)\|_{X_0\to X_1}\,ds \\ &\le \int_0^{t/2} C^v_1\Theta(t-s)\,C^w_{01}s^{-\alpha}\,ds + \int_{t/2}^t C^v_{01}(t-s)^{-\alpha}\,C^w_0\Theta(s)\,ds \\ &\le C^v_1\,C^w_{01}\,\Theta(t/2)\int_0^{t/2}s^{-\alpha}\,ds + C^v_{01}\,C^w_0\,\Theta(t/2)\int_{t/2}^t(t-s)^{-\alpha}\,ds \\ &= [C^v_1\,C^w_{01} + C^v_{01}\,C^w_0]\,\Theta(t/2)\frac{t^{1-\alpha}}{1-\alpha},\end{aligned}$$

with the same obvious notation and we have used again that Θ is a decaying function.

Step 4. Iterating $n := [\alpha]$ times steps 1 and 2, we get that $u^{(*n)}$ still satisfies estimate **(a)** and satisfies the estimate **(b)** for the exponent $\alpha - [\alpha] \in (0,1)$. We then conclude that **(c)** holds with $N := n+1$ and any $\tilde\zeta \in (0,\zeta^*/2)$ by using the third step. □

Proof of Proposition 7.8. We classically establish that $\mathcal{B}$ generates a positive semigroup $S_\mathcal{B}$ in both spaces X_i and we thus concentrate on the announced estimates. On the one hand, proceeding as for the proof of (7.64), we have

$$\int(\mathcal{B}f)(\operatorname{sign} f)m \le -a''\int|f|(\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c}|x|^{s+\gamma-2})m, \tag{7.65}$$

for any nice function f and any weight function $m = m_a$, with $m_a(x) := e^{a\langle x\rangle^\gamma}$, $a \in (a_1,a_2)$, $0 < a_1 < a_2 < 1/(\sqrt{2}\gamma)$, where we define $a'' := a\gamma/2 - (a\gamma)^2$. That exactly means that $\mathcal{B}$ is weakly dissipative in L^1_m as defined in (3.19). From the discussion in Section 3.3, we deduce that $S_\mathcal{B}$ is a semigroup of contractions and satisfies the associated decay estimate (3.23), (3.24), and more precisely

$$\|S_\mathcal{B}(t)f\|_{L^1_{m_a}} \le \|f\|_{L^1_{m_a}}, \quad \|S_\mathcal{B}(t)f\|_{L^1_{m_a}} \le \Theta_\zeta(t)\|f\|_{L^1_{m_{a'}}}, \tag{7.66}$$

for any $a, a' \in (a_1,a_2)$, $a < a'$, $\zeta \in (0,\zeta_*)$, $\zeta_* := (a'-a)^{(2-2\gamma)/(2-\gamma)}(a'\gamma(1-a'\gamma))^{\gamma/(2-\gamma)}$. We refer to [231, Lem. 2.1] for details. Using that $\mathcal{A} : L^1 \to L^1_m$ is bounded, that establishes (ii) in X_0. Similarly, starting from (7.61) and proceeding as in the proof of (7.64), we get

$$(\mathcal{B}f,f)_{L^2_m} \le -\int|\nabla(fm)|^2 - a''\int f^2(\mathbf{1}_{B_1} + \mathbf{1}_{B_1^c}|x|^{s+\gamma-2})m^2, \tag{7.67}$$

for any nice function f and any weight function $m = m_a$ as above. Throwing away the first term at the RHS and arguing as we did in L^1_m, we obtain that $S_\mathcal{B}$ satisfies

$$\|S_\mathcal{B}(t)f\|_{L^2_{m_a}} \le \|f\|_{L^2_{m_a}}, \quad \|S_\mathcal{B}(t)f\|_{L^2_{m_a}} \le \Theta_\zeta(t)\|f\|_{L^2_{m_{a'}}}, \tag{7.68}$$

for any $a, a' \in (a_1,a_2)$. Using that $\mathcal{A} : L^2_{m_a} \to L^2_{m'_a}$ is bounded, that establishes (i) and (ii) in X_1.

On the other hand, throwing away the second term at the RHS in (7.67), for any trajectory $f_t = S_{\mathcal{B}}(t) f_0$, f_0 in the domain of $\mathcal{B}$ in L^2_m, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} f_t^2\, m^2 dx \le -\int_{\mathbb{R}^d} |\nabla(f_t m)|^2 dx.$$

Using Nash's inequality which for some constant $C_N \in (0,\infty)$ stipulates that

$$\int_{\mathbb{R}^d} g^2 dx \le C_N \left(\int_{\mathbb{R}^d} |\nabla g|^2 dx\right)^{\frac{d}{d+2}} \left(\int_{\mathbb{R}^d} |g| dx\right)^{\frac{4}{d+2}}, \quad \forall\, g,$$

with $g := f_t m$ and the first estimate in (7.68), we deduce

$$F'(t) \le -2\, C_N'\, F(t)^{-4/d}\, G(t)^{1+\frac{2}{d}} \le -2\, C_N'\, F(t)^{-4/d}\, G(0)^{1+\frac{2}{d}}, \tag{7.69}$$

with $C_N' := C_N^{-1-2/d}$ and where for brevity of notations we have set

$$F(t) := \|f_t\|^2_{L^2(m)}, \qquad G(t) := \|f_t\|_{L^1(m)}.$$

Integrate the differential inequality (7.69), we find

$$\|S_{\mathcal{B}}(t) f_0\|^2_{L^2_m} \lesssim t^{-d/4}\, \|f_0\|_{L^1_m}, \quad \forall\, t > 0,$$

and using that $\mathcal{A} : L^1 \to L^1_m$, we next obtain

$$S_{\mathcal{B}}(t)\mathcal{A} t^{d/4} \in L^\infty(0,\infty; \mathscr{B}(X_0, X_1)). \tag{7.70}$$

Setting with $u(t) := S_{\mathcal{B}}(t)\mathcal{A}$, we see that u satisfies **(a)** in Lemma 7.9 thanks to (ii) in X_0 and X_1. Furthermore, u satisfies **(b)** in Lemma 7.9 thanks to (7.70). Using Lemma 7.9, we conclude that condition (iii) holds. □

We come back to the proof of **(HS3)**. Gathering (i) and (ii) in X_1 in Proposition 7.8, we get that $(S_{\mathcal{B}}\mathcal{A})^{(*\ell)} * S_{\mathcal{B}} \in L^\infty_t(\mathscr{B}(X_1))$ for any $\ell \in \{0, \dots, N-1\}$, $N := [d/4] + 1$. Using that $\Theta \in L^1(0,\infty)$ and (iii) in Proposition 7.8, we deduce that $(S_{\mathcal{B}}\mathcal{A})^{(*N)} \in L^1(0,\infty; \mathscr{B}(X_0, X_1))$.

We may now handle the existence part of the first eigenvalue problem. On the one hand, recalling **(H2)**, we have $\mathcal{L}^*\phi_0 \ge 0$ with $\phi_0 = 1$ so that the condition (i) in Theorem 3.4 holds. On the other hand, the condition (ii) in Theorem 3.4 is an immediate consequence of **(HS3)** as emphasized in Remark 3.5-(1). As a conclusion, the hypotheses of Theorem 3.4 are thus met, and we deduce that there exists $(\lambda_1, f_1) \in \mathbb{R}_+ \times L^2_{m+}$ solution to the first eigenvalue problem. Because the strong maximum principle **(H4)** holds, we have $f_1 > 0$ on $\mathbb{R}^d$.

In order to prove the existence of a first positive eigenfunction for the dual problem, we argue in the following way. We start observing that we have the alternative: $\lambda_1 = 0$ or $\lambda_1 > 0$.
- In the first case, we may argue as in Remark 4.19. We indeed have in the same time $\mathcal{L}^*\phi_0 \ge 0$ and $\langle \mathcal{L}^*\phi_0, f_1\rangle = \langle \phi_0, \mathcal{L} f_1\rangle = 0$, so that $\mathcal{L}^*\phi_0 = 0$ because $f_1 > 0$. The function $\phi_1 := \phi_0$ is thus a solution to the first dual eigenvalue problem.
- In the second case $\lambda_1 > 0$, we may argue as in the case $\gamma = 1$ above. On the one hand, the operator $\mathcal{R}_{\mathcal{B}}(\alpha)$ is uniformly bounded in L^2_m for any $\alpha \ge \kappa_{\mathcal{B}} := \lambda_1/2 > 0$ and on the other hand the operator $\mathcal{W}(\alpha) := \mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A} : L^2_m \to H^1_m \cap L^2_{m^\sharp}$ is uniformly bounded for any $\alpha \ge \kappa_{\mathcal{B}}$ with $m = o(m^\sharp)$, so that $H^1_m \cap L^2_{m^\sharp} \subset L^2_m$ is compact. We may thus apply Theorem 2.21 and we conclude to the existence of a solution $(\lambda_1', f_1', \phi_1')$ to the eigentriplet problem.

The conditions **(H4)** and **(H5)** being true in a general situation as well as the conclusions of Lemma 7.1, as an intermediate conclusion, we have established under the general condition $\gamma > 0$ in (7.56) that yet the same conclusions as in section 7.2 hold true.

Quantitative stability. We now establish a quantitative stability estimate using the Doblin-Harris approach presented in Section 6 and yet used in the case of a bounded domain in Section 7.1. We first consider the more difficult case $\gamma \in (0,1)$, so that $\lambda_1 \ge \kappa_0 = 0$, and then explain the modifications to be made in order to deal with the case $\gamma \ge 1$. As explained just above, $\lambda_1 = 0$ corresponds to the conservative case $(\lambda_1, \phi_1) = (0, 1)$ which has been considered in [231]. We thus focus on the case $\lambda_1 > 0$ for which an adapted version of Theorem 7.3 already imply the exponential asymptotic stability **(E3$_1$)** in L^2_m with **non constructive rate**. We do not develop further this argument but rather establish a **a constructive sub-exponential asymptotic stability**.

Step1 - Lyapunov condition. We take $m = e^{a|x|^\gamma}$ with $0 < a < \gamma^{-1}$. From Lemma 7.7 or a direct computation, we have

$$\begin{aligned} \mathcal{L}^* m &= \Delta m + (c - \mathrm{div} b) m - b \cdot \nabla m \\ &\le (C\langle x\rangle^{\gamma-2} + |c| - 2a^*\langle x\rangle^{2\gamma-2}) m \\ &\le C_0 \mathbf{1}_{B_{\varrho_0}} - a^*\langle x\rangle^{2\gamma-2} m, \end{aligned}$$

for three positive constants $C = C(d)$, $C_0 = C_0(c, C, a, \gamma)$, $\varrho_0 = \varrho_0(c, C, a, \gamma)$ and with $a^* := (a\gamma - (a\gamma)^2)/2 > 0$. We now set $m_1 := m$ and $m_0 := a^*\langle x\rangle^{2\gamma-2} m$. We fix $T > 0$ and for $0 \le f_0 \in L^1_m$, we denote $f_t := \widetilde{S}_{\mathcal{L}}(t) f_0$. Recording that $\lambda_1 \ge 0$ and using the above pointwise estimate, we deduce

$$\int f_T m_1 + \int_0^T \int f_t m_0 dt \le \int f_0 m_1 + C_1 \int_0^T \int_{B_{\varrho_0}} f_t dt. \tag{7.71}$$

Because the same kind of pointwise estimate holds for $\mathcal{L}^* m_0$, we have

$$\int f_T m_0 \le \int f_t m_0 + C_1 \int_t^T \int_{B_{\varrho_0}} f_s ds$$

and integrating in time, we get

$$T \int f_T m_0 \le \int_0^T \int f_t m_0 dt + C_0 \int_0^T s \int_{B_{\varrho_0}} f_s ds.$$

Coming back to the first estimate, we deduce

$$\int f_T m_1 + T \int f_T m_0 \le \int f_0 m_1 + (C_1 + C_0 T) \int_0^T \int_{B_{\varrho_0}} f_t dt. \tag{7.72}$$

Step 2 - Pointwise estimates on ϕ_1. We define $\mathcal{B} := \mathcal{L} - C_0 \chi_{\varrho_0}$ which is the generator of a positive semigroup of contraction in L^1_m because of the above discussion. For $\lambda > 0$, $0 \le g \in L^1_m$ and $0 \le f \in L^1_m$ the solution to $(\lambda - \mathcal{B}) f = g$, we compute

$$\int g m = \int f (\lambda - \mathcal{B}^*) m \ge \int f (\lambda m + m_0) \ge \int f m_0,$$

from what we deduce

$$\|\mathcal{R}_{\mathcal{B}}(\lambda) f\|_{L^1_{m_0}} \le \|f\|_{L^1_m}, \quad \forall\, f \in L^1_m.$$

Now, we consider two weight functions m_1 and m_3 with $m_i := e^{a_i |x|^\gamma}$, $0 < a_1 < a_3 < \gamma^{-1}$, we denote $m_0 := a_1^* \langle x\rangle^{2\gamma-2} m_1$ and we compute

$$\begin{aligned} \|\mathcal{A}\mathcal{R}_{\mathcal{B}}(\lambda) f\|_{L^1_{m_3}} &\le C_0 \|\mathcal{R}_{\mathcal{B}}(\lambda) f\|_{L^1_{m_3}(B_{2\varrho_0})} \\ &\le C_1 \|\mathcal{R}_{\mathcal{B}}(\lambda) f\|_{L^1_{m_0}} \le C_1 \|f\|_{L^1_{m_1}}. \end{aligned}$$

By duality, we obtain

$$\|\mathcal{R}_{\mathcal{B}^*}(\lambda)\phi\|_{L^\infty_{m_1^{-1}}} \le \|\phi\|_{L^\infty_{m_0^{-1}}} \quad \text{and} \quad \|\mathcal{R}_{\mathcal{B}^*}\mathcal{A}^*(\lambda)\phi\|_{L^\infty_{m_1^{-1}}} \le C_1 \|\phi\|_{L^\infty_{m_3^{-1}}}, \tag{7.73}$$

for any $\lambda > 0$ and $\phi \in L^\infty_{m_0^{-1}}$. We also deduce from Proposition 7.8 the regularization estimate $(\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*})^N : L^2_{m^{-1}} \to L^\infty$. Let us now consider $0 \le \phi_1 \in L^2_{m_1^{-1}}$ the first eigenvector for the dual problem built in the preceding paragraph. From the eigenvalue equation

$$\mathcal{B}^*\phi_1 + \mathcal{A}^*\phi_1 = \mathcal{L}^*\phi_1 = \lambda_1 \phi_1,$$

we deduce that $\phi_1 = (\mathcal{R}_{\mathcal{B}^*}\mathcal{A}^*)\phi_1$, and iterating

$$\phi_1 = (\mathcal{R}_{\mathcal{B}^*}\mathcal{A}^*)^{N+1}\phi_1 = \mathcal{R}_{\mathcal{B}^*}(\mathcal{A}^*\mathcal{R}_{\mathcal{B}^*})^N \mathcal{A}^*\phi_1.$$

From the above regularization estimate and the first estimate in (7.73), we thus deduce that $\phi_1 \in L^\infty_{m_1^{-1}}$. Moreover, normalizing ϕ_1 and using the second estimate in (7.73), we may obtain

$$\|\phi_1\|_{L^\infty_{m_3{}^{-1}}} = 1 \ \text{ and } \ \|\phi\|_{L^\infty_{m_1^{-1}}} \le C_1. \tag{7.74}$$

We deduce

$$\begin{aligned} 1 &= \max\Big(\sup_{B_{\varrho_2}} \frac{|\phi_1|}{m_3}, \sup_{B^c_{\varrho_2}} \frac{|\phi_1|}{m_3}\Big) \\ &\le \max\Big(\sup_{B_{\varrho_2}} \frac{|\phi_1|}{m_3}, C_{0,1} \sup_{B^c_{\varrho_2}} \frac{m_1}{m_3}\Big), \end{aligned}$$

so that $\sup_{B_{\varrho_2}} \frac{|\phi_1|}{m_3} = 1$ by choosing $\varrho_2 := \max(\varrho_0, \varrho_1)$ with $C_1 e^{(a_1-a_3)\varrho_1^\gamma} = 1$. As a consequence, there exists $x_0 \in B_R$ such that $\phi_1(x_0) \ge 1$. On the other hand, using standard regularity result for elliptic equation in the ball B_{2R}, we obtain that $\phi_1 \in C^{0,1}(B_R) \cap W^{2,p}(B_R)$ for any $p \in [1, \infty)$ with constructive bound. Making use next of the Harnack inequality as at the end of Section 7.1 or using barrier functions as in in the proof of [231, Lem. 6.2], we classically deduce that

$$\phi_1 \ge z_\varrho \mathbf{1}_{B_\varrho}, \quad \forall\, \varrho > 0, \tag{7.75}$$

for a constructive constant $z_\varrho > 0$ (where we emphasize here and below the ϕ_1 always denote the normalized by (7.74) dual eigenvector).

Step 3 - Doblin-Harris estimate. We fix $T > 0$ (for instance $T := 1$) and $A > 0$ arbitrary. For $0 \le f_0 \in L^1_m$ such that $\|f_0\|_{L^1_m} \le A\|f_0\|_{L^1_{\phi_1}}$, we denote $f_t := \widetilde{S}_{\mathcal{L}}(t) f_0$. On the one hand, we have

$$\int f_t \phi_1 = \int f_0 \phi_1 \quad \text{and} \quad \int f_t m \le C_T \int f_0 m,$$

for any $t \in [0, T]$, the second estimate being an immediate consequence of (7.71). On the other hand, we define $\varepsilon(r) := \sup_{|x| \ge r}(m(x)/\phi_1(x))$ and we compute

$$\begin{aligned} \int_{B_\rho} f_t \phi_1 &= \int f_t \phi_1 - \int_{B^c_\rho} f_t \phi_1 \ge \int f_t \phi_1 - \varepsilon(\rho) \int f_t m \\ &\ge \int f_0 \phi_1 - C_T \varepsilon(\rho) \int f_0 m \ge \Big(1 - \frac{C_T}{A}\varepsilon(\rho)\Big) \int f_0 \phi_1 \ge \frac{1}{2} \int f_0 \phi_1, \end{aligned}$$

for any $t \in (0, T)$, by choosing $\rho := \rho(T, A) > 0$ large enough. In particular, there exists $x_0(t) \in B_\rho$ such that

$$f(t, x_0(t)) \ge \vartheta := \frac{1}{2} \frac{1}{\|\phi_1\|_{L^1(B_\rho)}} \int f_0 \phi_1.$$

Next, arguing exactly as in Section 7.1 or as in the proof of [231, Lem. 6.2], we deduce

$$\widetilde{S}_T f_0 \ge \eta_{T,A} \mathbf{1}_{B_1} [f_0]_{L^1_{\phi_1}}, \tag{7.76}$$

for some constructive constant $\eta_{T,A} > 0$.

8. Transport equations

The main aim of this part is to analysis the long time asymptotic of the solutions to the transport equation

$$\partial_t f + \mathrm{div}_y(af) = \mathscr{K}[f] - Kf \quad \text{in} \quad (0, \infty) \times \mathcal{O}, \tag{8.1}$$

on the function $f = f(t, y)$, $t \ge 0$, $y \in \mathcal{O}$, with $\mathcal{O} \subset \mathbb{R}^D$, $D \ge 1$, a smooth open connected set. We assume that $a = a(y)$, $a : \mathcal{O} \to \mathbb{R}^D$, $K = K(y)$, $K : \mathcal{O} \to \mathbb{R}_+$ and that the collision operator $\mathcal{K}$ is linear and defined by

$$(\mathscr{K} g)(y) := \int_{\mathcal{O}} k\, g_*\, dy_*, \tag{8.2}$$

for some kernel $k : \mathcal{O} \times \mathcal{O} \to \mathbb{R}_+$ and for any (conveniently) bounded function $g : \mathcal{O} \to \mathbb{R}$. Here and below, we use the common shorthands

$$g_* := g(y_*), \quad k := k(y, y_*), \quad k_* := k(y_*, y).$$

When $\mathcal{O} \ne \mathbb{R}^D$, the equation is complemented with a boundary condition which imposes the value of the trace $\gamma_- f$ of f on the incoming subsets of the boundary and takes the form

$$(\gamma_- f)(t, y) = \mathscr{R}[f(t, \cdot), \gamma_+ f(t, .)](y) \ \text{ on } \ (0, \infty) \times \Sigma_-. \tag{8.3}$$

Let us explain the meaning of the different terms involved in (8.3). We denote by $\Sigma := \partial\mathcal{O}$ the boundary set, by $d\sigma_y$ the Lebesgue measure on Σ, by $n : \Sigma \to \mathbb{S}^{D-1}$ the normal outward vector field, we write $n = n_y = n(y)$, and by Σ_- the incoming, Σ_+ the outgoing and Σ_0 the singular subsets of the boundary defined by

$$\Sigma_\pm := \{y \in \Sigma;\ \pm a(y) \cdot n_y > 0\}, \quad \Sigma_0 := \{y \in \Sigma;\ a(y) \cdot n_y = 0\}.$$

We denote $\gamma f = f_{|(0,\infty)\times\Sigma}$ the trace of f and $\gamma_\pm f := \mathbf{1}_{(0,\infty)\times\Sigma_\pm} \gamma f$ the trace restrictions on the incoming and outgoing sets. We then assume that the boundary operator $\mathscr{R}$ splits into two pieces $\mathscr{R}(g,h) = \mathscr{R}_{\mathcal{O}}(g) + \mathscr{R}_\Sigma(h)$, where

$$(\mathscr{R}_{\mathcal{O}} g)(y) = \int_{\mathcal{O}} g(y_*) r_{\mathcal{O}}(y, dy_*), \quad (\mathscr{R}_\Sigma h)(y) = \int_{\Sigma_+} h(y_*) r_\Sigma(y, dy_*), \tag{8.4}$$

for a domain transition kernel $r_{\mathcal{O}} : \Sigma_- \times \mathscr{B}_{\mathcal{O}} \to [0,\infty]$, a boundary transition kernel $r_\Sigma : \Sigma_- \times \mathscr{B}_{\Sigma_+} \to [0,\infty]$ and for any (conveniently) bounded functions $g : \mathcal{O} \to \mathbb{R}$ and $h : \Sigma_+ \to \mathbb{R}$, where $\mathscr{B}_E$ stands for the set of Borel subsets of E.

In the next sections we will first consider the trace problem for a general force field a and next the well-posedness for the transport equation with given inflow at the boundary and with reflection condition at the boundary. We will also revisit the characteristic method for general force field a. We will next consider the Krein-Rutman problem still for a general force field a, but making strong simplification assumptions on the kernel operators $\mathscr{K}$ and $\mathscr{R}$. We will next explain how the classical age structured equation falls into the present framework. We will come back to more specific physical situations concerning the growth-fragmentation equation and the kinetic relaxation equation with more general and physically relevant hypothesis on the kernel in parts 9 and 10.

8.1. **The trace problem.**

In this section, we are concerned with the trace problem associated to a (mainly stationary) transport equation for a general vector field $a : \mathcal{O} \to \mathbb{R}^D$ for which we only assume

$$a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}}), \tag{8.5}$$

where we recall that $\mathcal{O} \subset \mathbb{R}^D$, $D \geq 1$, is a smooth open connected set. The regularity needed on the domain is formulated in the following way: we assume that there exists $n : \mathcal{O} \to \mathbb{R}^D$, $y \mapsto n(y)$ a vector field belonging to $W^{1,\infty}(\mathcal{O})$ and which coincides with the previously defined unit outgoing normal vector field on Σ and satisfies $\|n\|_{L^\infty} = 1$. In that situation, it is well-known that the above vector field is the restriction of a vector field $a \in W^{1,1}_{\rm loc}(\mathbb{R}^D)$ (where we abuse notations denoting the restriction and the extension in the same way). We also consider the associated differential equation

$$\frac{dY}{dt} = a(Y), \qquad Y(0) = y, \tag{8.6}$$

and then define the characteristic flow $Y_t = Y(t,y)$, for any $y \in \mathcal{O}$, which is the solution to (8.6) on a maximal time interval $(t_-(y), t_+(y))$ where $t_-(y) < 0 < t_+(y)$ are defined by $t_- := -\,\mathrm{t}_{\mathbf{b}}$ and $t_+ := t_{\mathbf{f}}$, the backward exit time is defined by

$$\mathrm{t}_{\mathbf{b}}(y) := \sup\{\tau > 0;\ Y_{-t}(y) \in \mathcal{O},\ \forall\, t \in [0,\tau]\} \in (0,+\infty] \tag{8.7}$$

and the forward exit time is defined by

$$t_{\mathbf{f}}(y) := \sup\{\tau > 0;\ Y_t(y) \in \mathcal{O},\ \forall\, t \in [0,\tau]\} \in (0,+\infty]. \tag{8.8}$$

The real number $t_{\ell t}(y) := \mathrm{t}_{\mathbf{b}}(y) + t_{\mathbf{f}}(y) \in (0,\infty]$ corresponds to the *"life time"* of the characteristic flow in $\mathcal{O}$ going by y. The construction of the flow (Y_t) is classical when a is a Lipschitz function and we refer to [139, Thm. II.3] for a more general situation which corresponds to the assumptions we will make in the present work (see also Lemma 8.14 below).

For a solution $g : \mathcal{O} \to \mathbb{R}$ to the transport equation

$$a \cdot \nabla_y g = G \quad \text{in} \quad \mathcal{O}, \tag{8.9}$$

for a given source term $G : \mathcal{O} \to \mathbb{R}$, we wish to define the trace γg of g on the boundary set Σ. Similarly, for a solution $g : (0,T) \times \mathcal{O} \to \mathbb{R}$, $T \in (0,+\infty]$, to the transport equation

$$\partial_t g + a \cdot \nabla_y g = G \quad \text{in} \quad (0,T) \times \mathcal{O}, \tag{8.10}$$

for a given source term $G : (0,T) \times \mathcal{O} \to \mathbb{R}$, we wish to define the trace γg of g on the boundary set $(0,T) \times \Sigma$. It is worth emphasizing that the trace will be in fact only defined out of the singular set Σ_0 and thus only on the boundary set $\Sigma \backslash \Sigma_0$.

We start by recalling several possible definitions of the trace of a function g satisfying (8.9) when

$$a \in W^{1,s}_{\rm loc}(\bar{\mathcal{O}}), \quad g \in L^p_{\rm loc}(\bar{\mathcal{O}}), \quad G \in L^q_{\rm loc}(\bar{\mathcal{O}}), \quad s,p,q \in [1,\infty]. \tag{8.11}$$

Here and below, we denote by $L(E)$ the Lebesgue space of measurable functions $g : E \to \bar{\mathbb{R}} := [-\infty,+\infty]$ with typically $E = \mathcal{O}$ or $E \subset \Sigma$, and by $L^0(E) = L^0(E,\mu) \subset L(E)$ the subset of almost everywhere finite measurable functions on a measurable space $(E,\mathcal{A},\mu)$.

Definition 8.1. *We say that a function g on $\mathcal{O}$ satisfying* (8.9) *and* (8.11) *admits a trace if one of the following assertions holds true:*

- **Extension of the restriction on the boundary.** *There exists $\gamma g \in L^r_{\rm loc}(\Sigma\backslash\Sigma_0)$, $r \in [1,\infty]$, such that*
$$g_{n|\Sigma\backslash\Sigma_0} \to \gamma g \quad in \quad L^r_{\rm loc}(\Sigma\backslash\Sigma_0)$$
for any sequence (g_n) satisfying
$$g_n \in C^1_c(\bar{\mathcal{O}}), \qquad g_n \to g \quad in \quad L^p_{\rm loc}(\bar{\mathcal{O}}), \qquad a(y)\cdot\nabla_y g_n \to G \quad in \quad L^q_{\rm loc}(\bar{\mathcal{O}}). \tag{8.12}$$
- **Characteristics.** *There exists a measurable function γg on $\Sigma\backslash\Sigma_0$ such that for a.e. $y \in \mathcal{O}$ satisfying $t_-(y) > -\infty$, there holds*
$$g(y) = \gamma g(Y(t_-(y),y)) + \int_{t_-(y)}^0 G(Y(t,y))\,dt, \tag{8.13}$$
and for a.e. $y \in \mathcal{O}$ satisfying $t_+(y) < \infty$, there holds
$$g(y) = \gamma g(Y(t_+(y),y)) - \int_0^{t_+(y)} G(Y(t,y))\,dt. \tag{8.14}$$
- **Green formula.** *There exists $\gamma g \in L^r_{\rm loc}(\Sigma\backslash\Sigma_0)$, $r \in [1,\infty]$, such that*
$$\int_{\mathcal{O}} \big(G\,\varphi + g\,{\rm div}(a\varphi)\big)\,dy = \int_\Sigma \gamma g\,\varphi\,a(y)\cdot n(y)\,d\sigma_y, \tag{8.15}$$
for any $\varphi \in C^1_c(\bar{\mathcal{O}}\backslash\Sigma_0)$.
- **Renormalized Green formula.** *There exists a measurable function γg on $\Sigma\backslash\Sigma_0$ such that*
$$\int_{\mathcal{O}} \big(\beta'(g)\,G\,\varphi + \beta(g)\,{\rm div}(a\varphi)\big)\,dy = \int_\Sigma \beta(\gamma g)\,\varphi\,a\cdot n\,d\sigma_y, \tag{8.16}$$
for any $\varphi \in C^1_c(\bar{\mathcal{O}})$ and any $\beta \in C^1(\mathbb{R})$ such that $\beta' \in L^\infty(\mathbb{R})$.

Remark 8.2. *(1) In order that the first definition makes sense, we implicitly assume that there exists at least one sequence (g_n) which satisfies* (8.12)*. That last fact corresponds to the density of $C^1_c(\bar{\mathcal{O}})$ in the Sobolev space $\{g \in L^p(\mathcal{O});\ a(y)\cdot\nabla_y g \in L^q(\mathcal{O})\}$, which is true as we will see in Lemma 8.5 below under the regularity assumptions made on a and $\mathcal{O}$. It is worth emphasizing that the last convergence in* (8.12) *may require additional integrability assumption, typically $a \in W^{1,s}(\mathcal{O})$ with $1/r \ge 1/p + 1/s$. Such a definition has been introduced in* [37] *for a C^1 vector field a. It is also the point of view adopted dy Cessenat in* [99, 100] *in the case of the neutronic operator, see also* [126, chap. XXI] *or Agoshkov* [2, 3, 4].

(2) In order that the second definition makes sense, we implicitly assume that the set of points $y \in \mathcal{O}$ such that the characteristic $Y_t(y)$ hits the boundary on Σ_0 has zero measure in $\mathcal{O}$. It is indeed the case thanks to the Sard theorem under enough regularity assumption on a and $\mathcal{O}$, see [37, Prop. 2.3]*. It is worth emphasizing that what we really need in order to write* (8.13) *and* (8.14) *is that $t \mapsto G(Y(t,y)) \in L^1(t_-(y),t_+(y))$ for a.e. $y \in \mathcal{O}$. We also mention that this characteristics description leads to a layer cake formula linking the integral of a function on the domain to the integral of its trace on the boundary. Such a definition has been widely used in kinetic theory for constructing DiPerna-Lions renormalized solution, see* [138, 199, 18] *and the references therein. For the classical kinetic operator this trace approach is developed by Arkeryd, Cercignani and co-authors in* [86, 96, 16, 17] *while for more general (but still regular) vector fields, the approach has been developed in* [37, 348, 43, 184] *and more recently by Arlotti et al. in* [20, 21, 22, 23, 24].

(3) In order that the third definition makes sense, we need that $a \cdot n\,\gamma g \in L^1_{\rm loc}(\Sigma)$ *and* $p \ge s'$. *In some situation, this third definition is in some sense the weakest: it makes sense also when* $\gamma g \in M^1_{\rm loc}(\Sigma\backslash\Sigma_0)$ *for instance and can be relevant under the weak assumption* $a, {\rm div} a \in L^{p'}_{\rm loc}(\bar{\mathcal O})$ *as it is the case in the early works on weak solution to the Vlasov-Poisson equation in* [322, 194, 1, 361]. *It is also easier to handle than the two first definitions because of the way it connects the function* g *and its trace.*

(4) We will adopt the last definition which extends up to the boundary the renormalization technique introduced in [139]. *It is more general and adapted to the weak regularity assumption made on the vector field* a *than the two first definitions and we recover the third definition by just letting* $\beta(s) \to s$ *when the conditions of integrability make the limit well defined. Such a kind of definition has been introduced in* [270, 272] *for kinetic equations and in* [70, 10] *for transport equations.*

We start with a trace result in a L^∞ framework. We denote by $C^1_{\rm pw}(\mathbb R)$ the space of continuous functions $\beta : \mathbb R \to \mathbb R$ with piecewise continuous derivative.

Theorem 8.3. *Assume that* $g \in L^\infty(\mathcal O)$, $a \in W^{1,1}_{\rm loc}(\bar{\mathcal O})$ *and* $G \in L^1_{\rm loc}(\bar{\mathcal O})$ *satisfy the transport equation* (8.9) *in the distributional sense. Then, there exists a unique function*

$$\gamma g \in L^\infty(\Sigma\backslash\Sigma_0; d\sigma_y), \quad \|\gamma g\|_{L^\infty} \le \|g\|_{L^\infty},$$

which satisfies the renormalized Green formula

$$\int_{\mathcal O} \big(\beta'(g)\, G\, \varphi + \beta(g)\, {\rm div}(a\varphi)\big)\, dy = \int_\Sigma \beta(\gamma g)\, \varphi\, a\cdot n\, d\sigma_y, \tag{8.17}$$

for any $\varphi \in C^1_c(\bar{\mathcal O})$ *and any* $\beta \in C^1_{\rm pw}(\mathbb R)$. *As a consequence, renormalization and trace operations commute:*

$$\gamma\, \beta(g) = \beta(\gamma\, g), \quad \forall\, \beta \in C^1_{\rm pw}(\mathbb R). \tag{8.18}$$

Remark 8.4. *(1) Because of the very general assumption* (8.5) *made on the vector field* $a : \mathcal O \to \mathbb R^D$ *which is exactly the one made in the DiPerna-Lions theory for transport equation in the whole space developed in* [139], *the above trace result slightly improves the similar trace result established by Boyer in* [70, Thm. 3.1], *where an additional assumption* $a \cdot n \in L^\zeta(\partial\mathcal O)$, $\zeta > 1$, *is made.*

(2) An alternative approach has been developed by Ambrosio and co-authors by assuming weaker bound on Da *but stronger bound on* a. *More precisely, denoting by* $\mathcal M_\infty$ *the set of vector fields* $a \in L^\infty(\mathcal O)$ *such that* ${\rm div} a \in M^1(\mathcal O)$, *it is established in* [10, Prop. 3.2] *that there exists a linear and bounded mapping* ${\rm Tr} : \mathcal M_\infty(\mathcal O) \to L^\infty(\partial\mathcal O)$ *such that* ${\rm Tr} a = n \cdot a_{|\partial\mathcal O}$ *when* $a \in C^1(\bar{\mathcal O})$. *The proof relies on Ambrosio's extension to a BV framework in* [8] *of the famous Di Perna-Lions improvement* [139, Lem. II.1] *of Friedrichs' type Lemma on the estimate of the commutator between directional derivative and convolution (see Lemma 8.5 below). Moreover, it is also established in* [10] *(see in particular* [10, Thm. 4.2]*) that*

$${\rm Tr}(a\beta(g)) = \beta\Big(\frac{{\rm Tr}(ag)}{{\rm Tr} a}\Big){\rm Tr}(a), \quad \forall\, \beta \in C^1(\mathbb R),$$

for any $a \in BV(\mathcal O) \cap L^\infty(\mathcal O)$ *and* $g \in L^\infty(\mathcal O)$ *such that* $ag \in \mathcal M_\infty$. *The above formula is then nothing but* (8.18) *when* $a \in W^{1,1}(\mathcal O) \cap L^\infty(\mathcal O)$.

Before coming to the proof of Theorem 8.3, we state one technical but fundamental result. We define the mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ by

$$\rho_\varepsilon(z) = \tfrac{1}{\varepsilon^d}\, \rho(z/\varepsilon), \quad 0 \le \rho \in \mathcal D(\mathbb R^d), \quad {\rm supp}\, \rho \subset B_1, \quad \int_{\mathbb R^N} \rho(z)\, dz = 1, \tag{8.19}$$

and for any $u \in L^1_{\rm loc}(\bar{\mathcal O})$, $v_\varepsilon \in C_c(\mathbb R^D)$, ${\rm supp}\, v_\varepsilon \subset B_\varepsilon$, we introduce the convolution-translation function $u *_\varepsilon v_\varepsilon$ defined by

$$(u *_\varepsilon v_\varepsilon)(y) := \int_{\mathcal O} u(z)\, v_\varepsilon(y - 2\varepsilon\, n(y) - z)\, dz. \tag{8.20}$$

Lemma 8.5. *For* $g \in L^p_{\rm loc}(\bar{\mathcal O})$, $p \in [1,\infty]$, $a \in W^{1,p'}_{\rm loc}(\bar{\mathcal O})$ *and* $G \in L^1_{\rm loc}(\bar{\mathcal O})$ *satisfying* (8.9) *in the distributional sense, the sequence* (g_ε) *defined by* $g_\varepsilon := g *_\varepsilon \rho_\varepsilon$ *satisfies*

$$g_\varepsilon \in W^{1,\infty}_{\rm loc}(\bar{\mathcal O}), \quad G_\varepsilon := a \cdot \nabla g_\varepsilon \to a \cdot \nabla g \ \ in\ L^1_{\rm loc}(\bar{\mathcal O}),$$

as $\varepsilon \to 0$, *and*

$$g_\varepsilon \to g \quad in \quad L^p_{\rm loc}(\bar{\mathcal O}), \quad if\ p<\infty,$$
$$g_\varepsilon \to g \quad in \quad L^1_{\rm loc}(\bar{\mathcal O}), \quad (g_\varepsilon)\ bounded\ in\ L^\infty_{\rm loc}(\bar{\mathcal O}), \quad if\ p=\infty.$$

We skip the proof of Lemma 8.5 since it follows by just repeating the proofs of [139, Lem. II.1], [271, Lem. 1] or [70, Lem. 3.1].

Proof of Theorem 8.3. Let us fix $\chi \in \mathcal D(\bar{\mathcal O})$ such that $0 \le \chi \le 1$ and denote $R>0$ a real number such that $\operatorname{supp}\chi \subset B_R$. We observe that $\chi{\rm sign}(a\cdot n) \in L^1(\Sigma)$. From Gagliardo trace theorem [170, Teor. 1.II], there exists $\psi \in W^{1,1}(\mathcal O)$ such that $\gamma\psi = \chi{\rm sign}(a\cdot n)$ and $\operatorname{supp}\psi \subset B_R$. Denoting $T_1 : \mathbb R \to [-1,1]$ the truncation function which is odd and is defined by $T_1(\sigma) = \sigma \wedge 1$ for any $\sigma \ge 0$, we see that $\gamma T_1(\psi) = T_1(\gamma\psi) = \gamma\psi$, and thus we may assume $\psi \in L^\infty(\mathcal O)$ up to replacing ψ by $T_1(\psi)$. As a consequence, there exists a sequence (ψ_k) of $W^{1,\infty}(\mathcal O)$ such that $\psi_k \to \psi$ in $W^{1,1}(\mathcal O)$, with (ψ_k) bounded in $L^\infty(\mathcal O)$, ${\rm supp}\psi_k \subset B_R$, and $\gamma\psi_k \to \chi\,{\rm sign}(a\cdot n)$ in $L^1(\Sigma)$, with $(\gamma\psi_k)$ bounded in $L^\infty(\Sigma)$.

Let us then consider the sequences (g_ε) and (G_ε) defined in Lemma 8.5. The classical Green formula for Lipschitz functions writes

$$\begin{aligned}
&\int_\Sigma (g_{\varepsilon|\Sigma} - g_{\varepsilon'|\Sigma})^2\, |a\cdot n|\, \chi\, d\sigma_y \\
&= \int_\Sigma (g_{\varepsilon|\Sigma} - g_{\varepsilon'|\Sigma})^2\, a\cdot n\, \psi_k\, d\sigma_y + \int_\Sigma (g_{\varepsilon|\Sigma} - g_{\varepsilon'|\Sigma})^2\, [|a\cdot n|\,\chi - a\cdot n\,\psi_k]\, d\sigma_y \\
&= \int_{\mathcal O} [2\,(g_\varepsilon - g_{\varepsilon'})\,(G_\varepsilon - G_{\varepsilon'})\,\psi_k\, dy + (g_\varepsilon - g_{\varepsilon'})^2 {\rm div}(a\psi_k)]\, dy \\
&\quad + \int_\Sigma (g_{\varepsilon|\Sigma} - g_{\varepsilon'|\Sigma})^2\, [|a\cdot n|\,\chi - a\cdot n\,\psi_k]\, d\sigma_y \\
&\le 4\|\psi_k\|_{L^\infty} \|g\|_{L^\infty} \|G_\varepsilon - G_{\varepsilon'}\|_{L^1(B_R)} + \|\psi_k\|_{W^{1,\infty}} \int_{B_R} (|a| + |{\rm div}a|)(g_\varepsilon - g_{\varepsilon'})^2\, dy \\
&\quad + 2\|g\|^2_{L^\infty} \|(a\cdot n)\gamma\psi_k - \chi\,|a\cdot n|\|_{L^1(\Sigma)},
\end{aligned}$$

for any $\varepsilon>0$ and $k\ge 1$. We deduce that $(g_{\varepsilon|\Sigma})$ is a Cauchy sequence in $L^2(|a\cdot n|\,\chi\, d\sigma)$. From the fact that (g_ε) is bounded in $L^\infty(\mathcal O)$, we deduce that the sequence (γg_ε) is also bounded in $L^\infty(\Sigma)$. As a consequence, there exists a function $\gamma g \in L^\infty(\Sigma)$ such that $\gamma g_\varepsilon \to \gamma g$ in $L^2(|a\cdot n|\,\chi\, d\sigma)$. Next, we may write the Green formula

$$\int_{\mathcal O} \big[G_\varepsilon\, \varphi + g_\varepsilon {\rm div}(a\varphi)\big]\, dy = \int_\Sigma \gamma\, g_\varepsilon\, \varphi\, a\cdot n\, d\sigma_y,$$

for any test function $\varphi \in C^1_c(\bar{\mathcal O})$, and we may pass to the limit as $\varepsilon \to 0$. We deduce that the Green formula

$$\int_{\mathcal O} \big(G\,\varphi + g\,{\rm div}(a\varphi)\big)\, dy = \int_\Sigma \gamma g\, \varphi\, a(y)\cdot n(y)\, d\sigma_y, \tag{8.21}$$

holds for any $\varphi \in C^1_c(\bar{\mathcal O})$. That clearly uniquely defines the trace function γg on $\Sigma\backslash\Sigma_0$.

Now, on the one hand, from the DiPerna-Lions renormalizing theory [139, proof of Corollary II.1], we know that $\beta(g) \in L^\infty(\mathcal O)$ satisfies the transport equation

$$a(y)\cdot\nabla_y \beta(g) = \beta'(g) G \quad \text{in} \quad \mathcal D'(\mathcal O), \tag{8.22}$$

for any renormalizing function $\beta \in {\rm Lip}(\mathbb R)$ and any test function $\varphi \in C^1_c(\mathcal O)$. Using the already established trace result, we know that there exists $\gamma\beta(g) \in L^\infty(\Sigma\backslash\Sigma_0)$ such that

$$\int_{\mathcal O} \big[\beta'(g) G\,\varphi + \beta(g){\rm div}(a\varphi)\big]\, dy = \int_\Sigma \gamma\beta(g)\,\varphi\, a\cdot n\, d\sigma_y, \tag{8.23}$$

for any test function $\varphi \in C^1_c(\bar{\mathcal O})$. On the other hand, from the classical Green formula for Lipschitz functions and because $\beta(g_\varepsilon)_{|\Sigma} = \beta(g_{\varepsilon|\Sigma})$, we have

$$\int_{\mathcal O} \big[\beta'(g_\varepsilon) G_\varepsilon\,\varphi + \beta(g_\varepsilon){\rm div}(a\varphi)\big]\, dy = \int_\Sigma \beta(g_{\varepsilon|\Sigma})\,\varphi\, a\cdot n\, d\sigma_y,$$

for any renormalizing function $\beta \in \mathrm{Lip}(\mathbb{R})$ and any test function $\varphi \in C^1_c(\bar{\mathcal{O}})$. Using that

$$\beta'(g_\varepsilon)G_\varepsilon \to \beta'(g)G, \quad \beta(g_\varepsilon) \to \beta(g), \quad \beta(g_{\varepsilon|\Sigma}) \to \beta(\gamma g)$$

respectively in $L^1_{\rm loc}(\bar{\mathcal{O}})$ and in $L^1_{\rm loc}(\Sigma)$, and that the two last sequences are bounded in L^∞, we may pass to the limit $\varepsilon \to 0$ in the last Green formula, and we thus get

$$\int_{\mathcal{O}} \big[\beta'(g)G\,\varphi + \beta(g)\mathrm{div}(a\varphi)\big]\,dy = \int_\Sigma \beta(\gamma\, g)\,\varphi\, a\cdot n\,d\sigma_y.$$

Together with (8.23) and by uniqueness of the trace function, we conclude to $\gamma\beta(g) = \beta(\gamma g)$. □

Let us state several variants of the preceding trace result. For the transport evolution equation (8.10) a first possible trace result writes as follows.

Theorem 8.6. *Assume that $g \in L^\infty((0,T)\times\mathcal{O})$, $a \in L^1(0,T;W^{1,1}_{\rm loc}(\bar{\mathcal{O}}))$ and $G \in L^1_{\rm loc}([0,T]\times\bar{\mathcal{O}})$ satisfy the evolution transport equation (8.10) in the distributional sense. Then,*

$$g \in C([0,T];L^1_{\rm loc}(\bar{\mathcal{O}}))$$

and there exists a unique function

$$\gamma g \in L^\infty((0,T)\times\Sigma\backslash\Sigma_0;\,dt\otimes d\sigma_y), \quad \|\gamma g\|_{L^\infty} \le \|g\|_{L^\infty},$$

which satisfies the renormalized Green formula

$$\int_{t_0}^{t_1}\int_{\mathcal{O}} \big(\beta'(g)\,G\,\varphi + \beta(g)\,[\partial_t\varphi + \mathrm{div}(a\varphi)]\big)\,dydt \tag{8.24}$$
$$= \Big[\int_{\mathcal{O}} \beta(g(t,\cdot))\varphi dy\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1}\int_\Sigma \beta(\gamma g)\,\varphi\, a\cdot n\,d\sigma_y dt,$$

for any $\varphi \in C^1_c([0,T]\times\bar{\mathcal{O}})$, any $t_0,t_1 \in [0,T]$ and any $\beta \in C^1_{\rm pw}(\mathbb{R})$. In particular renormalization and trace operations commute: (8.18) holds.

We skip the proof of Theorem 8.6 which is very similar to the proof Theorem 8.3 using the slight modifications that one can find in [271, Thm. 2] or [70, Thm. 3.1]. Under the slightly more regularity assumption $a \in W^{1,1}_{\rm loc}([0,T]\times\bar{\mathcal{O}})$, Theorem 8.6 is a direct corollary of Theorem 8.3 applied to the for field $(1,a(t,y))$ on the open set $(0,T)\times\mathcal{O}$.

For some additional function $b:\mathcal{O}\to\mathbb{R}$, another possible variant is the following trace result for the stationary transport equation

$$a\cdot\nabla_y g + bg = G \quad \text{in} \quad \mathcal{O} \tag{8.25}$$

in the renormalized framework as introduced by DiPerna and Lions in [139]. Assuming $a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, $b,G \in L^1_{\rm loc}(\bar{\mathcal{O}})$, we say that $g \in L^1_{\rm loc}(\bar{\mathcal{O}})$ is a renormalized solution to the transport equation (8.25) if

$$a\cdot\nabla_y\beta(g) + b\beta'(g)g = \beta'(g)G, \tag{8.26}$$

in the distributional sense for any renormalizing function $\beta \in C^1_*(\mathbb{R})$ the set of $C^1(\mathbb{R})$ functions such that β admits some finite limits in $\pm\infty$ and $s\mapsto \langle s\rangle\beta'(s)$ is bounded on $\mathbb{R}$, in particular $C^1_*(\mathbb{R}) \subset C^1_b\mathbb{R})$. We also denote by $\beta \in C^1_{{\rm pw},*}(\mathbb{R})$ the C^1 piecewise variant of $C^1_*(\mathbb{R})$. We will repeatedly use the family of functions $\beta_\delta \in C^1_*(\mathbb{R})$ defined by $\beta_\delta(s) := s/(1+\delta s^2)^{1/2}$ for any $\delta \in (0,1]$. We observe that $\beta'_\delta(s) = (1+\delta s^2)^{-3/2}$, so that $s\beta'(s)\to 0$ as $s\to\pm\infty$.

Let us start formulating some basic facts on renormalized solutions to equation (8.25).

Lemma 8.7. *Assume $a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, $b,G \in L^1_{\rm loc}(\bar{\mathcal{O}})$.*
(1) If $g \in L^1_{\rm loc}(\bar{\mathcal{O}})$ and $\alpha(g)$ satisfies equation (8.26) for one renormalizing function $\alpha:\mathbb{R}\to(-1,1)$ which is bijective and belongs to $C^1_{{\rm pw},*}(\mathbb{R})$ then $\beta(g)$ satisfies equation (8.26) for any renormalizing function $\beta \in C^1_{{\rm pw},*}(\mathbb{R})$.
(2) If $g_1,g_2 \in L^1_{\rm loc}(\bar{\mathcal{O}})$ are two renormalized solutions to the transport equations

$$a\cdot\nabla_y g_i + bg_i = G_i \in L^1_{\rm loc}(\bar{\mathcal{O}}),$$

then $g := g_1+g_2$ is a renormalized solution to the transport equation (8.25) with $G := G_1+G_2$.

(3) If g is a renormalized solution to the transport equation (8.25) *and* $\Phi, c \in L^\infty(\mathcal{O})$ *satisfy*

$$a \cdot \nabla_y \Phi = c$$

in the distributional sense, then $h := g e^{-\Phi}$ *satisfies*

$$a \cdot \nabla_y h + (b + c) h = G e^{-\Phi} \tag{8.27}$$

in the renormalized sense.

Proof of Lemma 8.7. We briefly sketch the proof and for more details we refer to [139], in particular to [139, Lem. II.2]. It is worth mentioning that only the case $b \in L^\infty(\mathcal{O})$ is considered in [139], but it readily extends to our framework. Assertion (1) is just a consequence of the chain rule $\beta'(s) = (\beta \circ \alpha^{-1})'(\alpha(s))\alpha'(s)$ for smooth enough solutions and thus for any solution thanks to Lemma 8.5 (see the proof of [139, Cor. II.1]) and to a standard approximation procedure in order to deal with piecewise C^1 functions. In order to establish (2), we consider two renormalized solutions g_i, a renormalized function $\beta \in C^1_*(\mathbb{R})$ and we write

$$a \cdot \nabla_y \beta(\beta_\delta(g_1) + \beta_\delta(g_2)) = \beta'(\beta_\delta(g_1) + \beta_\delta(g_2))[(G_1 - b g_1)\beta_\delta'(g_1) + (G_2 - b g_2)\beta_\delta'(g_2)],$$

where we have added the two renormalized formulations (8.26) associated to $\beta_\delta(g_i)$ and renormalized once more the resulting solution using (1). Letting $\delta \to 0$, we immediately obtain

$$a \cdot \nabla_y \beta(g_1 + g_2) = \beta'(g_1 + g_2)[G_1 + G_2 - b(g_1 + g_2)]$$

in the distributional sense. For proving (3), we introduce the mollified sequence (g_ε) and (Φ_ε) defined as in the statement of Lemma 8.5 so that

$$a \cdot \nabla g_\varepsilon + b g_\varepsilon = G_\varepsilon, \quad a \cdot \nabla \Phi_\varepsilon = c_\varepsilon$$

with $G_\varepsilon \to G$ and $c_\varepsilon \to c$ in $L^1_{\rm loc}(\bar{\mathcal{O}})$ as $\varepsilon \to 0$. The smooth function $h_\varepsilon := g_\varepsilon e^{-\Phi_\varepsilon}$ satisfies

$$a \cdot \nabla_y h_\varepsilon + (b + c_\varepsilon) h_\varepsilon = G_\varepsilon e^{-\Phi_\varepsilon}$$

and then

$$a \cdot \nabla_y \beta(h_\varepsilon) + \beta'(h_\varepsilon)(b + c_\varepsilon) h_\varepsilon = \beta'(h_\varepsilon) G_\varepsilon e^{-\Phi_\varepsilon}$$

for any $\beta \in C^1_*(\mathbb{R})$. Passing to the limit $\varepsilon \to 0$, we obtain the renormalized formulation of (8.27). ☐

We now generalize the trace result to the framework of renormalized solutions.

Theorem 8.8. *Assume that* $a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, $b, G \in L^1_{\rm loc}(\bar{\mathcal{O}})$ *and that* $g \in L^1_{\rm loc}(\bar{\mathcal{O}})$ *is a renormalized solution to the transport equation* (8.25). *Then there exists a unique function*

$$\gamma g \in L(\Sigma \backslash \Sigma_0;\, d\sigma_y)$$

which satisfies the renormalized Green formula

$$\int_{\mathcal{O}} \big(\beta'(g)\,(G - bg)\,\varphi + \beta(g)\,\mathrm{div}(a\varphi)\big)\,dy = \int_\Sigma \beta(\gamma g)\,\varphi\, a \cdot n\, d\sigma_y, \tag{8.28}$$

for any $\varphi \in C^1_c(\bar{\mathcal{O}})$ *and any* $\beta \in C^1_{{\rm pw},*}(\mathbb{R})$.

Proof of Theorem 8.8. We fix $\beta_1 : \mathbb{R} \to \mathbb{R}$ defined by $\beta_1(s) := s(1 + s^2)^{-1/2}$, so that $\beta_1 \in C^1_b(\mathbb{R})$ and $\beta_1 : \mathbb{R} \to (-1, 1)$ is a bijection. Since then $\beta_1(g) \in L^\infty(\mathcal{O})$ and $\beta'(g)(G - bg) \in L^1_{\rm loc}(\bar{\mathcal{O}})$, we know from Theorem 8.3 that $\gamma\beta_1(g)$ is well defined in $L^\infty(\Sigma\backslash\Sigma_0)$ through the Green formula

$$\int_{\mathcal{O}} \big[\beta_1'(g)(G - bg)\,\varphi + \beta_1(g)\,\mathrm{div}(a\varphi)\big]\,dy = \int_\Sigma \gamma\beta_1(g)\,\varphi\, a \cdot n\, d\sigma_y,$$

for any test function $\varphi \in C^1_c(\bar{\mathcal{O}})$. We set $\gamma g := \beta_1^{-1}(\gamma\beta_1(g)) \in L(\Sigma\backslash\Sigma_0;\, d\sigma_y)$, with the convention $\beta_1^{-1}(\pm 1) = \pm\infty$. For any $\beta \in C^1_{{\rm pw},*}(\mathbb{R})$, we then deduce

$$\gamma\beta(g) = \gamma[(\beta \circ \beta_1^{-1})(\beta_1(g))] = \beta \circ \beta_1^{-1}(\gamma\beta_1(g)) = \beta(\gamma g),$$

where we have used the renormalization result stated in Theorem 8.3 and the chain rule (1) stated in Lemma 8.7 in the second equality and the very definition of γg in the third equality. In other words, the renormalized Green formula (8.28) holds. ☐

Remark 8.9. *(1) We will see in Section 8.4 that under the same conditions as in Theorem 8.8 the information on γg can be slightly improved, in particular $\gamma g \in L^0(\Sigma \backslash \Sigma_0)$.*

(2) Theorem 8.8 in particular holds when we assume $a \in W^{1,p'}_{\rm loc}(\bar{\mathcal O})$, $b \in L^{p'}_{\rm loc}(\bar{\mathcal O})$, $G \in L^1_{\rm loc}(\bar{\mathcal O})$ and $g \in L^p_{\rm loc}(\bar{\mathcal O})$ satisfy the transport equation (8.26) *in the distributional sense. Indeed, in that situation one knows from the DiPerna-Lions renormalizing theory* [139, Cor. II.2] *that g is also a renormalized solution to the transport equation* (8.26) *(in the above sense).*

(3) Assuming more interior integrability on the functions g, b, G and a, we may deduce more accurate information on γg. A typical example, is that

$$\int_{\Sigma \cap B_R} |\gamma g|^r (|a \cdot n| \wedge 1)^2 d\sigma_y < \infty,$$

for some $r \in [1, \infty)$ and any $R > 0$, under the additional assumption

$$|g|^r (|{\rm div} a| + |a \cdot \nabla T_1(a \cdot n)| + |b|) \in L^1_{\rm loc}(\bar{\mathcal O}), \quad |g|^{r-1}|G| \in L^1_{\rm loc}(\bar{\mathcal O}).$$

The proof follows by choosing $\varphi := T_1(a \cdot n)\,\chi$, $\chi \in C^1_c(\bar{\mathcal O})$, $0 \le \chi \le 1$, and $\beta_k(s) = (|s| \wedge k)^r$ in the associated Green formula (8.16)*, and then to pass to the limit $k \to \infty$.*

(4) Even more integrability on $\gamma_\pm g$ is available on one part of the boundary if additional integrability assumption is made on $\gamma_\mp g$ on the other part of the boundary. A typically example, is that

$$\int_{\Sigma_\pm \cap B_R} |\gamma_\pm g|^r |a \cdot n| d\sigma_y < \infty,$$

under the additional assumption

$$|g|^r (|{\rm div} a| + |a| + |b|) \in L^1_{\rm loc}(\bar{\mathcal O}), \quad |g|^{r-1}|G| \in L^1_{\rm loc}(\bar{\mathcal O}), \quad |\gamma_\mp g|^r a \cdot n \in L^1_{\rm loc}(\bar\Sigma_\mp).$$

The proof follows by choosing $\varphi \in C^1_c(\bar{\mathcal O})$, $0 \le \varphi \le 1$, and $\beta_k(s) = (|s| \wedge k)^r$ in the associated Green formula (8.16)*, and then to pass to the limit $k \to \infty$.*

(5) The results stated in Lemma 8.7, in Theorem 8.8 and in points (1), (2), (3) and (4) above may be straightforwardly adapted to the evolution transport equation (8.10)*. We refer to* [271, 270, 272, 70] *where such results are established in a slightly less general framework. Let us emphasize again that when $a \in W^{1,1}_{\rm loc}([0,T] \times \bar{\mathcal O})$ (what it is the case in the time independent case when a satisfies* (8.5)*) this extension is directly implied by Theorem 8.8 applied to the vector field $(1, a(t, y))$ in the open set $(0,T) \times \mathcal O$.*

8.2. **Well-posedness for the transport equation with given inflow at the boundary.**

We deduce from the previous trace theorems and standard tools the well-posedness for the transport equation with several boundary conditions. In this section, we deal with the transport equation with given inflow at the boundary. We are first concerned with the stationary transport equation

$$\lambda g + a \cdot \nabla g + bg = G \ \text{ in } \mathcal O, \quad \gamma_- g = \mathfrak g \ \text{ on } \Sigma_-, \tag{8.29}$$

for a real number $\lambda \in \mathbb R$ large enough, a given source term $G : \mathcal O \to \mathbb R$ and a boundary term $\mathfrak g : \Sigma_- \to \mathbb R$. As we will see, our analysis also apply to the associated dual equation

$$\lambda \varphi - a \cdot \nabla \varphi + (b - {\rm div} a)\varphi = \Phi \ \text{ in } \mathcal D'(\mathcal O), \quad \gamma_+ \varphi = \psi \ \text{ on } \Sigma_+. \tag{8.30}$$

We will also consider the related evolution equation

$$\begin{cases} \dfrac{\partial g}{\partial t} + a \cdot \nabla g + bg = G \ \text{ on } \ (0,T) \times \mathcal O, \\ \gamma_- g = \mathfrak g \ \text{ on } \ (0,T) \times \Sigma_-, \quad g(0,\cdot) = g_0 \ \text{ on } \ \mathcal O, \end{cases} \tag{8.31}$$

with given source term $G : (0,T) \times \mathcal O \to \mathbb R$, boundary term $\mathfrak g : (0,T) \times \Sigma_- \to \mathbb R$ and initial datum $g_0 : \mathcal O \to \mathbb R$.

A possible simple framework consists in imposing the following conditions

$$a \in W^{1,1}_{\rm loc}(\bar{\mathcal O}), \quad b \in L^1_{\rm loc}(\bar{\mathcal O}), \tag{8.32}$$

and

$$b_-, {\rm div} a \in L^\infty(\mathcal O), \quad \frac{a}{\langle y \rangle \langle b_+ \rangle} \in L^1(\mathcal O) + L^\infty(\mathcal O). \tag{8.33}$$

The first condition on a is useful for the renormalization trick and the definition of the trace, the second condition is needed for the existence results in a L^p framework when $p \neq \infty$ and the last condition is used for proving the uniqueness result. In order to be able to apply our results to more general (and realistic) situations, we rather consider the following situation. We assume that a and b satisfy (8.32), and defining

$$\varpi = \varpi_p := b - \frac{1}{p}\mathrm{div}a - a \cdot \frac{\nabla m}{m}, \tag{8.34}$$

for some smooth enough weight function $m : \bar{\mathcal{O}} \to (0, \infty)$ and some exponent $p \in [1, \infty]$, we assume

$$\varpi_- \in L^\infty(\mathcal{O}), \quad b, \mathrm{div}a \in L^\infty_{\langle \varpi_+ \rangle^{-1}}(\mathcal{O}), \quad \frac{a}{\langle y \rangle} \in L^\infty_{\langle \varpi_+ \rangle^{-1}}(\mathcal{O}) + L^1(\mathcal{O}) \tag{8.35}$$

In the case $p = 1$ and $p = \infty$, we will additionnaly assume $(\varpi_q)_- \in L^\infty$ for any $q \in (1, \infty)$. It is worth emphasizing that condition (8.35) automatically holds when $m \equiv 1$ and a, b satisfy (8.33). We also define the critical real number

$$\lambda^*_p = \lambda^*_p(a, b, m) := \|\varpi_-\|_{L^\infty}, \tag{8.36}$$

and we may observe that

$$\lambda^*_{p'}(-a, b - \mathrm{div}a, m^{-1}) = \lambda^*_p(a, b, m), \tag{8.37}$$

what links up the primal and the dual problems. In order to shorten notations, we introduce the three weight functions

$$m_{\mathcal{O}} := m\langle \varpi_+ \rangle^{1/p}, \quad \widetilde{m}_{\mathcal{O}} := m\langle \varpi \rangle^{-1/p'}, \quad m_\Sigma := m|a \cdot n|^{1/p}. \tag{8.38}$$

We start with a general discussion about a priori bounds, formal representation formulas and general stability results.

A priori estimates. Consider a solution g to the stationary equation (8.29). For any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}_+$ and any function $\varphi : \bar{\mathcal{O}} \to (0, \infty)$, we (at least) formally have

$$\int_{\mathcal{O}} [(\lambda + b)g\beta'(g)\varphi - \beta(g)(\mathrm{div}(a\varphi))] + \int_{\Sigma_+} a \cdot n\beta(\gamma_+ g)\varphi = \int_{\mathcal{O}} \beta'(g)G\varphi + \int_{\Sigma_-} |a \cdot n|\beta(\mathfrak{g})\varphi.$$

Choosing $\beta(s) := |s|^p$, $1 \le p < \infty$, and $\varphi := m^p$, we get in particular

$$\int_{\mathcal{O}} |g|^p m^p (\lambda + \varpi) + \frac{1}{p}\int_{\Sigma_+} |\gamma_+ g|^p m^p a \cdot n = \int_{\mathcal{O}} Gg|g|^{p-2}m^p + \frac{1}{p}\int_{\Sigma_-} |\mathfrak{g}|^p m^p |a \cdot n|. \tag{8.39}$$

For $p = 1$ and any $\lambda > \lambda^*_1$, we get

$$\int_{\mathcal{O}} |g| m\{\lambda - \lambda^*_1 + \varpi_+\} + \int_{\Sigma_+} |\gamma_+ g| m_\Sigma \le \int_{\mathcal{O}} |G| m + \int_{\Sigma_-} |\mathfrak{g}| m_\Sigma.$$

For $p \in (1, \infty)$, we split $G = G_1 + G_2$ and using the Young inequality, we have

$$\int_{\mathcal{O}} Gg|g|^{p-2}m^p \le \varepsilon_1 \int g^p m^p + \varepsilon_2 \int g^p m^p_{\mathcal{O}} + \frac{1}{p}\frac{1}{(p'\varepsilon_1)^{p/p'}} \int G^p_1 m^p + \frac{1}{p}\frac{1}{(p'\varepsilon_2)^{p/p'}} \int G^p_2 \widetilde{m}^p_{\mathcal{O}},$$

for any $\varepsilon_i > 0$. For $\lambda > \lambda^*_p$, we choose $\varepsilon_1 := (\lambda - \lambda^*_p)/2$ and $\varepsilon_2 := 1/2$, we get

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{O}} |g|^p m^p_{\mathcal{O}} + \frac{\lambda - \lambda^*_p}{2}\int_{\mathcal{O}} |g|^p m^p + \frac{1}{p}\int_{\Sigma_+} |\gamma_+ g|^p m^p_\Sigma \\ \le \frac{2^{p-1}}{p(p')^{p/p'}}(\lambda - \lambda^*_p)^{1-p}\int_{\mathcal{O}} |G_1|^p m^p + \frac{1}{p(p'/2)^{p/p'}}\int_{\mathcal{O}} |G_2|^p \widetilde{m}^p_{\mathcal{O}} + \frac{1}{p}\int_{\Sigma_-} |\mathfrak{g}|^p m^p_\Sigma. \end{aligned}$$

We thus deduce

$$\|g\|_{L^p_m} \le \frac{C_p}{\lambda - \lambda^*_p}\|G_1\|_{L^p_m} + \frac{C_p}{(\lambda - \lambda^*_p)^{1/p}}(\|\mathfrak{g}\|_{L^p_{m_\Sigma}} + \|G_2\|_{L^p_{\widetilde{m}_{\mathcal{O}}}}) \tag{8.40}$$

and

$$\|g\|_{L^p_{m_{\mathcal{O}}}} + \|\gamma_+ g\|_{L^p_{m_\Sigma}} \le \frac{C_p}{(\lambda - \lambda^*_p)^{1/p'}}\|G_1\|_{L^p_m} + C_p(\|\mathfrak{g}\|_{L^p_{m_\Sigma}} + \|G_2\|_{L^p_{\widetilde{m}_{\mathcal{O}}}}), \tag{8.41}$$

for some numerical constant $C_p \in (0,\infty)$ and any $p \in (1,\infty)$ and also for $p = 1$ because of the previous estimate. Finally, for $\lambda > \lambda^*_\infty$ and $\alpha \in (0, \lambda - \lambda^*_\infty)$, we may proceed exactly as above, but throwing also away the contribution of ϖ_+, and we may thus first write

$$\Big(\lambda - \|\varpi_-\|_{L^\infty} - \frac{\alpha^{p'}}{p'}\Big)\|g\|^p_{L^p_m(\mathcal{O})} \le \frac{1}{p}\big(\|G/\alpha\|^p_{L^p_m(\mathcal{O})} + \|\mathfrak{g}|a\cdot n|^{1/p}\|^p_{L^p_m(\Sigma_-)}\big), \tag{8.42}$$

for any $p \in (1,\infty)$ large enough in such a way that the coefficient in front of $\|g\|^p_{L^p_m(\mathcal{O})}$ is positive. Taking the power $1/p$ in both sides and passing first to the limit $p \nearrow \infty$ and next to the limit $\alpha \nearrow \lambda - \lambda^*_\infty$, we end with

$$\|g\|_{L^\infty_m(\mathcal{O})} \le \max\Big(\frac{1}{\lambda - \lambda^*_\infty}\|G\|_{L^\infty_m(\mathcal{O})}, \|\mathfrak{g}\|_{L^\infty_m(\Sigma_-)}\Big). \tag{8.43}$$

Consider now a solution to the evolution equation (8.31). For any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}_+$ and any test function $\varphi : [0,\infty) \times \bar{\mathcal{O}} \to (0,\infty)$, we (at least) formally have

$$\begin{aligned}\int_0^t \int_{\mathcal{O}} \big(\beta'(g)G\varphi - \beta'(g)g\varphi + \beta(g)[\partial_t\varphi + \operatorname{div}(a\varphi)]\big)dyds \\ = \Big[\int_{\mathcal{O}} \beta(g(s,y))\varphi(s,y)dy\Big]_0^t + \int_0^t \int_\Sigma \beta(\gamma g)\varphi\, a\cdot n\, d\sigma_y ds.\end{aligned} \tag{8.44}$$

Choosing $\beta(s) := |s|^p$, $1 \le p < \infty$, and $\varphi(t,y) := m^p(y)$, we get in particular

$$\begin{aligned}\int_{\mathcal{O}} |g(t)|^p m^p + \int_0^t \int_{\Sigma_+} a\cdot n\, |\gamma_+ g|^p m^p + p\int_0^t \int_{\mathcal{O}} |g|^p m^p \varpi \\ = \int_{\mathcal{O}} |g_0|^p m^p + p\int_0^t \int_{\mathcal{O}} g|g|^{p-2} G m^p + \int_0^t \int_{\Sigma_-} |\mathfrak{g}|^p m^p |a\cdot n|.\end{aligned}$$

Using the Young inequality

$$p\int_{\mathcal{O}} g|g|^{p-2}Gm^p \le \frac{p}{p'}\int_{\mathcal{O}} |g|^p m^p_{\mathcal{O}} + \int_{\mathcal{O}} |G|^p \widetilde{m}^p_{\mathcal{O}}$$

and the Gronwall lemma, we then deduce

$$\begin{aligned}\|g(t)\|^p_{L^p_m} + \int_0^t e^{p\kappa(t-s)}(\|g_s\|^p_{L^p_{m_{\mathcal{O}}}} + \|\gamma_+ g_s\|^p_{L^p_{m_\Sigma}})\, ds \\ \le e^{p\kappa t}\|g_0\|^p_{L^p_m} + \int_0^t e^{p\kappa(t-s)}(\|G_s\|^p_{L^p_{\widetilde{m}_{\mathcal{O}}}} + \|\mathfrak{g}_s\|^p_{L^p_{m_\Sigma}})\, ds, \quad \forall\, t \ge 0,\end{aligned} \tag{8.45}$$

with $\kappa := \|\langle\varpi_-\rangle\|_{L^\infty}$. Passing to the limit $p \to \infty$, we also have

$$\max\big(\|g(t)\|_{L^\infty_m}, \|\gamma_+ g(t)\|_{L^\infty_m}\big) \le e^{\kappa t}\max\big(\|g_0\|_{L^\infty_m}, \sup_{[0,t]}(\|G_s\|_{L^\infty_m} + \|\mathfrak{g}_s\|_{L^\infty_m})\big), \tag{8.46}$$

for any $t \ge 0$.

Representation formulas. In a smooth functions framework or still formally, one classically knows that the solution g to the evolution transport equation (8.31) is given by

$$\begin{aligned} g(t,y) \;=\; & g_0(Y_{-t}(y))e^{-\int_0^t b(Y_{s-t}(y))ds}\mathbf{1}_{t<t_{\mathbf{b}}} + \mathfrak{g}(t-t_{\mathbf{b}}, y_{\mathbf{b}})e^{-\int_0^{t_{\mathbf{b}}} b(Y_{s-t_{\mathbf{b}}}(y))ds}\mathbf{1}_{t>t_{\mathbf{b}}} \\ & + \int_0^{t'_{\mathbf{b}}} G(s, Y_{s-t'_{\mathbf{b}}}(y))e^{-\int_0^{t'_{\mathbf{b}}-s} b(Y_{-\tau}(y))d\tau}du,\end{aligned} \tag{8.47}$$

where we recall that the characteristics Y and the backward exit time $t_{\mathbf{b}}$ are defined in (8.6)-(8.7) and we denote $t'_{\mathbf{b}} := \min(t, t_{\mathbf{b}})$. Similarly, the solution g to the stationary transport equation (8.29) is given by

$$g(y) = \mathfrak{g}(y_{\mathbf{i}}(y)))e^{-\int_0^{t_{\mathbf{b}}} b(Y_{-s}(y))ds} + \int_0^{t_{\mathbf{b}}} G(Y_{-s}(y))e^{-\int_0^s b(Y_{-\tau}(y))d\tau}du. \tag{8.48}$$

Alternatively, we may define a semigroup S_b (say on $L^\infty(\mathcal{O})$) by

$$(S_b(t)f_0)(y) := \begin{cases} f_0(Y_{-t}(y))\exp(-\int_0^t b(Y_{\tau-t}(y))d\tau), & \text{if } t \in (0, t_{\mathbf{b}}(y)), \\ 0 \ \ \text{otherwise}.\end{cases} \tag{8.49}$$

Given $f_0 : \mathcal{O} \to \mathbb{R}$, the function $f(t,y) := (S_b(t)f_0)(y)$ is thus a solution to the evolution equation

$$\partial_t f + a \cdot \nabla f + bf = 0 \text{ in } (0,\infty) \times \mathcal{O}, \quad \gamma_- f = 0 \text{ on } (0,\infty) \times \Sigma_-.$$

For $G, \tilde g : \mathcal{O} \to \mathbb{R}$, we next define

$$g := \tilde g + \int_0^\infty e^{-\lambda t} S_b(t) \tilde G \, dt, \tag{8.50}$$

with $\tilde G := G - \lambda \tilde g + a \cdot \nabla_x \tilde g - b \tilde g$. By construction, it is a solution to the stationary transport equation (8.29).

Stability. We present some stability and continuity results. Generalizing slightly [272, Definitions 2.6 and 3.1], we say that a sequence (g_n) of $L(E)$ converges in the renormalized sense to g, we note $g_n \overset{r}{\rightharpoonup} g$, if for any $\delta \in \Delta$, $\Delta \subset (0,1]$, $0 \in \overline{\Delta}$, there exists $\bar\beta_\delta \in L^\infty(E)$ such that

$$\beta_\delta(g_n) \rightharpoonup \bar\beta_\delta \ \ *\sigma(L^\infty, L^1) \text{ as } n \to \infty \ \text{ and } \beta_1(\bar\beta_\delta) \to \beta_1(g) \ L^1_{\rm loc}(\bar{\mathcal{O}}) \text{ as } \delta \to 0. \tag{8.51}$$

We may observe that in particular $g_n \rightharpoonup g$ weakly $L^1(E)$ or $g_n \to g$ a.e. in E implies $g_n \overset{r}{\rightharpoonup} g$. We refer to [272] and the references therein for more material about the subject.

Proposition 8.10. *Let us consider four sequences (g_k) of $L^1_{\rm loc}(\bar{\mathcal{O}})$, (a_k) of $W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, (b_k) and (G_k) of $L^1_{\rm loc}(\bar{\mathcal{O}})$ such that*

$$a_k \cdot \nabla \beta(g_k) + b_k \beta'(g_k) g_k = \beta'(g_k) G_k \quad \textit{in} \quad \mathcal{D}'(\mathcal{O}),$$

for any $k \geq 1$ and any $\beta \in C^1_*(\mathbb{R})$ and four functions $g \in L^1_{\rm loc}(\bar{\mathcal{O}})$, $a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, $b, G \in L^1_{\rm loc}(\bar{\mathcal{O}})$ such that $a_k \to a$ in $W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$ and $b_k \to b$, $G_k \to G$ in $L^1_{\rm loc}(\bar{\mathcal{O}})$. Let us denote by Σ_0 the boundary singular subset associated to a.
(1) If $g_k \to g$ a.e. in $\mathcal{O}$ then g satisfies (8.26) *for any $\beta \in C^1_*(\mathbb{R})$ and, up to the extraction of a subsequence, $\gamma g_k \to \gamma g$ a.e. on $\Sigma \backslash \Sigma_0$.*
(2) If $g_k \rightharpoonup g$ weakly in $L^1_{\rm loc}(\bar{\mathcal{O}})$ then g satisfies (8.26) *and, up to the extraction of a subsequence, $\gamma g_k \overset{r}{\rightharpoonup} \gamma g$ on $\Sigma \backslash \Sigma_0$.*

Remark 8.11. *Because of Remark 8.9-(5) and the time independence made on a and b in* (8.35), *exactly the same stability result holds for the evolution equation* (8.31) *as a consequence of Proposition 8.10.*

Proof of Proposition 8.10. We split the proof into two steps.
Step 1. We establish (1). We fix β_1 as in the proof of Theorem 8.8 and we write the Green formula

$$\int_{\mathcal{O}} \big[\beta_1'(g_k) G_k \, \varphi - \beta_1'(g_k) g_k b_k \, \varphi + \beta_1(g_k) \, \mathrm{div}(a_k \varphi)\big] \, dy = \int_\Sigma \beta_1(\gamma g_k) \, \varphi \, a_k \cdot n \, d\sigma_y,$$

for any test function $\varphi \in C^1_c(\bar{\mathcal{O}})$. There exists $\bar\beta_1 \in L^\infty(\Sigma \backslash \Sigma_0)$ and a subsequence (g_{n_k}) such that $\beta_1(\gamma g_{n_k}) \rightharpoonup \bar\beta_1$ weakly $\sigma(L^\infty, L^1)$. Passing to the limit in the above equation, we get

$$\int_{\mathcal{O}} \big[\beta_1'(g) G \, \varphi - \beta_1'(g) g b \, \varphi + \beta_1(g) \, \mathrm{div}(a \varphi)\big] \, dy = \int_\Sigma \bar\beta_1 \, \varphi \, a \cdot n \, d\sigma_y.$$

From Lemma 8.7 and Theorem 8.8, we deduce that $\bar\beta_1 = \beta_1(\gamma g)$, so that $\beta_1(\gamma g_n) \rightharpoonup \beta_1(\gamma g)$ weakly $\sigma(L^\infty, L^1)$. Fixing now $\beta_2 := \beta_1^2 \in C^1_*(\mathbb{R})$ and repeating the same argument, we get $\beta_2(\gamma g_n) \rightharpoonup \beta_2(\gamma g)$ weakly $\sigma(L^\infty, L^1)$. We then immediately deduce that

$$(\beta_1(\gamma g_n) - \beta_1(\gamma g))^2 \ \rightharpoonup \ 0 \quad \text{weakly } \sigma(L^\infty, L^1),$$

so that $\beta_1(\gamma g_n) \to \beta_1(\gamma g)$ in $L^1_{\rm loc}(\Sigma \backslash \Sigma_0)$. We conclude by using that β_1 is one-to-one.
Step 2. We establish (2). We fix β_δ as defined just before the statement of Lemma 8.7 and we write the Green formula

$$\int_{\mathcal{O}} \big[\beta_\delta'(g_k) G_k \, \varphi + \beta_\delta(g_k) \, \mathrm{div}(a \varphi)\big] \, dy = \int_\Sigma \beta_\delta(\gamma g_k) \, \varphi \, a \cdot n \, d\sigma_y,$$

for any test function $\varphi \in C^1_c(\bar{\mathcal{O}})$. There exist $\overline{\beta}_\delta, \widetilde{\beta}_\delta, \overline{\beta'}_\delta \in L^\infty(\mathcal{O})$, $\overline{\gamma\beta}_\delta \in L^\infty(\Sigma \backslash \Sigma_0)$ and a subsequence (g_{n_k}) such that $\beta_\delta(g_{n_k}) \rightharpoonup \overline{\beta}_\delta$, $g_{n_k} \beta_\delta'(g_{n_k}) \rightharpoonup \widetilde{\beta}_\delta$, $\beta_\delta'(g_{n_k}) \rightharpoonup \overline{\beta'}_\delta$ and $\beta_\delta(\gamma g_{n_k}) \rightharpoonup \overline{\gamma\beta}_\delta$ weakly $\sigma(L^\infty, L^1)$. Passing to the limit in the above equation, we get

$$a \cdot \nabla_y \overline{\beta}_\delta + b \widetilde{\beta}_\delta = \overline{\beta'}_\delta G \quad \text{in} \quad \mathcal{D}'(\mathcal{O}), \quad \gamma \overline{\beta}_\delta = \overline{\gamma\beta}_\delta \quad \text{on} \quad \Sigma \backslash \Sigma_0.$$

From the fact that (g_k) is locally uniformly integrable, we classically deduce that

$$\overline{\beta}_\delta, \widetilde{\beta}_\delta \to g \quad \text{in} \quad L^1_{\rm loc}(\mathcal{O}), \quad \overline{\beta'}_\delta G \to G \quad \text{in} \quad L^1_{\rm loc}(\mathcal{O}),$$

as $\delta \to 0$. More precisely, the two first convergences come from the elementary inequalities

$$\forall\, M > 0,\ \exists\, c_M > 0, \quad |s - \beta_\delta(s)| \le |s - s\beta'_\delta(s)| \le c_M \delta + |s| \mathbf{1}_{|s|>M},$$

for any $s \in \mathbb{R}$, $\delta \in (0,1)$, and the last convergence comes from the convexity inequality $\overline{\beta'}_\delta \ge \beta'_\delta(g)$ and the elementary inequalities

$$\forall\, M > 0,\ \exists\, c_M > 0, \quad 0 \le 1 - \beta'_\delta(s) \le c_M \delta + \mathbf{1}_{|s|>M}, \quad \forall\, s \in \mathbb{R},\ \forall \delta \in (0,1).$$

From Step 1, we deduce that g satisfies (8.26) and that, up to the extraction of a subsequence, $\overline{\gamma\beta}_\delta = \gamma\overline{\beta}_\delta \to \gamma g$ a.e. on $\Sigma \backslash \Sigma_0$. Using the Cantor diagonal process, we obtain that there exist two sequences (δ_m) and (g_{n_k}) such that $\delta_m \searrow 0$ and $\gamma g_{n_k} \stackrel{r}{\rightharpoonup} \gamma g$ in the renormalized sense associated to (δ_m). ☐

Existence. We establish two existence results of solutions to the transport equation (8.29).

Lemma 8.12 (Existence in L^∞_m)**.** *We assume that a and b satisfy* (8.35) *with $p = \infty$ and some weight function $m : \bar{\mathcal{O}} \to [1,\infty)$. For any $\lambda > \lambda^*_\infty$ and any given functions $G \in L^\infty_m(\mathcal{O})$ and $\mathfrak{g} \in L^\infty_m(\Sigma_-)$, there exists $g \in L^\infty_m(\mathcal{O})$ solution to* (8.29) *in the distributional sense. This solution satisfies* (8.29) *in the renormalized sense, the weak maximum principle, namely*

$$g \ge 0 \text{ in } \mathcal{O} \quad \text{if} \quad \mathfrak{g} \ge 0 \text{ on } \Sigma_- \text{ and } G \ge 0 \text{ in } \mathcal{O}, \tag{8.52}$$

and the L^∞_m estimate (8.43)*.*

Proof of Lemma 8.12. The proof follows [270, Lem. 3] using [37, Thm. 2.3]; we only sketch it. Under the stronger regularity assumption $a, b \in C^1_b(\bar{\mathcal{O}})$, $G \in C^1_c(\mathcal{O})$, $\mathfrak{g} := \tilde{g}_{|\Sigma_-}$ with $\tilde{g} \in C^1_c(\bar{\mathcal{O}})$, both definitions (8.48) and (8.50) provide a classical (and thus also renormalized) solution g to (8.29). In such a situation, we may justify the computations made in the above a priori estimates paragraph and we conclude that g satisfies the L^∞ estimate (8.43). In the general case for a, b, $\mathfrak{g}$ and G, we introduce some sequences (a^ε), (b^ε), $(\mathfrak{g}^\varepsilon)$ and (G^ε) of regular and approximating functions so that we may apply the first step above. In that way, we build a sequence (g^ε) of renormalized solutions to the approximated problem which is uniformly bounded and thus converges (up to the extraction of a subsequence) in the weakly $*\sigma(L^\infty, L^1)$ sense to a function $g \in L^\infty(\mathcal{O})$ satisfying (8.46). We then immediately conclude by passing to the limit $\varepsilon \to 0$ thanks to Proposition 8.10. ☐

We give a first version of an existence result in a L^p framework with strong assumption on the boundary condition.

Lemma 8.13 (Existence in L^p)**.** *We assume that a and b satisfy* (8.35) *for some $p \in [1,\infty)$ and some weight function $m : \bar{\mathcal{O}} \to [1,\infty)$. For any $\lambda > \lambda^*_p$, $G \in L^p_{\widetilde{m}_{\mathcal{O}}}(\mathcal{O})$ and $\mathfrak{g} \in L^p_{m_\Sigma}(\Sigma_-)$, there exists $g \in L^p_{m_{\mathcal{O}}}(\mathcal{O})$ a renormalized solution to the transport equation* (8.29)*. This one satisfies* (8.40)*,* (8.52) *and $\gamma_+ g \in L^p_{m_\Sigma}(\Sigma_+)$.*

Proof of Lemma 8.13. We argue similarly as during the proof of Lemma 8.12. When $\mathfrak{g} = \tilde{g}_{|\Sigma_-}$ with $\tilde{g}$, a, b, G smooth and with compact support, the classical solution built above satisfies (8.40), and thus

$$\|g\|_{L^p_{m_{\mathcal{O}}}(\mathcal{O})} \lesssim \|G\|_{L^p_{\widetilde{m}_{\mathcal{O}}}} + \|\mathfrak{g} m\|_{L^\infty} \|a\|^{1/p}_{L^1(\operatorname{supp}\mathfrak{g})}. \tag{8.53}$$

For $p > 1$, and under the general conditions (8.35) on a and b, but still assuming $\mathfrak{g} = \tilde{g}_{|\Sigma_-}$ and $G, \tilde{g} \in C^1_c(\bar{\mathcal{O}})$, we may introduce two sequences (a_ε) and (b_ε) of smooth functions approximating a and b. Since the resulting solution g_ε satisfies (8.53), so that the sequence (g_ε) is bounded in $L^p_{m_{\mathcal{O}}}$, we may argue with the same (compactness) argument as in the proof of Lemma 8.12. We then conclude to the existence of a (renormalized) solution $g \in L^p_{m_{\mathcal{O}}}$ to the transport equation (8.29) satisfying (8.40). Still for $p > 1$, but assuming $G \in L^p_{\widetilde{m}_{\mathcal{O}}}$ and $\mathfrak{g} \in L^p_{m_\Sigma}(\Sigma_-)$, we may introduce two sequences (G_ε) and $(\mathfrak{g}^\varepsilon)$ of smooth functions approximating G and $\mathfrak{g}$. Thanks to (8.40), the associated sequence of solutions (g_ε) is bounded (and better it is a Cauchy sequence) in $L^p_{m_{\mathcal{O}}}(\mathcal{O})$ and we conclude again to the existence of a (renormalized) solution $g \in L^p_{m_{\mathcal{O}}}(\mathcal{O})$ to the transport equation (8.29) satisfying (8.40). Finally, in the case $p = 1$ and $\lambda > \lambda^*_1$, we may find $q > 1$ small enough such that $\lambda > \lambda^*_q$. For $G, \mathfrak{g} \in L^1 \cap L^q$, the last step imply the existence of a renormalized solution $g \in L^q_{m_\mathcal{O}}(\mathcal{O})$ to the transport equation (8.29). Renormalizing the equation, we deduce that g satisfies (8.40) for $p = 1$. When $G, \mathfrak{g} \in L^1$, we introduce two sequences (G_ε) and $(\mathfrak{g}^\varepsilon)$ of $L^1 \cap L^q$ functions approximating G and $\mathfrak{g}$, and using (8.40) for $p = 1$, we deduce that the resulting sequence (g_ε) is a Cauchy sequence in $L^1_{m_\mathcal{O}}(\mathcal{O})$. We easily conclude again. Finally $\gamma_+ g \in L^p_{m_\Sigma}(\Sigma_+)$ from (8.41) (see also Remark 8.9-(4)). □

Uniqueness. We present now a uniqueness result.

Lemma 8.14 (Uniqueness)**.** *We assume that a and b satisfy* (8.35) *for some exponent $p \in [1, \infty]$ and some weight function $m : \bar{\mathcal{O}} \to [1, \infty)$ as well as for $p = 1$ and $m \equiv 1$. We additionally assume* $\operatorname{div} a \in L^\infty_{\rm loc}(\bar{\mathcal{O}})$ *(what is automatically true under assumption* (8.33)*). With obvious notations, for any $\lambda > \max(\lambda^*_p(m), \lambda^*_1(1))$, and any solution $g \in L^p_{m_\mathcal{O}}(\mathcal{O})$ to the transport equation*

$$\lambda g + a \cdot \nabla g + bg = 0 \quad in \quad \mathcal{D}'(\mathcal{O}), \quad \gamma_- g = 0 \quad on \quad \Sigma_-, \tag{8.54}$$

we have $g \equiv 0$.

Proof of Lemma 8.14. We main follow the proof of [139, Cor. II.1]. We fix $\beta \in W^{1,\infty}(\mathbb{R})$, $\beta(0) = 0$, in such a way that $\beta(g) \in L^p_{m_\mathcal{O}} \cap L^\infty$ is a solution to

$$(\lambda + b) g \beta'(g) + a \cdot \nabla \beta(g) = 0 \ \text{ in } \ \mathcal{D}'(\mathcal{O}), \quad \gamma_- \beta(g) = 0 \ \text{ on } \ \Sigma_-.$$

For any $\psi \in C_c(\mathcal{O})$ and any $\lambda > \lambda^*_{m,p}$, we solve in $L^{p'}_{m^{-1}\langle \varpi_+ \rangle^{1/p'}} \cap L^\infty$ the dual problem

$$\lambda \varphi - a \cdot \nabla \varphi + (b - \operatorname{div} a) \varphi = \psi \ \text{ in } \ \mathcal{D}'(\mathcal{O}), \quad \gamma_+ \varphi = 0 \ \text{ on } \ \Sigma_+, \tag{8.55}$$

thanks to Lemma 8.12 and Lemma 8.13, where we observe that, because of (8.37), the necessary condition on λ in these results is precisely the one made here. For $\chi \in C^1_c(\mathbb{R}^D)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$, and $R > 0$, we define $\chi_R(x) := \chi(x/R)$. Using the Green formula (8.21), we have

$$0 = \int_\mathcal{O} ((\lambda + b)\varphi - \psi) \beta(g) \chi_R - \int_\mathcal{O} (\lambda + b) \varphi g \beta'(g) \chi_R + \int_\mathcal{O} \varphi \beta(g) \frac{a}{R} \cdot (\nabla \chi)_R.$$

Because on the one hand $\varphi \beta(g) \in L^1_{\langle \varpi_+ \rangle} \cap L^\infty$ and on the other hand $a/R \cdot (\nabla \chi)_R \to 0$ a.e. and is bounded in $L^\infty_{\langle \varpi_+ \rangle^{-1}} + L^1$ we deduce that the last term vanishes when $R \to \infty$. Using also that $b \varphi g \in L^1$ thanks to (8.35), we may pass to the limit $R \to \infty$ in the above equation and we get

$$0 = \int_\mathcal{O} ((\lambda + b)\varphi - \psi) \beta(g) - \int_\mathcal{O} (\lambda + b) \varphi g \beta'(g).$$

We take $\beta := \beta_\delta$ and we observe that $\langle b \rangle |\varpi| |\beta_\delta(g) - g \beta'_\delta(g)| \le \langle b \rangle |\varphi g| \in L^1(\mathcal{O})$. We may then pass to the limit $\delta \to 0$ in the last equation, and we get

$$0 = - \int_\mathcal{O} \psi g, \quad \forall \psi \in C_c(\mathcal{O}),$$

from which we conclude that $g \equiv 0$. □

We come to the time dependent transport equation by formulating a general continuity result.

Proposition 8.15. *Assume that $a \in W^{1,1}_{\rm loc}(\bar{\mathcal{O}})$, $b \in L^1_{\rm loc}(\bar{\mathcal{O}})$, $G \in L^1_{\rm loc}([0,T] \times \bar{\mathcal{O}})$. Any renormalized solution $g \in L^1_{\rm loc}([0,T] \times \bar{\mathcal{O}})$ to the first equation in* (8.31)*, meaning*

$$\frac{\partial}{\partial t} \beta(g) + a \cdot \nabla \beta(g) + \beta'(g) b g = \beta'(g) G \quad in \quad \mathcal{D}'((0,T) \times \mathcal{O}),$$

for any renormalizing function $\beta \in C^1_*(\mathbb{R})$, satisfies $g \in C([0,T]; L^0(\mathcal{O}))$, meaning that $\beta(g) \in C([0,T]; L^1_{\rm loc}(\bar{\mathcal{O}}))$ for any $\beta \in C_b(\mathbb{R})$.

Proof of Proposition 8.15. The proof is a variant of the proof of [139, Thm. II.3] and we just allude it. Because $\beta(g) \in L^\infty((0,T) \times \mathcal{O})$ is a solution to the transport equation with source term $\beta'(g) G - \beta'(g) b g \in L^1_{\rm loc}([0,T] \times \bar{\mathcal{O}})$, we have $\beta(g) \in C([0,T]; \mathcal{D}'(\mathcal{O}))$ for any $\beta \in C^1_*(\mathbb{R})$. Fixing $\beta_0 \in C^1_*(\mathbb{R})$ strictly increasing, we deduce that $\beta_0(g), \beta_0(g)^2 \in C([0,T]; \mathcal{D}'(\mathcal{O}))$, so that $\beta_0(g) \in C([0,T]; L^2_{\rm loc}(\bar{\mathcal{O}}))$, and the conclusion. □

We consider now the time dependent transport equation (8.31).

Proposition 8.16 (Renormalized solutions). *We assume* (8.35) *for some* $p \in [1,\infty]$ *and some weight function* m. *For any* $g_0 \in L^p_m(\mathcal{O})$, $G \in L^p_{\widetilde m_{\mathcal{O}}}((0,T)\times\mathcal{O})$, $\mathfrak{g} \in L^p_{m_\Sigma}((0,T)\times\Sigma)$, *there exists a unique* $g \in C([0,T];L^1_{\rm loc}(\bar{\mathcal{O}}))$ *satisfying the estimate* (8.45) *or* (8.46) *and being a solution to the transport equation* (8.31) *in the renormalized sense, namely*

$$\begin{cases} \dfrac{\partial\beta(g)}{\partial t} + a\cdot\nabla\beta(g) + \beta'(g)bg = \beta'(g)G \quad on \quad (0,T)\times\mathcal{O}, \\ \gamma_-\beta(g) = \beta(\mathfrak{g}) \quad on \quad (0,T)\times\Sigma_-, \quad \beta(g)(0,\cdot) = \beta(g_0) \quad on \quad \mathcal{O}, \end{cases} \tag{8.56}$$

for any $\beta \in C^1_{{\rm pw},*}$. *Furthermore,* $g \in C([0,T];L^p_m)$ *when* $p \in [1,\infty)$ *and* $g \ge 0$ *if* $g_0, G, \mathfrak{g} \ge 0$.

Remark 8.17. *(1) The above result extends some previous results due to Bardos in* [37, Chap. III]*, Boyer in* [70, Thm. 4.1] *and Crippa et al in* [120, Thm. 1.1] *and* [119, Thm. 1.1]*, where the cases* $p = 2$ *or* $p = \infty$ *are considered with always the additional assumption* $a \in L^\infty$ *(in the last paper however the present* $W^{1,1}$ *bound on* a *is relaxed into a BV condition) by adapting the Di Perna-Lions theory developed in* [139, Sec. II].

(2) We immediately deduce from the above result and Lemma 8.7-(2) a weak maximum principle: $g_1 \le g_2$ *if* g_i *is renormalized solution to the transport equation* (8.31) *associated to the data* g_{0i}, G_i, $\mathfrak{g}_i$ *such that* $g_{01} \le g_{02}$, $G_{01} \le G_{02}$, $\mathfrak{g}_{01} \le \mathfrak{g}_{02}$.

Proof of Proposition 8.16. We proceed similarly as during the proof of Lemma 8.12.

Step 1. Characteristics. We assume first $a \in C^1(\mathbb{R}^D)$, $g_0 \in C_c(\mathcal{O})$, $b \in C_b(\bar{\mathcal{O}})$, $\mathfrak{g} \in C_c((0,T)\times\Sigma_0)$, $G \in C^1_c((0,T)\times\mathcal{O})$. We use the characteristics representation (8.47). We may verify that $\bar g$ both satisfies the transport equation in the renormalized sense and the boundary conditions in (8.56) and we may justify the computations leading to the a priori estimates (8.45) and (8.46).

Step 2. Existence. In the general case, we define some regularized sequence (a_ε), $(g_{0,\varepsilon})$, (b_ε) $(\mathfrak{g}_\varepsilon)$, (G_ε) and thanks to the first step we deduce the existence of an associated function $g_\varepsilon \in C([0,T];L^p_m)$ satisfies both the equation (8.56) in the renormalized sense and the a priori estimates (8.45) or (8.46). When $p > 1$, the sequence (g_ε) is bounded in $L^\infty(0,T;L^p_m)$ and (up to the extraction of a subsequence) we may pass to the limit $\varepsilon \to 0$ using Proposition 8.10-(2) and Remark 8.11. We have established the existence of a renormalized solution to the transport equation which satisfies the estimate (8.45) or (8.46). When $p = 1$, we may for instance proceed in the following way by first assuming $0 \le g_0 \in L^1_m$, $0 \le G \in L^1_m((0,T)\times\mathcal{O})$, $0 \le \mathfrak{g} \in L^1_{m_\Sigma}((0,T)\times\Sigma)$. We may thus consider some nonnegative approximating sequences $(g_{0,\varepsilon})$ in $L^p_m \cap L^1_m$, $G_\varepsilon \in L^p_{\widetilde m_{\mathcal{O}}} \cap L^1_m$, $\mathfrak{g}_\varepsilon \in L^p_{m_\Sigma} \cap L^p_{m_\Sigma}$ such that $g_{0,\varepsilon} \nearrow g_0$, $G_\varepsilon \nearrow G$ and $\mathfrak{g}_\varepsilon \nearrow \mathfrak{g}$. The same construction as above implies the existence of $0 \le g_\varepsilon \in L^\infty(0,T;L^1_m \cap L^p_m)$ renormalized solution to the transport equation associated to these data and such that (g_ε) is increasing and uniformly bounded in $L^\infty(0,T;L^1_m)$ thanks to the a priori L^1_m estimate (8.45). There thus exists $0 \le g \in L^\infty(0,T;L^1_m)$ such that $g_\varepsilon \nearrow g$, and we get that g is a renormalized solution to the transport equation by using again Proposition 8.10-(2) and Remark 8.11. We remove the nonnegative condition on g_0, G and $\mathfrak{g}$ by introducing the positive and negative parts of each function, using the preceding step in order to prove the existence of two solutions $0 \le g_\pm \in L^\infty(0,T;L^1_m)$ associated respectively to $(g_{0+}, G_+, \mathfrak{g}_+)$ and $(g_{0-}, G_-, \mathfrak{g}_-)$, and finally defining $g := g_+ - g_-$ which is a renormalized solution to the transport equation thanks to Lemma 8.7 and Remark 8.9-(5).

Step 3. Continuity. From Proposition 8.15, we already know that $g \in C([0,T];L^0(\mathcal{O}))$. Together with the a priori estimate (8.45) or (8.46), we also have $g \in C([0,T];L^1_{\rm loc}(\mathcal{O}))$ when $p > 1$. When $p \in [1,\infty)$, we may improve the above continuity properties by arguing in the following way. We define $\widetilde g := gm$ and we observe that it is a solution to the transport equation

$$\partial_t \widetilde g + a\cdot\nabla\widetilde g + \widetilde b\widetilde g = \widetilde G, \quad \gamma\widetilde g = \widetilde{\mathfrak{g}}, \quad \widetilde g(0) = \widetilde g_0,$$

with $\widetilde b := b - a\cdot\nabla m/m$, $\widetilde G := Gm$, $\widetilde{\mathfrak{g}} := \mathfrak{g}m$ and $\widetilde g_0 := g_0 m$. We write the associated renormalized equation (8.44) for the renormalizing function $\beta_M(s) := (|s|\wedge M)^p$, $M > 0$, and the test function $\varphi := \chi_R$, with $\chi \in C^1_c(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$ and $\chi_R(y) := \chi(y/R)$. Observing in particular that

$$\int_0^t \int_{\mathcal{O}} \beta_M(\widetilde g) a\cdot\nabla\chi_R \to 0 \text{ as } R \to \infty,$$

because of (8.45) and (8.35) by arguing as in the proof of Lemma 8.14, we may pass to the limit in the associated renormalized equation as $R \to \infty$, and we obtain

$$\Big[\int_{\mathcal{O}} \beta_M(\widetilde g) dy\Big]_0^t = \int_0^t\int_{\mathcal{O}} \beta_M'(\widetilde g)\widetilde G dyds - \int_0^t\int_\Sigma \beta_M(\gamma\widetilde g) a\cdot n\, d\sigma_y ds$$
$$+ \int_0^t\int_{\mathcal{O}} \beta_M(\widetilde g)\,\mathbf{1}_{\widetilde g > M}\mathrm{div} a\, dyds - \int_0^t\int_{\mathcal{O}} p\beta_M(\widetilde g)\,\mathbf{1}_{\widetilde g\le M}\varpi dyds.$$

Using again (8.45) and (8.35), we may next pass to the limit as $M\to\infty$ in the above equation, and we get

$$\frac{d}{dt}\int_{\mathcal{O}} |\widetilde g|^p = -p\int_{\mathcal{O}} |\widetilde g|^p \varpi + \int_{\mathcal{O}} p\widetilde G \widetilde g |\widetilde g|^{p-2} + \int_\Sigma |\gamma \widetilde g|^p a\cdot n \in L^1(0,T).$$

We deduce that $t\mapsto \|g(t)\|_{L^p_m} = \|\widetilde g(t)\|_{L^p}$ is continuous. Consider then $t\in[0,T]$ and $t_k\to t$, so that in particular $\|g_{t_k}\|_{L^p_m} \to \|g_t\|_{L^p_m}$ as $k\to\infty$. On the other hand, we have yet established that $\|\beta_0(g_{t_k}) - \beta_0(g_t)\|_{L^1(\mathcal{O}\cap B_R)} \to 0$ as $k\to\infty$ for any $R>0$. There exists thus a subsequence $(g_{t_{k'}})$ such that $g_{t_{k'}} \to g_t$ a.e. on $\mathcal{O}$. Thanks to Brézis-Lieb theorem [72], we deduce that $g_{t_{k'}}\to g_t$ in L^p_m and it is the whole sequence which converges by uniqueness of the limit. We have thus established $g\in C([0,T];L^p_m)$ when $p\in[1,\infty)$.

Step 4. Uniqueness. Because of Lemma 8.7 and Remark 8.9-(5), we just have to prove that $g\equiv 0$ if g is a renormalized solution associated to vanishing data $g_0=0$, $G=0$ and $\mathfrak{g}=0$. When $p\in[1,\infty)$, the previous step implies that

$$\frac{d}{dt}\int_{\mathcal{O}} |g|^p m^p = \int_{\mathcal{O}} |g|^p m^p \varpi \in L^1(0,T), \quad \int_{\mathcal{O}} |g(0)|^p m^p = 0,$$

and together with the Gronwall lemma, we deduce that $g=0$. The case $p=\infty$ may be tackled thanks to a duality argument exactly as in the proof of Lemma 8.14. ☐

Corollary 8.18. *The semigroup S_b defined by* (8.49) *extends to a positive semigroup of contractions in L^p_m.*

Proof of Corollary 8.18. We just apply Proposition 8.16 with $G=\mathfrak{g}=0$. When $p\in[1,\infty)$, we define in that way a mapping $L^p_m\to C(\mathbb{R}_+;L^p_m)$, $g_0\mapsto g$, where g denotes the unique renormalized solution. Defining then $S(t)g_0 := g(t)$ we have built a strongly continuous semigroup in L^p_m. The case $p=\infty$ is identical, except the fact that the semigroup is only weak $*\sigma(L^\infty_m, L^1_{m^{-1}})$ continuous. The positivity has been established in Proposition 8.16 and the contraction property comes from the estimates (8.45) and (8.46). ☐

Remark 8.19. *It is worth emphasizing that in Bardos* [37] *the semigroup is defined by its representation formula for smooth data and by Hille-Yosida theory for L^2 data. Here we proceed in another way, by rather following* [139, 271, 261].

8.3. **Optimal weighted trace theorem and transport equation with reflection at the boundary.**

We define the functions $\tau^\pm$ as the solutions to

(8.57) $$\lambda_0\tau^\pm \mp a\cdot\nabla\tau^\pm = 1 \ \text{ in } \mathcal{D}'(\mathcal{O}), \quad \gamma_\pm\tau^\pm = 0 \ \text{ on } \Sigma_\pm,$$

with $\lambda_0 := 1 + \|\mathrm{div} a\|_{L^\infty}$.

Lemma 8.20. *Each of the two equations* (8.57) *has a unique solution $\tau^\pm\in L^\infty(\mathcal{O})$ and*

$$0<\tau^\pm\le 1 \ \text{ a.e. in } \mathcal{O}, \quad 0<\gamma_\pm\tau^\pm \le 1 \ \text{ a.e. on } \Sigma_\mp.$$

Proof of Lemma 8.20. We follow a similar proof as in [70, Prop. 5.1] (see also [271, Sec. 5]). We only deals with τ^- since the case of τ^+ can be handled in the same way. The existence of $\tau^-\in L^\infty$, its non negativity and the upperbound are consequences of Lemma 8.12 while the uniqueness is ensured by Lemma 8.14. In order to prove the strict positivity we argue as follows. We first fix $A\in\mathcal{O}$, $|A|\in(0,\infty)$ and we solve

$$\lambda_0\varphi - \mathrm{div}(a\varphi) = \mathbf{1}_A \ \text{ in } \mathcal{D}'(\mathcal{O}), \quad \gamma_+\varphi = 0 \ \text{ on } \Sigma_+,$$

for which there exists a unique solution $\varphi \in L^1(\mathcal{O})$ thanks to Lemma 8.12 and Lemma 8.14, which furthermore satisfies $\varphi \geq 0$ and $\varphi \not\equiv 0$. We observe that $\tau^-\varphi \in L^1(\mathcal{O})$ satisfies

$$\mathrm{div}(a\tau^-\varphi) = \varphi - \tau \mathbf{1}_A \quad \text{in} \quad \mathcal{O}, \quad \gamma(\tau^-\varphi) = 0 \quad \text{on} \quad \Sigma \backslash \Sigma_0.$$

Thanks to the Green formula, first written for $\beta_\delta(\tau^-\varphi)$ and next passing to the limit $\delta \to 0$, we deduce

$$0 = \int_\Sigma \gamma(\tau^-\varphi)\, a \cdot n d\sigma_y = \int_\mathcal{O} \mathrm{div}(a\tau^-\varphi) dy = \int_\mathcal{O} \varphi - \int_A \tau_- dy,$$

so that the last integral does not vanish. This being true for any $A \subset \mathcal{O}$, we get $\tau^- > 0$ a.e. on $\mathcal{O}$. For $A \subset \Sigma_+$ such that

$$0 < \int_A (a \cdot n)_+ d\sigma_y < \infty,$$

we solve

$$\lambda_0 \varphi - \mathrm{div}(a\varphi) = 0 \quad \text{in } \mathcal{D}'(\mathcal{O}), \quad \gamma_+ \varphi = \mathbf{1}_A \quad \text{on } \Sigma_+,$$

thanks to Lemma 8.12 and Lemma 8.14, and we get a unique solution $0 \leq \varphi \in L^1(\mathcal{O})$ such that $\varphi \not\equiv 0$. The Green formula again implies

$$\int_A \gamma\tau^-(a \cdot n)_+ d\sigma_y = \int_\mathcal{O} \mathrm{div}(a\tau^-\varphi) dy = \int_\mathcal{O} \varphi,$$

so that the first integral does not vanish. This being true for any $A \subset \Sigma_+$, we conclude that $\gamma_+\tau^- > 0$ a.e. on Σ_+. □

Lemma 8.21 (Optimal weight)**.** *We assume that a satisfies* (8.33) *as well as* $a \in W^{1,p'}_{\mathrm{loc}}(\bar{\mathcal{O}})$ *for some* $1 \leq p < \infty$*. For any* $g \in L^p(\mathcal{O})$ *satisfying* (8.9) *in the distributional sense with* $G \in L^p(\mathcal{O})$*, the associated trace function* γg *defined in Theorem 8.8 satisfies*

$$\gamma\, g \in L^p\big(\Sigma, |n \cdot a|\, \tau d\sigma\big).$$

Proof of Lemma 8.21. One fixes $\beta_M(z) = (|z| \wedge M)^p$. From the DiPerna-Lions renormalizing theory, we have

$$a \cdot \nabla(\beta_M(g)\, \tau^+) = \beta_M'(g)\, G\, \tau^+ + \beta_M(g)(\tau^+ - 1) \quad \text{in } \mathcal{D}'(\mathcal{O}).$$

Because $\beta_M(g_\varepsilon)\, \tau^+ \in L^1(\mathcal{O}) \cap L^\infty(\mathcal{O})$ and $a/\langle y \rangle \in L^1 + L^\infty$, we may use the Green formula (8.21) with $\phi \equiv 1$, and we get

$$\begin{aligned} \int_{\Sigma_-} \beta_M(\gamma g)\, \tau^+\, |n \cdot a| d\sigma &= \int_\mathcal{O} \big\{ (\mathrm{div} a)\beta_M(g)\, \tau^+ - \beta_M'(g)\, G\, \tau^+ + \beta_M(g)(\tau^+ - 1) \big\} \\ &\lesssim \|g\|^{p-1}_{L^p} \, \big\{ \|g\|_{L^p} + \|G\|_{L^p} \big\}. \end{aligned}$$

Passing to the limit $M \to \infty$, we obtain $\gamma_-\, g \in L^p\big(\Sigma_-, |n \cdot a|\, \tau d\sigma\big)$. In a very same way, we prove $\gamma_+\, g \in L^p\big(\Sigma_+, |n \cdot a|\, \tau d\sigma\big)$. □

We give now a second version of an existence result in a L^p framework with optimal assumption on the boundary condition in the sense that it is reverse with respect to Lemma 8.21. That also a posteriori justifies that Lemma 8.21 provides the optimal trace result in term of weight function on the boundary.

Lemma 8.22 (Existence in L^p - optimal assumption)**.** *We make the same assumption on a, b and p as in Lemma 8.13. For any* $\lambda > \lambda_{a,b,p} + 1/p$ *and any given functions* $G \in L^p(\mathcal{O})$ *and* $\mathfrak{g} \in L^p(\Sigma_-, \tau |a \cdot n| d\sigma)$*, there exists* $g \in L^p(\mathcal{O})$ *solution to* (8.29).

Proof of Lemma 8.22. We only sketch the proof in the case of equation (8.29), arguing along the lines of Lemma 8.12. We start with an a priori estimate. Observing that

$$\mathrm{div}(a\tau^+|g|^p) = (\mathrm{div} a)\tau^+ g^p + (\tau^+ - 1) g^p + p\tau^+(Gg|g|^{p-2} - b|g|^p - \lambda|g|^p),$$

we have

$$\int_\mathcal{O} |g|^p \big\{ 1 + p\tau^+(\lambda + b - \tfrac{1}{p}\mathrm{div} a - \tfrac{1}{p} \big\} = \int_{\Sigma_-} |\gamma_- g|^p \tau |a \cdot n| d\sigma + \int_\mathcal{O} Gg|g|^{p-2}\tau^+.$$

Using the condition on λ, the property $0 \le \tau^+ \le 1$ and the Young inequality, we deduce

$$\frac{1}{p}\int_{\mathcal{O}} |g|^p \le \int_{\Sigma_-} |\gamma_- g|^p \tau |a\cdot n| d\sigma + \frac{1}{p}\int_{\mathcal{O}} |G|^p.$$

We conclude in a similar way as in the proof of Lemma 8.13. ◻

We consider now the time dependent transport equation with positive abstract kernels

$$\begin{cases} \dfrac{\partial g}{\partial t} + a\cdot\nabla g + bg = \mathscr{K}[g] + G \ \text{ on } \ (0,T)\times\mathcal{O}, \\ \gamma_- g = \mathscr{R}[g,\gamma_+ g] + \mathfrak{g} \ \text{ on } \ (0,T)\times\Sigma_-, \quad g(0,\cdot) = \mathfrak{g}_0 \ \text{ on } \ \mathcal{O}, \end{cases} \tag{8.58}$$

with notations introduced at the beginning of the Section. We will work in a weighted Lebesgue space L^p_m with the same conditions on p, m, a and b as introduced at the beginning of Section 8.2. On the other hand, we assume

$$\mathscr{K} : L^p_{m_{\mathcal{O}}}(\mathcal{O}) \to L^p_{\widetilde{m}_{\mathcal{O}}}(\mathcal{O}) \text{ linear and positive,} \tag{8.59}$$

$$\mathscr{R} : L^p_{m_{\mathcal{O}}}(\mathcal{O}) \times L^p_{m_\Sigma}(\Sigma_+) \to L^p_{m_\Sigma}(\Sigma_-) \text{ linear and positive in each variable,} \tag{8.60}$$

where we recall that the weight functions $m_{\mathcal{O}}$, $\widetilde{m}_{\mathcal{O}}$ and m_Σ have been defined in (8.38). More precisely, recalling that $\mathscr{R} = \mathscr{R}_{\mathcal{O}} + \mathscr{R}_\Sigma$ with $\mathscr{R}_{\mathcal{O}}$ and $\mathscr{R}_\Sigma$ defined by (8.4), we assume

$$\|\mathscr{K}[g]\|^p_{L^p_{\widetilde{m}_{\mathcal{O}}}} \le \alpha_{\mathscr{K}} \|g\|^p_{L^p_{m_{\mathcal{O}}}} + M_{\mathscr{K}} \|g\|^p_{L^p_m}, \tag{8.61}$$

$$\|\mathscr{R}_{\mathcal{O}}[g]\|^p_{L^p_{m_\Sigma}} \le \alpha_{\mathscr{R}} \|g\|^p_{L^p_{m_{\mathcal{O}}}} + M_{\mathscr{R}} \|g\|^p_{L^p_m}, \quad \|\mathscr{R}_\Sigma[h]\|^p_{L^p_{m_\Sigma}} \le \beta_{\mathscr{R}} \|h\|^p_{L^p_{m_\Sigma}}, \tag{8.62}$$

with $\alpha_{\mathscr{K}}$, $\alpha_{\mathscr{R}}$, $\beta_{\mathscr{R}} \in [0,1]$, $M_{\mathscr{K}}$, $M_{\mathscr{R}} \ge 0$ and

$$\vartheta_{\mathcal{O}} := (1 - \alpha_{\mathscr{R}} - \alpha_{\mathscr{K}})/2 > 0, \quad \vartheta_\Sigma := 1 - \beta_{\mathscr{R}} \ge 0. \tag{8.63}$$

Let us emphasize that when $p = 1$, the assumption (8.61) is equivalent to the Lyapunov type condition

$$\mathscr{K}^*[m] \le \alpha_{\mathscr{K}} \varpi_+ m + M_{\mathscr{K}} m.$$

Proposition 8.23. *We assume that a, b, $\mathscr{K}$ and $\mathscr{R}$ satisfy the conditions* (8.35), (8.59), (8.60), (8.61), (8.62), *and* (8.63) *for some weight function $m : \bar{\mathcal{O}} \to [1,\infty)$ and some exponent $p \in [1,\infty)$. We consider some data $g_0 \in L^p_m(\mathcal{O})$, $G \in L^p_m((0,T)\times\mathcal{O})$ and $\mathfrak{g} \in L^p_{m_\Sigma}((0,T)\times\Sigma_+)$ with either*

(1) $\beta_{\mathscr{R}} \in [0,1)$;

or $\beta_{\mathscr{R}} = 1$. In the latter case, we assume that $\mathfrak{g} = 0$ and we make one of the following additional structural assumption

(2) there exist an exponent $p_0 \in [1,p]$ and a weight function m_0 such that $\mathscr{K}$ and $\mathscr{R}$ satisfy (8.59) *in $L^{p_0}_{m_0}$ and* (8.60) *in $L^{p_0}_{m_{0\mathcal{O}}}(\mathcal{O}) \times L^{p_0}_{m_{0\Sigma}}(\Sigma_+)$, with obvious definitions for the weight functions $m_{0\mathcal{O}}$ and $m_{0\Sigma}$, and with $L^p_m \subset L^{p_0}_{m_0}$, $L^p_{m_{\mathcal{O}}} \subset L^{p_0}_{m_{0\mathcal{O}}}$, $L^p_{\hat{m}_\Sigma} \subset L^{p_0}_{m_{0\Sigma}}$, where $\hat{m}_\Sigma := m(\tau^+ a\cdot n)^{2/p}$;*

(3) $p = 1$ and $\mathscr{R}_\Sigma$ is diffusive, namely $\mathscr{R}^*_\Sigma[\tau^+ m_\Sigma] \ge c_\Sigma m_\Sigma$ a.e. on Σ_+ with $c_\Sigma > 0$.

In the above three cases, there exists a unique solution $g \in L^\infty(0,T; L^p_m(\mathcal{O})) \cap C([0,T]; L^{p_0}_{m_0}(\mathcal{O}))$ satisfying the transport equation (8.58) *in the renormalized sense as well as $g \in L^{p_0}(0,T; L^{p_0}_{m_{0\mathcal{O}}}(\mathcal{O}))$ and $\gamma g \in L^{p_0}(0,T; L^{p_0}_{m_{0\mathcal{O}}}(\Sigma))$, with $p_0 = p$ and $m_0 = m$ in the first and the third cases.*

Remark 8.24. *(1) The above result extends some previous results initiated by Bardos in* [37, Chap. III] *and Beals et al in* [43, Thm. 1&7]*, where however only the kinetic case were considered. We refer to Section 10 for a discussion about that important model.*

(2) When $\beta_{\mathscr{R}} = 1$, the existence part of the above result still holds (without any additional structural assumption).

(3) Similarly as observed in Remark 8.17, a weak maximum principle holds: $g_1 \le g_2$ if g_1 and g_2 are the renormalized solutions to two transport equations (8.58) *such that (with obvious notations) $b_1 \ge b_2$, $\mathscr{K}_1 \le \mathscr{K}_2$, $\mathscr{R}_1 \le \mathscr{R}_2$, $g_{01} \le g_{02}$, $G_{01} \le G_{02}$ and $\mathfrak{g}_{01} \le \mathfrak{g}_{02}$. That is an immediate consequence of the way we build the solutions g_i thanks to the iterative scheme we present in Step 2 of the proof of Proposition 8.23.*

(4) Another immediate consequence of the iterative way of building the solution, together with the fact that the characteristics representation (8.47) *is the very first step of the construction, is the validity of the Duhamel formula*

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}}$$

if we denote by $S_{\mathcal{L}}$ the semigroup generated by the transport equation (8.58) *with* $G = \mathfrak{g} = 0$, *by* $S_{\mathcal{B}}$ *the semigroup when additionally* $\mathscr{K} = 0$, *and* $\mathcal{A}f = \mathscr{K}[f]$.

Proof of Proposition 8.23. We split the proof into five steps.

Step 1. A priori estimates. For a positive solution, we formally compute

$$\frac{1}{p}\frac{d}{dt}\int g^p m^p = \frac{1}{p}\int_{\Sigma_-} (\mathscr{R}[g,\gamma_+ g] + \mathfrak{g})^p m_\Sigma^p d\sigma_y - \frac{1}{p}\int_{\Sigma_+} (\gamma_+ g)^p m_\Sigma^p d\sigma_y + \int_{\mathcal{O}} \big\{ g^{p-1}(\mathscr{K}[g] + G) - g^p \varpi \big\} m^p. \tag{8.64}$$

Using the Young inequality and (8.61), we have

$$\begin{aligned}\int_{\mathcal{O}} g^{p-1}\mathscr{K}[g] m^p &\le \frac{1}{p'}\int_{\mathcal{O}} g^p \langle \varpi_+\rangle m^p + \frac{1}{p}\int_{\mathcal{O}} \mathscr{K}[g]^p \langle\varpi_+\rangle^{-p/p'} m^p \\ &\le \big(\frac{1}{p'} + \frac{\alpha_{\mathscr{K}}}{p}\big)\int_{\mathcal{O}} g^p\langle\varpi_+\rangle m^p + \frac{M_{\mathscr{K}}}{p}\int_{\mathcal{O}} g^p m^p.\end{aligned}$$

• When $\mathfrak{g} = 0$, using also (8.62) and once more the Young inequality, we then have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int g^p m^p &\le \frac{1}{p}(\beta_{\mathscr{R}} - 1)\int_{\Sigma_+} (\gamma_+ g)^p m^p a\cdot n + \big(\frac{M_{\mathscr{R}}}{p} + \frac{M_{\mathscr{K}}}{p} + \|\langle\varpi_-\rangle\|_{L^\infty}\big)\int_{\mathcal{O}} g^p m^p \\ &\quad + \big(\frac{\alpha_{\mathscr{R}}}{p} + \frac{1}{p'} + \frac{\alpha_{\mathscr{K}}}{p} + \frac{\varepsilon}{p'} - 1\big)\int_{\mathcal{O}} g^p\langle\varpi_+\rangle m^p + \frac{\varepsilon^{-p/p'}}{p}\int_{\mathcal{O}} G^p m^p \langle\varpi_+\rangle^{-p/p'}\end{aligned}$$

for any $\varepsilon > 0$. Making the choice $\varepsilon := \vartheta_{\mathcal{O}} p'/p$, we deduce

$$\frac{d}{dt}\|g\|_{L^p_m} + \vartheta_{\mathcal{O}}\|g\|^p_{L^p_{m_{\mathcal{O}}}} + \vartheta_\Sigma \|\gamma_+ g\|^p_{L^p_{m_\Sigma}} \le p\kappa\|g\|^p_{L^p_m} + C_{\mathcal{O}}\|G\|^p_{L^p_m},$$

with

$$\kappa := \frac{M_{\mathscr{R}}}{p} + \frac{M_{\mathscr{K}}}{p} + \|\langle\varpi_-\rangle\|_{L^\infty}, \quad C_{\mathcal{O}} := (\vartheta_{\mathcal{O}} p'/p)^{-p'/p}.$$

Using the Gronwall lemma, we then obtain

$$\begin{aligned}\|g(t)\|^p_{L^p_m} &+ \int_0^t e^{p\kappa(t-s)}(\vartheta_{\mathcal{O}}\|g_s\|^p_{L^p_{m_{\mathcal{O}}}} + \vartheta_\Sigma\|\gamma_+ g_s\|^p_{L^p_{m_\Sigma}})\, ds \\ &\le e^{p\kappa t}\|\mathfrak{g}_0\|^p_{L^p_m} + C_{\mathcal{O}}\int_0^t e^{p\kappa(t-s)}\|G_s\|^p_{L^p_{\widetilde{m}_{\mathcal{O}}}}\, ds, \quad \forall\, t \ge 0.\end{aligned} \tag{8.65}$$

• When $\mathfrak{g} \neq 0$ and thus $\vartheta_\Sigma > 0$, we control the ingoing boundary term by

$$\int_{\Sigma_-} (\mathscr{R}[g,\gamma_+ g] + \mathfrak{g})^p m_\Sigma^p \le (1+\varepsilon_1)\int_{\Sigma_-} \mathscr{R}[g,\gamma_+ g]^p m_\Sigma^p + C_{\varepsilon_1}\int_{\Sigma_-} \mathfrak{g}^p m_\Sigma^p, \quad \forall\, \varepsilon_1 > 0,$$

and a very similar computation as above leads to the a priori estimate

$$\begin{aligned}\|g(t)\|^p_{L^p_m} &+ \int_0^t e^{p\kappa(t-s)}(\vartheta'_{\mathcal{O}}\|g_s\|^p_{L^p_{m_{\mathcal{O}}}} + \vartheta'_\Sigma\|\gamma_+ g_s\|^p_{L^p_{m_\Sigma}})\, ds \\ &\le e^{p\kappa t}\|\mathfrak{g}_0\|^p_{L^p_m} + \int_0^t e^{p\kappa(t-s)}(C_{\mathcal{O}}\|G_s\|^p_{L^p_{\widetilde{m}_{\mathcal{O}}}} + C_\Sigma\|\mathfrak{g}_s\|^p_{L^p_{m_\Sigma}})\, ds,\end{aligned} \tag{8.66}$$

for any $t \ge 0$, with

$$\vartheta'_\Sigma := 1 - \beta_{\mathscr{R}}(1+\varepsilon_1), \quad \vartheta'_{\mathcal{O}} := 1 - \alpha_{\mathscr{R}}(1+\varepsilon_1) - \alpha_{\mathscr{K}} - \varepsilon\frac{p}{p'},$$

$$\kappa := \frac{M_{\mathscr{R}}}{p}(1+\varepsilon_1) + \frac{M_{\mathscr{K}}}{p} + \|\langle\varpi_-\rangle\|_{L^\infty}, \quad C_{\mathcal{O}} := \varepsilon^{-p'/p}, \quad C_\Sigma := C_{\varepsilon_1},$$

and where we have chosen $\varepsilon, \varepsilon_1 > 0$ small enough in such a way that $\vartheta'_\Sigma > 0$ and $\vartheta'_{\mathcal{O}} > 0$.

• When $\vartheta_\Sigma = 0$ and thus $\mathfrak{g} = 0$, we further multiply the equation by $m^p \tau^\pm$, where $\tau^\pm$ is defined in (8.57), and integrating, we deduce

$$\int_0^T \int_{\Sigma_\mp} \tau^\pm a \cdot n (\gamma g)^p m^p \, d\sigma dt = \Big[\int_{\mathcal{O}} g^p m^p \tau^\pm \Big]_T^0$$
$$+ \int_0^T \int_{\mathcal{O}} p g^{p-1} (\mathscr{K}[g] + G) m^p \tau^\pm + \int_0^T \int_{\mathcal{O}} g^p \big(\frac{\operatorname{div}(a m^p)}{m^p} + 1 - \lambda_0 \tau^\pm - pK m^p \tau^\pm \big).$$

Together with (8.65) and $\tau^\pm \in L^\infty(\mathcal{O})$, we obtain

$$\int_0^T \int_{\Sigma_+} (\gamma_+ g)^p \tau^- m_\Sigma^p \le C_T \Big(\|g_0\|^p_{L^p_m} + \|G\|^p_{L^p(0,T;L^p_m)} \Big) \tag{8.67}$$

and

$$\int_0^T \int_{\Sigma_-} [\mathscr{R}_\Sigma(\gamma_+ g)]^p \tau^+ m_\Sigma^p \le C_T \Big(\|g_0\|^p_{L^p_m} + \|G\|^p_{L^p(0,T;L^p_m)} \Big), \tag{8.68}$$

for some constant $C_T \in (0,\infty)$. In particular, when $p = 1$ and $\mathscr{R}_\Sigma$ is diffusive, we have

$$c_\Sigma \int_0^T \int_{\Sigma_+} (\gamma_+ g) m_\Sigma \le \int_0^T \int_{\Sigma_+} (\gamma_+ g) \mathscr{R}^*_\Sigma(\tau^+ m_\Sigma) = \int_0^T \int_{\Sigma_-} \mathscr{R}_\Sigma(\gamma_+ g) \tau^+ m_\Sigma,$$

and together with (8.68), we deduce the additional estimate

$$c_\Sigma \int_0^T \int_{\Sigma_+} (\gamma_+ g) m_\Sigma \le C_T \Big(\|g_0\|_{L^1_m} + \|G\|_{L^1(0,T;L^1_m)} \Big). \tag{8.69}$$

Step 2. Existence. As a consequence of these a priori estimates, we may classically build a solution through an iterative scheme. For the sake of brevity, we only consider the (more interesting and more difficult) case $b_\Sigma = 1$ (so that $\vartheta_\Sigma = 0$ and $\mathfrak{g} = 0$) and $G = 0$. For a given $0 \le g_0 \in L^p_m(\mathcal{O})$, we define a sequence of solution (h_n) starting from $h_0 \equiv 0$ thanks to the recursive definition

$$\begin{cases} \dfrac{\partial h_{n+1}}{\partial t} + a \cdot \nabla h_{n+1} + K h_{n+1} = \mathscr{K}[h_n] \ \text{ on } \ (0,T) \times \mathcal{O}, \\ \gamma_- h_{n+1} = \mathscr{R}[h_n, \gamma_+ h_n] \ \text{ on } \ (0,T) \times \Sigma_-, \quad h_{n+1}(0,\cdot) = \mathfrak{g}_0 \ \text{ on } \ \mathcal{O}. \end{cases}$$

From Proposition 8.16, there exists a unique renormalized solution $h_{n+1} \in C([0,T); L^p_m(\mathcal{O}))$ to the above equation satisfying the estimate (8.45) with $g := h_{n+1}$, $G := \mathscr{K}[h_n]$ and $\mathfrak{g} := \mathscr{R}[h_n, \gamma_+ h_n] \in L^p_{m_\Sigma}$. We observe that $0 \le h_n \le h_{n+1}$ thanks to the weak maximum principle (see Remark 8.17) and that h_n satisfies the estimates (8.65) and (8.68) where g is replaced by h_n. Thanks to the monotonous convergence theorem of Beppo Levi, there exists g satisfying estimates (8.65) and $h_n \to g$ in $L^p_{m_{\mathcal{O}}}((0,T) \times \mathcal{O})$. We may pass to the limit in the equation satisfied by (h_n) and we deduce that g is a renormalized solution to

$$\frac{\partial g}{\partial t} + a \cdot \nabla g + b g = \mathscr{K}[g] \ \text{ on } \ (0,T) \times \mathcal{O}.$$

From Theorem 8.8 and Remark 8.9-(5), the function g admits a trace γg and thanks to Proposition 8.10, we have $\gamma h_n \to \gamma g$ a.e. on $\Sigma \backslash \Sigma_0$. Because of (8.4) and the Beppo Levi theorem again we deduce that $\mathscr{R}[h_n, \gamma_+ h_n] \to \mathscr{R}[g, \gamma_+ g]$ a.e. on Σ_-. Together with the fact that $\gamma_- h_n \to \gamma_- g$ a.e. on Σ_-, we have established that the boundary condition in (8.58) holds true. It is worth emphasizing here that $\gamma_+ g \in L^1(\Sigma_+; dr_\Sigma(y,\cdot))$ for a.e. $y \in \Sigma_-$ because of (8.68). For $\mathfrak{g}_0 \in L^p_m(\mathcal{O})$, we separate the positive and the negative parts $\mathfrak{g}_0 = \mathfrak{g}_{0+} - \mathfrak{g}_{0-}$ and we obtain two renormalized solutions $g^\pm \in L^\infty(0,T; L^p_m)$ associated to $\mathfrak{g}_{0\pm}$ respectively. By linearity, the function $g := g^+ - g^- \in L^\infty(0,T; L^p_m)$ is a renormalized solution to the transport equation and the boundary condition is

$$\begin{aligned} \gamma_- g &= \gamma_- g^+ - \gamma_- g^- = \mathcal{R}_{\mathcal{O}}[g^+] - \mathcal{R}_{\mathcal{O}}[g^-] + \mathcal{R}_\Sigma[\gamma_+ g^+] - \mathcal{R}_\Sigma[\gamma_+ g^-] \\ &= \mathcal{R}_{\mathcal{O}}[g] + \mathcal{R}_\Sigma[\gamma_+ g], \end{aligned}$$

where the last term is indeed well defined a.e. from the fact that $\gamma_+ g^\pm \in L^1(\Sigma_+; dr_\Sigma(y,\cdot))$ for a.e. $y \in \Sigma_-$ and thus $\gamma_+ g = \gamma_+ g^+ - \gamma_+ g^-$ belongs to the same spaces. From Proposition 8.15, we already know that $g \in C([0,T]; L^0(\mathcal{O}))$ and thus using an interpolation argument $g \in C([0,T]; L^{p_1}_{m_1}(\mathcal{O}))$ for any $p_1 \in [1,p)$ and any weight function m_1 such that $m_1/m \in L^{pp_1/(p-p_1)}$ when $p > 1$.

Step 3. When $\beta_{\mathscr{R}} < 1$ and $p \in [1,\infty)$, we have (8.66), and we may just repeat the proof of Proposition 8.16 in order to get $g \in C([0,T]; L^p_m(\mathcal{O}))$ and the uniqueness of the solution.

Step 4. We assume $\beta_{\mathscr{R}} = 1$ and the structural assumption (2). From the estimate (8.64) on a solution g and the renormalized formulation of the equation, we deduce that

$$\begin{aligned} \frac{1}{p_0}\frac{d}{dt}\int |g|^{p_0} m^{p_0} &= \frac{1}{p_0}\int_{\Sigma_-} |\mathscr{R}[g,\gamma_+ g]|^{p_0} m_{0\Sigma}^{p_0} - \frac{1}{p_0}\int_{\Sigma_+} |\gamma_+ g|^{p_0} m_{0\Sigma}^{p_0} \\ &\quad + \int_{\mathcal{O}} \big\{ g|g|^{p_0-2}(\mathscr{K}[g]+G)m^{p_0} - |g|^{p_0} m_{0\mathcal{O}}^{p_0}\big\}, \end{aligned}$$

with a RHS term in $L^1(0,T)$. As above, we thus deduce $g \in C([0,T]; L^{p_0}_{m_0}(\mathcal{O}))$ and next the uniqueness of the solution.

Step 5. We assume $\beta_{\mathscr{R}} = 1$ and the structural assumption (3). In that case, we have $p = 1$, $\gamma_+ g \in L^1_{m_\Sigma}((0,T)\times\Sigma_+)$ from (8.69) and then $\gamma_- g \in L^1_{m_\Sigma}((0,T)\times\Sigma_-)$ from (8.62). We may thus justify the same computation as in Step 4 with $p_0 = 1$, and we deduce $g \in C([0,T]; L^1_m(\mathcal{O}))$ and next the uniqueness of the solution. □

As an immediate consequence of the above analysis, we may associate to the transport equation (8.58) a semigroup.

Corollary 8.25. *Under the assumptions of Proposition 8.23, there exists a positive semigroup S on L^p_m such that for any $\mathfrak{g}_0 \in L^p_m(\mathcal{O})$, the function $t \mapsto g(t) := S(t)\mathfrak{g}_0 \in C(\mathbb{R}_+; L^{p_0}_{m_0}(\mathcal{O})) \cap L^\infty_{\rm loc}(\mathbb{R}_+; L^p_m(\mathcal{O}))$ is the unique renormalized solution to the transport equation* (8.58) *associated to the initial datum $\mathfrak{g}_0$ (and with $G = \mathfrak{g} = 0$). Furthermore the growth bound satisfies $\omega(S) \le \kappa$.*

We end this section by formulating the counterpart of the above result for the associated stationary problem

$$\tag{8.70} \begin{cases} \lambda g + a\cdot\nabla g + bg = \mathscr{K}[g] + G & \text{on } \mathcal{O}, \\ \gamma_- g = \mathscr{R}[g,\gamma_+ g] + \mathfrak{g} & \text{on } \Sigma_-. \end{cases}$$

Proposition 8.26. *We make exactly the same assumptions as in Proposition 8.23 on a, b, $\mathscr{K}$ and $\mathscr{R}$ for some weight function $m : \bar{\mathcal{O}} \to [1,\infty)$ and some exponent $p \in [1,\infty)$ as well as either $\beta_{\mathscr{R}} < 1$ holds or $\beta_{\mathscr{R}} = 1$ holds with $\mathfrak{g} = 0$ and one of the additional structure assumptions (1) or (2). There exists $\lambda^{**} \in \mathbb{R}$ such that for any $\lambda > \lambda^{**}$, $G \in L^p_m(\mathcal{O})$ and $\mathfrak{g} \in L^p_{m_\Sigma}(\Sigma_+)$, there exists a unique solution $g \in L^p_{m_\mathcal{O}}(\mathcal{O})$ satisfying the transport equation* (8.70) *in the renormalized sense and some additional a priori estimates listed during the proof.*

Proof of Proposition 8.26. We just explain the main steps. We first establish an a priori estimate. We observe that any solution g to the stationary problem (8.70) (at least formally) satisfies

$$\tag{8.71} \int_{\mathcal{O}} |g|^p m^p(\lambda+\varpi) + \frac{1}{p}\int_{\Sigma_+} |\gamma_+ g|^p m_\Sigma^p = \int_{\mathcal{O}} (\mathscr{K}[g]+G)g|g|^{p-2}m^p + \frac{1}{p}\int_{\Sigma_-} |\mathscr{R}[g,\gamma_+ g]+\mathfrak{g}|^p m_\Sigma^p.$$

We then only consider the case $\mathfrak{g} = 0$. Repeating the same computations as in Step 1 of the proof of Proposition 8.23 and with the same notations, we get

$$\tag{8.72} p(\lambda-\kappa)\|g\|^p_{L^p_m} + \vartheta_{\mathcal{O}}\|g\|^p_{L^p_{m_\mathcal{O}}} + \vartheta_\Sigma\|\gamma_+ g\|^p_{L^p_{m_\Sigma}} \le C_{\mathcal{O}}\|G\|^p_{L^p_m}.$$

For $\lambda > \lambda^{**} := \max(\kappa, \lambda^*_p)$ and $G \ge 0$, we next consider the sequence (h_k) in $L^p_{m_\mathcal{O}}$ defined iteratively as the solution given by Lemma 8.13 to

$$\begin{cases} \lambda h_k + a\cdot\nabla h_k + bh_k = \mathscr{K}[h_{k-1}] + G & \text{on } \mathcal{O}, \\ \gamma_- h_k = \mathscr{R}[h_{k-1},\gamma_+ h_{k-1}] & \text{on } \Sigma_-, \end{cases}$$

for $k \ge 1$ and starting from $h_0 \equiv 0$. We observe that (h_k) is increasing and satisfies the estimate (8.72) where g is replaced by h_k. We may pass to the limit in the above equation and estimate and we obtain a renormalized solution $g \in L^p_{m_\mathcal{O}}$ to the transport equation (8.70) and satisfying the estimate (8.72). By linearity, the same holds without sign condition on G. Finally, considering the three different cases as in Steps 3, 4 and 5 in Proposition 8.23, we similarly show that $g \in L^{p_0}_{m_{0\mathcal{O}}}$

and $\gamma_+ g \in L^{p_0}_{m_0\Sigma}$ for suitable exponent $p_0 \in [1,p]$ and weight function m_0. For two such solutions g_i to (8.70), the function $g := g_2 - g_1$ is also a renormalized solution to (8.70) for which we may justify the identity (8.71) with $p = p_0$, $m = m_0$, $G = 0$. We thus deduce that (8.72) holds with $p = p_0$, $m = m_0$, $G = 0$, and we conclude that $g = 0$, what ends the proof of the uniqueness. □

8.4. **Characteristics.**
In this section we come back to the characteristics method for the evolution and the stationary transport equation. Our aim is in particular to discuss the representation formula (8.47).
We consider a vector field $a : \mathcal{O} \to \mathbb{R}^D$ which extends to $\mathbb{R}^D$ and, denoting by the same letter a its extension, we at least assume

$$a \in W^{1,1}_{\rm loc}(\mathbb{R}^D), \qquad \mathrm{div} a \in L^\infty(\mathbb{R}^D), \quad a/\langle y\rangle \in L^1(\mathbb{R}^D) + L^\infty(\mathbb{R}^D). \tag{8.73}$$

After DiPerna and Lions [139, 255] (see also [204, Def. 1] or [203, Def. 1]), we introduce the following notion of flow.

Definition 8.27. *We name almost everywhere flow associated to* (8.6) *a measurable function* $Y : \mathbb{R} \times \mathbb{R}^D \to \mathbb{R}^D$, $(t,y) \mapsto Y_t(y)$, *such that*
(i) for a.e. $y \in \mathbb{R}^D$, *the map* $t \mapsto Y_t(y)$ *is continuous and*

$$\dot Y_t(y) = a(Y_t(y)) \ \textit{in}\ \mathcal{D}'(\mathbb{R}), \quad Y_0(y) = y;$$

(ii) for a.e. $y \in \mathbb{R}^D$ *and for any* $s,t \in \mathbb{R}$, *there holds* $Y_{s+t}(y) = Y_s(Y_t(y))$;
(iii) there exists $C \ge 0$ *such that*

$$\forall\, t \in \mathbb{R}, \quad e^{-CT}\lambda \le Y(t,\cdot)_\sharp \lambda \le e^{CT}\lambda, \tag{8.74}$$

where $(Y(t,\cdot)_\sharp\lambda)(A) = \lambda(Y(-t,A))$, $A \subset \mathbb{R}^D$, *is the pushforward of the Lebesgue measure* λ.

From [139, Thm. III.1], [9, Thm. 31 & Remark 32] and [204, Sec. 3] (see also Theorem 8.33 below), we know the existence and uniqueness of such an a.e. flow for a satisfying (8.73). In the incompressible case ($\mathrm{div} a = 0$), this one furthermore satisfies:

$$\int_{\mathbb{R}^D} \varphi(Y_t(y)) e^{\int_0^t (\mathrm{div} a)(Y_\tau(y))d\tau} dy = \int_{\mathbb{R}^D} \varphi(y)dy, \tag{8.75}$$

for any $t \in \mathbb{R}$ and any $\varphi \in L^\infty_c(\mathbb{R}^D)$, the space of L^∞ functions with compact support. In the compressible case ($\mathrm{div} a \neq 0$) and when a only satisfies (8.73), it seems not clear that [139, Thm. III.2] or [9, Thm. 31 & Remark 32] provides an a.e. flow such that (8.75) holds. In that general case, the volume identity (8.75) must be replaced by the volume two sides estimate (or nearly-incompressible condition):

$$e^{-t\|\mathrm{div} a\|_\infty} \int_{\mathbb{R}^D} \varphi(y)dy \le \int_{\mathbb{R}^D} \varphi(Y_t(y))dy \le e^{t\|\mathrm{div} a\|_\infty} \int_{\mathbb{R}^D} \varphi(y)dy,$$

for any $0 \le \varphi \in L^\infty_c(\mathbb{R}^D)$ and $t \in \mathbb{R}$, what is nothing but (8.74) with $C := \|\mathrm{div} a\|_\infty$. It is however quite straightforward to prove from [139, 9] that the a.e. flows Y satisfies (8.75) when a additionally satisfies $\mathrm{div} a \in C(\mathbb{R}^D)$. One possible way to construct the a.e. flow Y is to define $Y = Y_t(y)$ as the unique renormalized solution in $C(\mathbb{R}; L)$ to the transport equation

$$\partial_t Y - a \cdot \nabla_y Y = 0 \ \text{ on } \ \mathbb{R} \times \mathbb{R}^D, \quad Y_0(y) = y \ \text{ on } \ \mathbb{R}^D,$$

or more explicitly

$$\frac{\partial}{\partial t}\beta(Y) - a(Y) \cdot \nabla \beta(Y) = 0 \ \text{ on } \ \mathbb{R} \times \mathbb{R}^D, \quad \beta(Y_0) = \beta(y) \ \text{ on } \ \mathbb{R}^D,$$

in the distributional sense for any $\beta \in C^1(\mathbb{R}^D, \mathbb{R})$ such that β and $|\nabla\beta(z)|(1+|z|)$ are uniformly bounded on $\mathbb{R}^D$. In particular, for any $g_0 \in C^1(\mathbb{R}^D)$ and next for any $g_0 \in L^0(\mathbb{R}^D)$ the function $g^\sharp(t,y) := g_0(Y_{-t}(y))$ is the unique renormalized solution to the transport equation

$$\partial_t g^\sharp + a \cdot \nabla g^\sharp = 0 \ \text{ on } \ \mathbb{R} \times \mathbb{R}^D, \quad g^\sharp(0,\cdot) = g_0 \ \text{ on } \ \mathbb{R}^D.$$

We introduce some notations. We denote $y \in \mathcal{Y}$ if (i) and (ii) hold. In particular, $\mathcal{Y}$ is a measurable subset of $\mathbb{R}^D$ and $|\mathbb{R}^D \backslash \mathcal{Y}| = 0$. Because of (i), for $y \in \mathcal{V} := \mathcal{O} \cap \mathcal{Y}$, we may define the backward exit time

$$t_{\mathbf{b}}(y) := \sup\big\{\tau > 0 \mid Y_{-t}(y) \in \mathcal{O},\ \forall\, s \in [0,\tau]\big\} \in (0,+\infty],$$

the subset $\mathcal{V}_{\mathbf{b}} := \{y \in \mathcal{V};\ t_{\mathbf{b}}(y) < +\infty\}$ and the associated *entering position*

$$y_{\mathbf{b}}(y) := Y_{-t_{\mathbf{b}}(y)}(y) \quad \text{when} \quad y \in \mathcal{V}_{\mathbf{b}}.$$

We observe that the function $t_{\mathbf{b}} : \mathcal{V} \to (0, +\infty]$ is measurable, $\mathcal{V}_{\mathbf{b}}$ is a mesurable subset of $\mathcal{O}$ and

$$t_{\mathbf{b}}(Y_s(y)) = t_{\mathbf{b}}(y) + s, \quad y_{\mathbf{b}}(Y_s(y)) = y_{\mathbf{b}}(y), \quad \forall\, y \in \mathcal{V}_{\mathbf{b}},\ \forall\, s \in [0, t_{\mathbf{b}}(y)). \tag{8.76}$$

$$\text{meas}(\{y \in \mathcal{O};\ t_{\mathbf{b}}(y) = t\}) = 0, \quad \forall\, t > 0. \tag{8.77}$$

The properties (8.76) are straightforward while (8.77) is a consequence of the fact that $\{y \in \mathcal{V};\ t_{\mathbf{b}}(y) = t\} \subset Y_t(\Sigma)$ and of the nearly-incompressible condition (8.74).

We now introduce the following first regularity assumption on a at the boundary

$$\forall\, y_0 \in \Sigma,\ y \mapsto a(y) \cdot n(y_0) \text{ is continuous on } \bar{\mathcal{O}}. \tag{8.78}$$

Let us present two examples.

- It may happen that $\mathcal{V}_{\mathbf{b}} = \emptyset$. For instance, choosing $\mathcal{O} := \{y \in \mathbb{R}^2;\ |y| < 1\}$ the unit disk of the plane and $a(y) := |y| y^\perp \in C^{0,1}(\mathbb{R}^2; \mathbb{R}^2)$, $y^\perp := (y_2, -y_1)$, we have $\text{div} a \equiv 0$ and $a(y) \cdot n(y) = y^\perp \cdot y = 0$ for any $y \in \mathbb{R}^2$, so that the flows do not encounter the boundary set $\Sigma = \{y \in \mathbb{R}^2;\ |y| = 1\}$. In that situation $\mathcal{V} = \mathcal{O}$ and $\mathcal{V}_{\mathbf{b}} = \emptyset$.
- In the kinetic case, namely $y = (x, v) \in \mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ an open set with smooth boundary with unit normal outward ν_x, so that $n(y) = (\nu_x, 0)$, and $a(y) = (v, F(x, v))$, we have $\text{div} a = \text{div}_v F$ and $\text{div}_v F = 0$ when $F = E(x) + v \wedge B(x)$, and we have $a(y) \cdot n(y_0) = v \cdot \nu_{x_0}$ which is a smooth function on $\bar{\mathcal{O}} \times \Sigma$.

Lemma 8.28. *Under the condition* (8.78)*, the mapping* $y_{\mathbf{b}} : \mathcal{V}_{\mathbf{b}} \to \Sigma_- \cup \Sigma_0$ *is measurable.*

Proof of Lemma 8.28. From the very definitions and composition rules, we have $y_{\mathbf{b}} : \mathcal{V}_{\mathbf{b}} \to \Sigma \cap \mathcal{Y}$ is mesurable. Take $y \in \mathcal{V}_{\mathbf{b}}$, denote $y_0 := y_{\mathbf{b}}(y)$ and consider a sequence $t_k \searrow 0$ so that $Y_{t_k}(y_0) \to y_0$ and $Y_{t_k}(y_0) \in \mathcal{O}$ for any $k \geq 1$. From (8.78), we deduce

$$0 \geq \limsup_{k \to \infty} \frac{Y_{t_k}(y_0) - y_0}{t_k} \cdot n(y_0) = \lim_{k \to 0} \frac{1}{t_k} \int_0^{t_k} a(Y_s(y_0)) \cdot n(y_0)\, ds = a(y_0) \cdot n(y_0),$$

which means that $y_0 \in \Sigma_- \cup \Sigma_0$. □

For further references, we introduce the following second additional mild regularity assumption on a in the domain

$$a \in L^\infty_{\text{loc}}(\bar{\mathcal{O}}), \quad (a(y) \cdot y)_+ \lesssim \langle y \rangle^2. \tag{8.79}$$

This one may in fact replace the last boundedness condition on a in (8.73). We establish a technical result which will be useful in the sequel.

Lemma 8.29. *Under assumptions* (8.78) *and* (8.79)*, the following hold:*

(1) For any $R_0, T > 0$*, there exists* $R_T, L_T > 0$ *such that for any* $y \in B_{R_0}$ *there hold*

$$\sup_{[-T,T]} |Y_t(y)| \leq R_T \quad \text{and} \quad |Y_{t_2}(y) - Y_{t_1}(y)| \leq L_T |t_2 - t_1|, \quad \forall\, t_1, t_2 \in [-T, T]. \tag{8.80}$$

(2) For a sequence (y_ε) *of* $\mathcal{V}$ *such that* $y_\varepsilon \to y_0 \in \Sigma_- \cap \mathcal{Y}$*, we have* $t_{\mathbf{b}}(y_\varepsilon) \to 0$ *and* $y_{\mathbf{b}}(y_\varepsilon) \to y_0$*.*

Proof of Lemma 8.29. Proof of (1). Take $y \in \mathcal{V} \cap B_{R_0}$. On the one hand, from (8.79), we have

$$\langle Y_t(y) \rangle^2 - \langle y \rangle^2 = 2 \int_0^t a(Y_s(y)) \cdot Y_s(y)) ds \leq C \int_0^t \langle Y_s(y) \rangle^2 ds,$$

and we conclude to $Y_t(y) \in B_{R_T}$ thanks to the Gronwall lemma. As a consequence, we have

$$|Y_{t_2}(y) - Y_{t_1}(y)| \leq |t_2 - t_1| \|a\|_{L^\infty(B_{R_T})}, \quad \forall\, t_i \in [-T, T].$$

Proof of (2). Assume by contradiction that $\limsup t_{\mathbf{b}}(y_\varepsilon) \geq \tau > 0$ and set $T := \tau + 1$. By assumption, there exists $R_0 > 0$ such that $y_\varepsilon \in B_{R_0}$ and thus by step 1, (8.80) holds uniformly in $\varepsilon \in (0, 1]$. Thanks to the Ascoli Theorem and the contradiction hypothesis, there exists $\varepsilon_k \to 0$ such that $t_{\mathbf{b}}(y_{\varepsilon_k}) \geq \tau$ and a there exists $Y \in C([-T, T])$ such that $Y_\bullet(y_{\varepsilon_k}) \to Y_\bullet$ in $C([-T, T])$. Next, passing to the limit in the conditions

$$Y_{-t}(y_{\varepsilon_k}) \in \mathcal{O} \quad \text{and} \quad Y_{-t}(y_{\varepsilon_k}) - y_{\varepsilon_k} = -\int_0^t a(Y_{-s}(y_{\varepsilon_k}))\, ds,$$

for any $k \ge 1$ and any $t \in [0,\tau]$, we get

$$Y_{-t} \in \bar{\mathcal{O}} \quad \text{and} \quad Y_{-t} - y_0 = -\int_0^t a(Y_{-s}) ds.$$

We deduce

$$0 \ge \limsup_{t\to 0} \frac{Y_{-t} - y_0}{t} \cdot n(y_0) = \lim_{t\to 0} -\frac{1}{t} \int_0^t a(Y_{-s}) \cdot n(y_0)\, ds = -a(y_0) \cdot n(y_0),$$

which is in contradiction with the hypothesis $y_0 \in \Sigma_-$. Now, we may estimate

$$|y_{\mathbf{b}}(y_\varepsilon) - y_0| \quad \le \quad |y_\varepsilon - y_0| + \int_0^{t_{\mathbf{b}}(y_\varepsilon)} |a(Y_s(y_\varepsilon))| ds \to 0,$$

as $\varepsilon \to 0$, as a consequence of the convergence $t_{\mathbf{b}}(y_\varepsilon) \to 0$, the first estimate in (8.80) and the first condition in (8.79). $\square$

We reformulate some *"space continuity"* of solution to the transport equation results picked up in [70, Sec. 7]. Defining

$$\mathcal{O}_\alpha := \{y \in \mathcal{O};\, \delta(y) > \alpha\}, \quad \Sigma_\alpha := \{y \in \mathcal{O};\, \delta(y) = \alpha\} = \partial\mathcal{O}_\alpha,$$

we know from [70, Sec. 2], that there exists $\alpha_{\mathcal{O}} > 0$ such that for any $\alpha \in (0, \alpha_{\mathcal{O}})$, the mapping

$$\theta_\alpha : \Sigma \to \Sigma_\alpha, \quad \theta_\alpha(z) := z - \alpha n(z)$$

is an isomorphim with associated jacobian function J_α and

$$\int_{\Sigma_\alpha} h(z') d\sigma_\alpha(z') = \int_\Sigma h \circ \theta_\alpha(z) J_\alpha(z) d\sigma_z, \tag{8.81}$$

$$\int_{\mathcal{O}\backslash\mathcal{O}_\alpha} g(y) dy = \int_0^\alpha \int_\Sigma g \circ \theta_{\alpha'}(z) J_{\alpha'}(z) d\sigma_z d\alpha', \tag{8.82}$$

for any $h \in L^1(\Sigma_\alpha)$ and $g \in L^1(\mathcal{O}\backslash\mathcal{O}_\alpha)$, where $d\sigma_\alpha$ denotes the Lebesgue measure on Σ_α and where the jacobian function J_α satisfies $1/2 \le J_\alpha \le 3/2$ as well as $J_\alpha \to I$ as $\alpha \to 0$.

Lemma 8.30. *For any $g \in L^\infty(\mathcal{O})$ satisying $a \cdot \nabla g \in L^1(\mathcal{O})$, we have*

$$g = \gamma_\alpha g \quad \text{a.e. on } \Sigma_\alpha \backslash \{a \cdot n = 0\} \text{ for a.e. } \alpha \in [0, \alpha_{\mathcal{O}}], \tag{8.83}$$

where we denote by $\gamma_\alpha g$ the trace of g on Σ_α, and

$$\gamma_\alpha g \circ \theta_\alpha \to \gamma g \text{ as } \alpha \to 0, \quad \text{a.e. on } \Sigma\backslash\Sigma_0. \tag{8.84}$$

Proof of Lemma 8.30. For $\varphi \in C_c(\bar{\mathcal{O}}) \cap W^{1,1}(\mathcal{O})$ and $\beta \in C^1(\mathbb{R})$, the renormalized Green formula (8.17) writes

$$\int_\Sigma \varphi\beta(\gamma g)\, a \cdot n\, d\sigma = \int_{\Sigma_\alpha} \varphi\beta(\gamma_\alpha g)\, a \cdot n\, d\sigma_\alpha + \int_{\mathcal{O}\backslash\mathcal{O}_\alpha} [\mathrm{div}(a\varphi)\beta(g) + \varphi\beta'(g) a \cdot \nabla g]\, dy,$$

and thus

$$\int_\Sigma \varphi\beta(\gamma g)\, a \cdot n\, d\sigma = \lim_{\alpha\to 0} \int_{\Sigma_\alpha} \varphi\beta(\gamma_\alpha g)\, a \cdot n\, d\sigma_\alpha. \tag{8.85}$$

Denoting $\psi := (1 - (\delta(x) - \alpha)/s)_+$, $\alpha + s \in (\alpha, \alpha_{\mathcal{O}})$, observing that $\psi_{|\Sigma_\alpha} = 1$, recalling that $n = -\nabla\delta$ and using $\varphi\psi$ as a test function, we similarly have

$$\int_{\Sigma_\alpha} \varphi\beta(\gamma_\alpha g)\, a \cdot n\, d\sigma_\alpha = \frac{1}{h} \int_{\mathcal{O}_\alpha\backslash\mathcal{O}_{\alpha+s}} \varphi\beta(g)\, a \cdot n\, dy + \int_{\mathcal{O}_\alpha\backslash\mathcal{O}_{\alpha+s}} [\mathrm{div}(a\varphi)\beta(g) + \varphi\beta'(g) a \cdot \nabla g]\psi\, dy,$$

so that

$$\int_{\Sigma_\alpha} \varphi\beta(\gamma_\alpha g)\, a \cdot n\, d\sigma_\alpha = \lim_{s\to 0} \frac{1}{s} \int_{\mathcal{O}_\alpha\backslash\mathcal{O}_{\alpha+s}} \varphi\beta(g)\, a \cdot n\, dy.$$

We immediately deduce

$$\int_{\Sigma_\alpha} \varphi\beta(\gamma_\alpha g)\, a \cdot n\, d\sigma_\alpha = \frac{d}{d\alpha} \int_{\mathcal{O}_\alpha} \varphi\beta(g)\, a \cdot n\, dy$$

and next

$$\int_0^\alpha \int_{\Sigma_{\alpha'}} \varphi\beta(\gamma_{\alpha'} g)\, a\cdot n\, d\sigma_{\alpha'} d\alpha' = \int_{\mathcal{O}\backslash\mathcal{O}_\alpha} \varphi\beta(g)\, a\cdot n\, dy.$$

Together with (8.82) and (8.81), we have established

$$\int_0^\alpha \int_\Sigma [\varphi\beta(\gamma_{\alpha'} g)\, a\cdot n]\circ\theta_{\alpha'}(z) J_{\alpha'}(z) d\sigma_z d\alpha' = \int_0^\alpha \int_\Sigma [\varphi\beta(g)\, a\cdot n]\circ\theta_{\alpha'}(z) J_{\alpha'}(z) d\sigma d\alpha',$$

so that

$$\beta(\gamma_{\alpha'} g)\, a\cdot n = \beta(g)\, a\cdot n \ \text{ a.e. on } \Sigma_{\alpha'} \text{ for a.e. } \alpha' \in (0,\alpha),$$

from what (8.83) follows. On the other hand, using (8.81) and (8.85) together, we have

$$\int_\Sigma \varphi\beta(\gamma g)\, a\cdot n\, d\sigma = \lim_{\alpha\to 0} \int_\Sigma [\varphi\beta(\gamma_\alpha g)\, a\cdot n]\circ\theta_\alpha J_\alpha\, d\sigma_\alpha,$$

which implies

$$[\beta(\gamma_\alpha g)\, a\cdot n]\circ\theta_\alpha J_\alpha \rightharpoonup \beta(\gamma g)\, a\cdot n \quad *\,\sigma(L^\infty_{\rm loc}(\Sigma), L^1_{\rm c}(\Sigma)). \tag{8.86}$$

Repeating the same argument with $g := a\cdot n$ and using that $J_\alpha \to I$ uniformly, we get

$$[\beta(a\cdot n)\, a\cdot n]\circ\theta_\alpha \rightharpoonup \beta(a\cdot n)\, a\cdot n \quad *\,\sigma(L^\infty_{\rm loc}(\Sigma), L^1_{\rm c}(\Sigma)),$$

for any $\beta \in C^1(\mathbb{R})$. Choosing $\beta(s) = 1$ and $\beta(s) = s$, we classically deduce that $[a\cdot n]\circ\theta_\alpha \to a\cdot n$ a.e. on Σ. We finally conclude to (8.84) by gathering that last information with (8.86) written for $\beta(s) = s$ and $\beta(s) = s^2$. □

Remark 8.31. *During the proof, we have in fact established that*

$$[0,\alpha_{\mathcal{O}}] \to L^1(\Sigma); \quad \alpha \mapsto [\gamma_\alpha g\, a\cdot n]\circ\theta_\alpha$$

is continuous.

Lemma 8.32. *We make the additional assumptions* (8.78) *and* (8.79)*. If Y is an almost everywhere flow associated to* (8.6)*, then the function $t_{\mathbf{b}} \in L(\mathcal{O})$ is a renormalized solution to the equation*

$$a\cdot\nabla t_{\mathbf{b}} = 1 \ \ \text{in} \ \ \mathcal{O}, \quad \gamma_- t_{\mathbf{b}} = 0 \ \ \text{on} \ \ \Sigma_-\cap\mathcal{Y}. \tag{8.87}$$

Proof of Lemma 8.32. Step 1. We fix $\beta \in C^1_*(\mathbb{R})$ and we recall that $\beta(t_{\mathbf{b}}(Y_s(y))) = \beta(t_{\mathbf{b}}(y)+s)$ for any $s\in\mathbb{R}$ for a.e. $y\in\mathcal{O}$. For any $\varphi\in C^1_c(\mathcal{O})$, we may compute

$$\begin{aligned}\int_{\mathbb{R}^d} \beta'(t_{\mathbf{b}}(y)+s)\varphi(y)\, dy &= \frac{d}{ds}\int_{\mathbb{R}^d} \beta(t_{\mathbf{b}}(y)+s)\varphi(y)\, dy \\ &= \frac{d}{ds}\int_{\mathbb{R}^d} \beta(t_{\mathbf{b}}(Y_s(y)))\varphi(y)\, dy \\ &= \frac{d}{ds}\int_{\mathbb{R}^d} \beta(t_{\mathbf{b}}(y))\varphi(Y_{-s}(y)) e^{-\int_{-s}^0 ({\rm div} a)(Y_\tau(y))d\tau}\, dy \\ &= \int_{\mathbb{R}^d} \beta(t_{\mathbf{b}}(y))[-a\cdot\nabla\varphi - ({\rm div} a)\varphi](Y_{-s}(y)) e^{-\int_{-s}^0 ({\rm div} a)(Y_\tau(y))d\tau}\, dy.\end{aligned}$$

Taking $s = 0$, we conclude to

$$\int_{\mathbb{R}^d} \beta'(t_{\mathbf{b}})\varphi\, dy = \int_{\mathbb{R}^d} \beta(t_{\mathbf{b}})[-a\cdot\nabla\varphi - ({\rm div} a)\varphi]\, dy,$$

which is nothing but the distributional formulation of the equation

$$a\cdot\nabla\beta(t_{\mathbf{b}}) = \beta'(t_{\mathbf{b}}).$$

That last family of equations is the renormalized formulation of equation (8.87) in the domain.
Step 2. Using lemma 8.30 with $g := \beta(t_{\mathbf{b}})$, we have

$$\gamma_\alpha\beta(t_{\mathbf{b}})\circ\theta_\alpha \to \beta(\gamma t_{\mathbf{b}}) \ \text{ a.e. on } \ \Sigma.$$

Using Lemma 8.29, we also have

$$\gamma_\alpha\beta(t_{\mathbf{b}})\circ\theta_\alpha \to 0 \ \text{ a.e. on } \ \Sigma_-\cap\mathcal{Y}.$$

Both together, we find $\gamma_- t_{\mathbf{b}} = 0$ on $\Sigma_-\cap\mathcal{Y}$. □

We establish the main result of this section.

Theorem 8.33 (characteristics method). *Assume that Y is an almost everywhere flow associated to* (8.6) *with a satisfying* (8.73). *For any $g_0 \in L^\infty(\mathcal{O})$, $\mathfrak{g} \in L^\infty((0,T)\times\Sigma_-)$, $T>0$, $b \in L^\infty(\mathcal{O})$, the function*

$$
\begin{aligned}
\bar g(t,y) \quad := \quad & g_0(Y_{-t}(y)) e^{-\int_0^t b(Y_{\tau-t}(y))\,d\tau} \mathbf{1}_{t<t_{\mathbf{b}}(y)} \\
& + \mathfrak{g}(t-t_{\mathbf{b}}(y), y_{\mathbf{b}}(y)) e^{-\int_0^{t_{\mathbf{b}}(y)} b(Y_{\tau-t_{\mathbf{b}}(y)}(y))\,d\tau} \mathbf{1}_{t>t_{\mathbf{b}}(y)}
\end{aligned}
\tag{8.88}
$$

is the unique solution in $C([0,T];L^1_{\mathrm{loc}}(\mathcal{O}))\cap L^\infty((0,T)\times\mathcal{O})$ to the evolution transport equation

$$
\begin{cases}
\dfrac{\partial g}{\partial t} + a\cdot\nabla g + bg = 0 \quad \textit{on} \quad (0,T)\times\mathcal{O}, \\
\gamma_- g = \mathfrak{g} \quad \textit{on} \quad (0,T)\times\Sigma_-, \quad g(0,\cdot) = g_0 \quad \textit{on} \quad \mathcal{O}.
\end{cases}
\tag{8.89}
$$

First proof of Theorem 8.33. We additionally assume that (8.75), (8.78) and (8.79) hold, that $\mathfrak{g} \in C((0,T)\times\Sigma_-)$, $\operatorname{supp}\mathfrak{g} \subset (0,T)\times(\Sigma_-\cap\mathcal{Y})$ and $b \in C(\bar{\mathcal{O}})$, and we mostly repeat the proof of [255, Prop. 1]. From the above definition, for a.e. $y \in \mathcal{O}$ and any $t \in (0,\infty)$, we have

$$
\bar g(t+s, Y_s(y)) e^{\int_0^s b(Y_\tau(y))d\tau} = \bar g(t,y), \quad \forall\, s \ge -t.
\tag{8.90}
$$

Let us then fix $\varphi \in \mathcal{D}((0,T)\times\mathcal{O})$ and let us extend $\bar g$ and φ by 0 outside of $\mathcal{O}$. We compute

$$
\begin{aligned}
0 \quad &= \quad \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar g(t,y)\varphi(t,y)\,dydt \\
&= \quad \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar g(t+s,Y_s(y)) e^{\int_0^s b(Y_\tau(y))d\tau}\varphi(t,y)\,dydt \\
&= \quad \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar g(t,y)\varphi(t-s,Y_{-s}(y)) e^{\int_0^s (b-\mathrm{div} a)(Y_\tau(y))ds}\,dydt \\
&= \quad \int_0^T\int_{\mathbb{R}^d} \bar g(t,y)\frac{d}{ds}[\varphi(t-s,Y_{-s}(y)) e^{\int_0^s (b-\mathrm{div} a)(Y_\tau(y))ds}]\,dydt \\
&= \quad \int_0^T\int_{\mathbb{R}^d} \bar g(t,y)[-\partial_t\varphi - a\cdot\nabla\varphi + (b-\mathrm{div} a)\varphi](t-s,Y_{-s}(y))\,dydt,
\end{aligned}
$$

where we have used the relation (8.90) in the second line and the change of variables property (8.75) in the third line. Taking $s=0$, we get

$$
0 = \int_0^T\int_{\mathbb{R}^d} \bar g(t,y)[-\partial_t\varphi - a\cdot\nabla\varphi + (b-\mathrm{div} a)\varphi](t,y)\,dydt,
$$

which exactly means that $\bar g$ is a solution to equation (8.89) in the distributional sense. Now, because $t_{\mathbf{b}}(y) > 0$ for any $y \in \mathcal{Y}$, we have $\bar g(0,y) = g_0(y)$. Take $t>0$, and for $\alpha \in (0,\alpha_{\mathcal{O}})$, let us denote $A_\alpha := \{y \in \Sigma_-\cap\mathcal{Y};\ t_{\mathbf{b}}\circ\theta_\alpha(y) < t\}$, so that

$$
\bar g(t,\theta_\alpha(y)) = \mathfrak{g}(t-t_{\mathbf{b},\alpha}(y), y_{\mathbf{b},\alpha}(y)) e^{-\int_0^{t_{\mathbf{b},\alpha}(y)} b(Y_{\tau-t_{\mathbf{b},\alpha}(y)}(\theta_\alpha(y)))\,d\tau} \quad \text{on} \quad A_\alpha,
$$

where we use the shorthands $t_{\mathbf{b},\alpha} := t_{\mathbf{b}}\circ\theta_\alpha$ and $y_{\mathbf{b},\alpha} := y_{\mathbf{b}}\circ\theta_\alpha$. From $\theta_{\alpha'}(y) \to y$ as $\alpha' \to 0$ when $y \in \Sigma$ and Lemma 8.29, we deduce

$$
\bar g(t,\theta_{\alpha'}(\cdot)) \to \mathfrak{g}(t,\cdot) \quad \text{as } \alpha' \to 0,
$$

on A_α for any fixed $\alpha>0$. Because $\cup_{\alpha>0}A_\alpha = \Sigma_-\cap\mathcal{Y}$, the same convergence holds on $\Sigma_-\cap\mathcal{Y}$. On the other hand, we have $\gamma_\alpha\bar g\circ\theta_\alpha \to \gamma\bar g$ as $\alpha\to 0$ a.e. on $\Sigma\backslash\Sigma_0$, from Lemma 8.30. We deduce that $\gamma\bar g = \mathfrak{g}$ on $\Sigma_-\cap\mathcal{Y}$.

Second proof of Theorem 8.33. We do not make any additional assumption and we mainly repeat the proof presented in [204, Sec. 3]. Consider the unique solution $g \in C([0,T];L^1_{\mathrm{loc}}(\bar{\mathcal{O}}))\cap L^\infty((0,T)\times\mathcal{O})$ to the transport equation (8.89). Regularizing by convolution

$$
g_\varepsilon := g *_{t,x,\varepsilon} \rho_\varepsilon,
$$

for a time and space dependent mollifier sequence ρ_ε similarly as in (8.20), we have $g_\varepsilon \in C^1([0,T] \times \bar{\mathcal{O}})$ and

$$\frac{\partial g_\varepsilon}{\partial t} + a \cdot \nabla g_\varepsilon + b g_\varepsilon = R_\varepsilon \ \text{ on } \ (0,T) \times \mathcal{O},$$

with the usual commutator $R_\varepsilon := [a \cdot \nabla g + bg, \rho_\varepsilon]$. We know from a classical time and space variant of Lemma 8.5 (see also [139, Lem. II.1], [271, Lem. 1], [70, Lem. 3.1] and the proof in [204, Sec. 3]) that $R_\varepsilon \to 0$ in L^1. Because g_ε is smooth and Y is an almost everywhere flow, we may set $H^\sharp(s,y) := H(t+s, Y_s(y))$, $\mathfrak{B}(s,y) := \int_0^s b(Y_\tau(y))d\tau$ and compute

$$\frac{d}{ds}[(g_\varepsilon^\sharp e^{\mathfrak{B}})(s,y)] = (R_\varepsilon^\sharp e^{\mathfrak{B}})(s,y),$$

from what we get

$$\begin{aligned}\widetilde{g}_\varepsilon(t,y) &:= g_\varepsilon(t,y) - g_\varepsilon(0, Y_{-t}(y)) e^{\mathfrak{B}(-t,y)} \mathbf{1}_{t<t_{\mathbf{b}}(y)} - g_\varepsilon(t - t_{\mathbf{b}}(y), y_{\mathbf{b}}(y)) e^{\mathfrak{B}(-t_{\mathbf{b}(y)},y)} \mathbf{1}_{t>t_{\mathbf{b}}(y)} \\ &= \int_0^t (R_\varepsilon^\sharp e^{\mathfrak{B}})(s,y) ds \mathbf{1}_{t<t_{\mathbf{b}}(y)} + \int_{t-t_{\mathbf{b}}(y)}^t (R_\varepsilon^\sharp e^{\mathfrak{B}})(s,y) ds \mathbf{1}_{t>t_{\mathbf{b}}(y)},\end{aligned}$$

for a.e. $y \in \mathcal{O}$ and any $t > 0$ and where we use (8.77) for getting rid of the set $\{y \in \mathcal{O}; t_{\mathbf{b}}(y) = t\}$. For $T, \varrho > 0$ and setting $\mathcal{U}_{T,\varrho} := (0,T) \times (\mathcal{O} \cap B_\varrho)$, we deduce

$$\begin{aligned}\int_{\mathcal{U}_{T,\varrho}} |\widetilde{g}_\varepsilon(t,y)|\, dydt &\le T \int_{\mathcal{U}_{T,\varrho}} |(R_\varepsilon^\sharp e^{\mathfrak{B}})(s,y)| dyds \\ &\le T e^{(C + \|b\|_{L^\infty})T} \int_{\mathcal{U}_{T,\varrho}} |R_\varepsilon(s,y)| dyds,\end{aligned}$$

from the near-incompressibility condition (8.74) of the flow. From Proposition 8.10 and Remark 8.11, we know that

$$\begin{aligned}&g_\varepsilon(t,\cdot) \to g(t,\cdot) \ \text{ in } \ L^1_{\rm loc}(\bar{\mathcal{O}}) \ \text{ as } \ \varepsilon \to 0, \ \text{ for any } \ t \in [0,T]; \\ &g_{\varepsilon|\Sigma_-} \to \gamma_- g = \mathfrak{g} \ \text{ in } \ L^1_{\rm loc}(\Sigma_-) \ \text{ as } \ \varepsilon \to 0.\end{aligned}$$

Passing to the limit, we get

$$\int_{\mathcal{U}_{T,\varrho}} |g(t,y) - g_0(Y_{-t}(y)) e^{\mathfrak{B}(-t,y)} \mathbf{1}_{t<t_{\mathbf{b}}(y)} - \mathfrak{g}(t - t_{\mathbf{b}}(y), y_{\mathbf{b}}(y) e^{\mathfrak{B}(-t_{\mathbf{b}(y)},y)} \mathbf{1}_{t>t_{\mathbf{b}}(y)}| dtdy = 0,$$

for any $T, \varrho > 0$, what is nothing but (8.88). □

Corollary 8.34 (Representation formula)**.** *Under the assumptions of Lemma 8.14 in a smooth domain $\mathcal{O} \neq \mathbb{R}^D$, for any $\lambda > \lambda_{a,b,p}$, $G \in L^p(\mathcal{O})$, the unique solution $g \in L^p(\mathcal{O})$ to the stationary transport equation* (8.29) *(with $\mathfrak{g} = 0$) satisfies*

(8.91) $$g(y) = \int_0^\infty e^{-\lambda t} (S_b(t) G)(y) dt, \quad \textit{for a.e } \ y \in \mathcal{O},$$

with S_b is defined by (8.49) *in which formula Y and $t_{\mathbf{b}}$ stand for the characteristics and backward exit time defined just as above.*

Proof of Corollary 8.34. That is nothing but (2.13). □

Adapting the second proof of Theorem 8.33, we obtain a more accurate characterization of the backward exit time $t_{\mathbf{b}}$ with more general assumptions on the vector field.

Lemma 8.35. *Assume that Y is an almost everywhere flow associated to* (8.6) *with a satisfying* (8.73)*. The backward exit time $t_{\mathbf{b}}$ is the unique renormalized solution in $L(\mathcal{O})$ to the backward exit time problem*

(8.92) $$a \cdot \nabla \tau = 1 \ \textit{ in } \ \mathcal{O}, \quad \gamma_- \tau = 0 \ \textit{ on } \ \Sigma_-.$$

We also have

(8.93) $$t_{\mathbf{b}} = \int_0^\infty S_0(t) \mathbf{1} dt,$$

where $\mathbf{1}$ stands for the unit function in $\mathcal{O}$ and S_0 is defined by (8.49) *with $b = 0$ and in which formula Y and $t_{\mathbf{b}}$ stand for the characteristics and backward exit time defined just as above.*

Proof of Lemma 8.35. Step 1. Existence. From Lemma 8.12, for any $\lambda > \lambda^*_\infty := 0$, there exists a (unique) solution $\tau_\lambda \in L^\infty(\mathcal{O})$ to the truncated backward exit time problem

$$\lambda\tau_\lambda + a\cdot\nabla\tau_\lambda = 1 \ \text{ in } \ \mathcal{O}, \quad \gamma_-\tau_\lambda = 0 \ \text{ on } \ \Sigma_-.$$

From the weak maximum principle, we have $\tau_\lambda \geq 0$. As a consequence, for $0 < \lambda < \lambda'$, we have

$$\lambda(\tau_\lambda - \tau_{\lambda'} + a\cdot\nabla(\tau_\lambda - \tau_{\lambda'} = (\lambda' - \lambda)\tau_{\lambda'} \geq 0 \ \text{ in } \ \mathcal{O}, \quad \gamma_-(\tau_\lambda - \tau_{\lambda'}) = 0 \ \text{ on } \ \Sigma_-.$$

From the weak maximum principle again, we deduce that $\tau_\lambda - \tau_{\lambda'} \geq 0$, and (τ_{λ_n}) is an increasing sequence when $\lambda_n \searrow 0$. We set $\tau := \lim_{n\to\infty}\tau_{\lambda_n}$, so that $\tau \in L(\mathcal{O})$ is a nonnegative renormalized solution to the backward exit time problem.

Step 2. Characterization. By definition, for any $\beta \in C^1_*(\mathbb{R})$, $\beta(0) = 0$, the function τ satisfies

$$a\cdot\nabla\beta(\tau) = \beta'(\tau) \ \text{ in } \ \mathcal{O}, \quad \gamma_-\beta(\tau) = 0 \ \text{ on } \ \Sigma_-.$$

With the notations of (8.20), we define $b_\varepsilon := \beta(\tau)*_\varepsilon\rho_\varepsilon$, $B_\varepsilon := \beta'(\tau)*_\varepsilon\rho_\varepsilon$, and thanks to Lemma 8.5, we have thus

$$a\cdot\nabla b_\varepsilon = B_\varepsilon + r_\varepsilon \ \text{ in } \ \mathcal{D}'(\mathcal{O}),$$

with

$$b_\varepsilon \to \beta(\tau), \quad B_\varepsilon \to \beta'(\tau), \quad r_\varepsilon \to 0,$$

respectively in $L^p_{\rm loc}(\bar{\mathcal{O}})$, $L^p_{\rm loc}(\bar{\mathcal{O}})$ and $L^1_{\rm loc}(\bar{\mathcal{O}})$ for any $p \in [1,\infty)$. Because then

$$\frac{d}{ds}b_\varepsilon \circ Y_s = (a\cdot\nabla b_\varepsilon)(Y_s) = (B_\varepsilon + r_\varepsilon)(Y_s),$$

and defining

$$\begin{aligned}\widetilde{b}_{t,\varepsilon}(y) \quad &:= \quad \Big\{b_\varepsilon(Y_{-t}(y)) + \int_{-t}^0 B_\varepsilon(Y_s(y))\,ds\Big\}\mathbf{1}_{t<t_{\mathbf{b}}(y)} \\ &\qquad + \Big\{(\gamma_-b_\varepsilon)(y_{\mathbf{b}}(y))) + \int_{-t_{\mathbf{b}}(y)}^0 B_\varepsilon(Y_s(y))\,ds\Big\}\mathbf{1}_{t>t_{\mathbf{b}}(y)},\end{aligned}$$

we have

$$b_\varepsilon(y) - \widetilde{b}_{t,\varepsilon}(y) = R_{t,\varepsilon} := \int_{-t}^0 r_\varepsilon(Y_s(y))\,ds\mathbf{1}_{t<t_{\mathbf{b}}(y)} + \int_{-t_{\mathbf{b}}(y)}^0 r_\varepsilon(Y_s(y))\,ds\mathbf{1}_{t>t_{\mathbf{b}}(y)}.$$

Arguing similarly as in the second proof of Theorem 8.33, we have $\widetilde{b}_{t,\varepsilon} \to \widetilde{b}_t$ and $R_{t,\varepsilon} \to 0$ in $L^1(\mathcal{U}_{\rho,T})$ as $\varepsilon \to 0$, with

$$\widetilde{b}_t(y) := \Big\{\beta(\tau(Y_{-t}(y))) + \int_{-t}^0 \beta'(\tau(Y_s(y)))\,ds\Big\}\mathbf{1}_{t<t_{\mathbf{b}}(y)} + \int_{-t_{\mathbf{b}}(y)}^0 \beta'(\tau(Y_s(y)))\,ds\Big\}\mathbf{1}_{t>t_{\mathbf{b}}(y)}.$$

We deduce that

$$\beta(\tau(y)) = \widetilde{b}_t(y), \quad \text{for a.e. } t > 0,\ y \in \mathcal{O}.$$

Choosing a sequence (β_n) of renormalizing function in $C^1_*(\mathbb{R})$ such that $0 \leq \beta_n(s) \nearrow s$ and $0 \leq \beta_n'(s) \nearrow 1$ locally uniformly, writing the above equation for $\beta = \beta_n$ and passing to the limit $n \to \infty$, we obtain

$$\tau(y) = \Big\{\tau(Y_{-t}(y)) + t\Big\}\mathbf{1}_{t<t_{\mathbf{b}}(y)} + t_{\mathbf{b}}(y)\mathbf{1}_{t>t_{\mathbf{b}}(y)},$$

and in particular $\tau = t_{\mathbf{b}}$ a.e. on $\mathcal{O}$. That implies that $t_{\mathbf{b}}$ is the unique renormalized solution to the backward exit time problem (8.92). From Corollary 8.34, for any $\lambda > 0$, we have

$$\tau_\lambda = \int_0^\infty e^{-\lambda t}S_0(t)\mathbf{1}dt \quad \text{a.e on } \ \mathcal{O},$$

and we deduce (8.93) by passing to the limit $\lambda \searrow 0$ in that identity. $\square$

8.5. **On the Krein-Rutman theorem for the transport equation with kernel terms.** In this section we carry on our analysis of the transport equation with kernel term (8.1)-(8.3) for which we establish a Krein-Rutman result under strong positivity assumption on the kernel acting on the domain. As in section 8.3, we assume that a, b, $\mathscr{K}$ and $\mathscr{R}$ satisfy the conditions (8.35), (8.59), (8.60), (8.61), (8.62) and (8.63) for some weight function $m : \bar{\mathcal{O}} \to [1, \infty)$ and some exponent $p \in [1, \infty)$.

On the kernel $\mathscr{K}$, we make the additional strong positivity hypothesis: for any $x \in \mathcal{O}$, there exist $r_x, \eta_x > 0$ such that

(8.94) $$\forall f \geq 0,\ \forall y \in B(x, r_x), \quad \mathscr{K}[f](y) \geq \eta_x \int_{B(x,r_x)} f_* dy_*$$

and

(8.95) $$\exists\, x_0, \quad a, b \in L^\infty(B(x_0, r_0)), \quad r_0 := r_{x_0},$$

as well as one of the two following regularity assumptions

(8.96) $$\mathscr{K} \in \mathcal{K}(L^p_m(\mathcal{O})) \ \text{ or } \ \mathscr{K} : L^p_m(\mathcal{O}) \to L^{p_1}(\mathcal{O}) \cap L^p_{m_1}(\mathcal{O}),$$

with $p_1 > p$ and $m_1/m \to \infty$ when $y \to \infty$.
We thus consider the operator

(8.97) $$\mathcal{L}f = -a \cdot \nabla f - bf + \mathscr{K}[f] = -\text{div}(af) - Kf + \mathscr{K}[f]$$

with $K := b - \text{div}a \geq 0$, which is complemented with the boundary condition

(8.98) $$\gamma_- f = \mathscr{R}_{\mathcal{O}}[f] + \mathscr{R}_\Sigma[\gamma_+ f] \ \text{ on } \Sigma_-.$$

More precisely, we define $\mathcal{L}$ in the Banach space $L^p_m(\mathcal{O})$ with domain

$$D(\mathcal{L}) \subset \{f \in L^1(\mathcal{O});\, a \cdot \nabla f \in L^1_{\text{loc}}(\bar{\mathcal{O}}),\ \gamma_- f = \mathscr{R}[f, \gamma_+ f]\}.$$

Notice that because of Section 8.1 the trace function is well defined.

Example. The nonlocal operator with a drift

(8.99) $$\partial_t f = -a\, \partial_x f - b + \mathscr{K}[f] \ \text{ in } \mathcal{O}, \quad \gamma_- f = 0 \ \text{ on } \Sigma_-,$$

with $\mathcal{O} \subset \mathbb{R}$ a bounded interval, $a \in W^{1,1}_{\text{loc}}(\mathcal{O})$, $a' \in L^\infty(\mathcal{O})$, $b \in L^\infty(\mathcal{O})$, and thus the boundary kernel is $\mathscr{R} \equiv 0$. Motivated by some non-local reaction-diffusion models, this problem was recently investigated in [109, 118, 249]. It is also used in the study of selection-mutation models in changing environment, see the even newer works [162, 207].

We start by checking that with the above assumptions, the conditions **(H1)**–**(H5)** presented in the abstract part hold true.

Condition (H1). From Proposition 8.26, we know that for any $\lambda > \lambda^{**}$ the stationary problem

$$(\lambda - \mathcal{L})g = G \text{ in } \mathcal{O}, \quad \gamma_- g = \mathscr{R}[g, \gamma_+ g] \text{ on } \Sigma_-,$$

has a unique solution. More precisely, the associated inverse operator denoted by $\mathcal{R}_\mathcal{L}$ (without reference to the boundary operator $\mathscr{R}$) satisfies $\mathcal{R}_\mathcal{L} : L^p_m \to L^p_m$ and $\mathcal{R}_\mathcal{L} G \geq 0$ if $G \geq 0$.

Condition (H2). We first consider the case when $\mathscr{R}_\mathcal{O} \equiv 0$ and we denote by $\mathcal{L}_0$ the associated generator. We fix $f_0 \in C^2_c(\mathcal{O})$, such that $\int_{\mathcal{B}_0} f_0 dy = 1$, $f_0 > 0$ on $\mathcal{B}_0$, $\text{supp} f_0 = \bar{\mathcal{B}}_0$, as well as

$$\|f_0^{-1}\|_{L^\infty(\mathcal{B}_\varepsilon)} = \varepsilon^{-2}, \quad \|\nabla f_0\|_{L^\infty(\mathcal{B}_0 \backslash \mathcal{B}_\varepsilon)} = \varepsilon, \quad \forall \varepsilon \in (0, 1/2),$$

where we denote $\mathcal{B}_\varepsilon := B(x_0, (1-\varepsilon)r_0)$. We also define $C_0 := \|f_0\|_{L^\infty(\mathcal{B}_0)}$ and $C_1 := \|\nabla f_0\|_{L^\infty(\mathcal{B}_0)}$, both may be bounded by a constant which only depends on r_0 and d. Because of (8.94), we have

$$\mathscr{K}[f_0](y) \geq \eta_0 \mathbf{1}_{\mathcal{B}_0} \geq \frac{\eta_0}{C_0} f_0.$$

We observe that $f_0 \in D(\mathcal{L}_0)$ and we compute

(8.100) $$\mathcal{L}_0 f_0 \geq -\|a\|_{L^\infty(\mathcal{B}_0)} C_1 \mathbf{1}_{\mathcal{B}_0} - \|b\|_{L^\infty(\mathcal{B}_0)} C_0 \mathbf{1}_{\mathcal{B}_0} + \eta_0 \mathbf{1}_{\mathcal{B}_0} \geq \kappa_0 f_0,$$

if $\kappa_0 := \eta_0/C_0 - \|a\|_{L^\infty(\mathcal{B}_0)} C_1/C_0 - \|b\|_{L^\infty(\mathcal{B}_0)} \geq 0$. More generally, we have

(8.101) $$\mathcal{L}_0 f_0 \geq -\|a\|_{L^\infty(\mathcal{B}_0)} C_1 \mathbf{1}_{\mathcal{B}_\varepsilon} - \|a\|_{L^\infty} \varepsilon \mathbf{1}_{\mathcal{B}_0 \backslash \mathcal{B}_\varepsilon} - \|b\|_{L^\infty(\mathcal{B}_0)} f_0 + \eta_0 \mathbf{1}_{\mathcal{B}_0} \geq \kappa_0 f_0,$$

with $\kappa_0 := -\|a\|_{L^\infty(\mathcal{B}_0)} C_1 \varepsilon^{-2} - \|b\|_{L^\infty(\mathcal{B}_0)} \in \mathbb{R}$ when $\|a\|_{L^\infty}\varepsilon \le \eta_0$. Depending on how $\eta_0 > 0$ is large, we obtain in that way two constructive lower bounds of $\mathcal{I}$ thanks to Lemma 2.4-**(ii)** and we have thus established that $\mathcal{L}_0$ satisfies **(H2)**. Because $f_0 \in D(\mathcal{L}_0)$, we have $S_{\mathcal{L}_0}(t) f_0 \ge e^{\kappa_0 t} f_0$ for any $t \ge 0$, from Remark 2.5-(2). On the other hand, we observe that $S_{\mathcal{L}}(t) \ge S_{\mathcal{L}_0}(t)$ for any $t \ge 0$, from the weak maximum principle mentioned in Remark 8.24-(3). These two last observations together imply $S_{\mathcal{L}}(t) f_0 \ge e^{\kappa_0 t} f_0$, for any $t \ge 0$. We deduce from Lemma 2.4-**(iv)** that **(H2)** holds.

Condition (H3). We introduce the semigroup $S_{\mathcal{B}}$ associated to the transport equation

$$\frac{\partial g}{\partial t} + a \cdot \nabla g + bg = 0, \quad \gamma_- g = \mathscr{R}[\gamma_+ g],$$

which is well defined thanks to Corollary 8.25 and satisfies $\|S_{\mathcal{B}}(t) g_0\|_{L^p_m} \le e^{\kappa_{\mathcal{B}} t} \|g_0\|_{L^p_m}$ for any $t \ge 0$ and $g_0 \in L^p_m$ with $\kappa_{\mathcal{B}} := \|\langle \varpi_- \rangle\|_{L^\infty} + M_{\mathscr{R}}/p$ because of the a priori estimate (8.65) particularized to the present case (in particular where we can take $\varepsilon_1 = 0$ because the influx function is $\mathfrak{g} = 0$ here). We formulate the first hypothesis

$$\eta_0 > \|\langle \varpi_- \rangle\|_{L^\infty} C_0 + M_{\mathscr{R}} C_0/p + \|a\|_{L^\infty(\mathcal{B}_0)} C_1 - \|b\|_{L^\infty(\mathcal{B}_0)} C_0, \tag{8.102}$$

with the same definitions as above for $\mathcal{B}_0$, C_0 and C_1, so that $\kappa_0 > \kappa_{\mathcal{B}}$ because of (8.100). In a second case, we assume

$$\mathscr{R} \equiv 0,\ \mathcal{O} \text{ is bounded and there exists } T_{\mathcal{O}} \text{ such that } t_{\mathbf{b}}(y) \le T_{\mathcal{O}} \text{ for a.e. } y \in \mathcal{O}. \tag{8.103}$$

In that case, the semigroup $S_{\mathcal{B}}$ is explicitly given by

$$(S_{\mathcal{B}}(t) f_0)(y) = \begin{cases} f_0(Y_{-t}(y)) \exp(-\int_0^t K(Y_{\tau - t}(y)) d\tau), & \text{if } t \in (0, t_{\mathbf{b}}(y)) \\ 0 \quad \text{otherwise,} \end{cases}$$

and in particular $S_{\mathcal{B}}(t) f = 0$ for any f and any $t > T_{\mathcal{O}}$. We immediately deduce $\kappa_{\mathcal{B}} = -\infty$ and thus $\kappa_0 > \kappa_{\mathcal{B}}$ because we have established that $\kappa_0 \in \mathbb{R}$. We next define $\mathcal{A} f := \mathscr{K}[f]$. Using Lemma 2.8 and Remark 2.9-(2) or Lemma 2.13 and Remark 2.14-(1) depending on the assumption (8.96) made on $\mathscr{K}$, we deduce that the condition **(H3)** holds in both cases discussed above. Under the first condition in (8.96), we conclude to the existence of eigenvalue triplet $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times L^p_m \times L^{p'}_{m^{-1}}$. Under the second condition in (8.96), we may also get the same conclusion by by using [337, Cor. 1 of Thm. II.9.9] when $p = 1$ or by observing that the dual problem is similar to the primal problem when $p > 1$ and thus we may apply the same arguments for the dual problem as those explained above for the primal problem.

Condition (H4). Let us consider $\lambda > \lambda^{**}$ and $0 \le f \in L^p_{m_{\mathcal{O}}}(\mathcal{O})$ a (renormalized) solution to

$$\lambda f + a \cdot \nabla f + bf - \mathscr{K}[f] = F \quad \text{in} \quad \mathcal{O}, \quad \gamma_- f = \mathscr{R}[f, \gamma_+ f] \quad \text{on} \quad \Sigma_-,$$

with $0 \le F \in L^p_m(\mathcal{O})$. If $f \not\equiv 0$, there exists $x_1 \in \mathcal{O}$ such that $\int_{B(x_1, r_1)} f(z)\, dz > 0$. From (8.94), we deduce

$$\mathscr{K}[f](y) \ge \int_{B(x_1, r_1)} \kappa(y, z) f(z)\, dz > 0, \quad \forall\, y \in B(x_1, r_1).$$

Now, we argue similarly as during the proof of Lemma 8.14 and in particular we use the same notations. For $A \subset B(x_1, r_1)$, we define the solution $0 \le \varphi \in L^{p'}_{m^{-1}} \cap L^\infty$ to the equation

$$\lambda \varphi - \mathrm{div}(a \varphi) + b \varphi = \mathbf{1}_A \ \text{ in } \mathcal{O}, \quad \gamma_+ \varphi = 0 \ \text{ on } \Sigma_+,$$

thanks to Lemma 8.12 and Lemma 8.13, and we observe that $\varphi \not\equiv 0$ on $B(x_1, r_1)$ if $|A| > 0$. For the renormalizing function β_δ and a truncation function χ_R, we compute

$$\begin{aligned} 0 \quad &\ge \quad \int_\Sigma a \cdot n \beta_\delta(\gamma f) \gamma \varphi \chi_R \\ &= \quad \int_{\mathcal{O}} [\beta'_\delta(f)(F + \mathscr{K}[f]) \varphi - \beta_\delta(f) \mathbf{1}_A] \chi_R \\ &\qquad + \int_{\mathcal{O}} (\beta_\delta(f) - f \beta'_\delta(f))(\lambda + b) \varphi \chi_R + \int_{\mathcal{O}} \varphi \beta(f) \frac{a}{R} \cdot (\nabla \chi)_R. \end{aligned}$$

Passing first to the limit $R \to \infty$ and next to the limit $\delta \to 0$, we deduce

$$0 \ge \int_{\mathcal{O}} [(F + \mathscr{K}[f])\varphi - f\mathbf{1}_A],$$

so that in particular

$$\int_A f\,dy \ge \int_{B(x_1,r_1)} \varphi \mathscr{K}[f] > 0.$$

This being true for any $A \subset B(x_1, r_1)$, we deduce $f > 0$ a.e. on $B(x_1, r_1)$. By a classical continuity argument, we conclude that $f > 0$ a.e. on $\mathcal{O}$. We have thus established **(H4)** for $\lambda > \lambda^{**}$ from what we immediately and classically deduce the general case $\lambda \in \mathbb{R}$.

Condition (H5). Assume that $(\lambda, f) \in \mathbb{C} \times D(\mathcal{L})$ satisfies

$$\mathcal{L}|f| = (\Re e\lambda)|f| \quad \text{in} \quad \mathcal{O}, \quad \mathscr{R}[|f|, \gamma_+|f|] = \gamma_-|f| \quad \text{on} \quad \Sigma_-,$$

and

$$\mathcal{L}|f| = \Re e(\text{sign} f)\mathcal{L}f \quad \text{in} \quad \mathcal{O}, \quad \mathscr{R}[|f|, \gamma_+|f|] = \Re e(\text{sign}\gamma_- f)\mathscr{R}[f, \gamma_+ f] \quad \text{on} \quad \Sigma_-.$$

From **(H4)** and the first identity, we know that $|f| > 0$ a.e. on $\mathcal{O}$. Using the second identity, we get

$$\mathscr{K}[|f|] = \Re e(\text{sign} f)\mathscr{K}[f].$$

Writing $f = e^{i\alpha}|f|$, we deduce

$$\int_{\mathcal{O}} k|f_*|(1 - \cos(\alpha - \alpha_*))dy_* = 0 \quad \text{a.e. on } \mathcal{O}.$$

Using (8.94), we deduce

$$\int_{B(y,r_{\mathcal{O}})} |f_*|(1 - \cos(\alpha - \alpha_*))dy_* = 0,$$

and thus $\alpha = \alpha_*$ a.e. on $\mathcal{O} \times \mathcal{O}$. That means $f = u|f|$, for a constant $u = \mathbb{S}^1$, that completes the proof of the fact that $\mathcal{L}$ satisfies the reverse Kato's inequality condition **(H5)**.

We summarize our analysis in the following result which is a straightforward consequence of the above checked conditions together with Theorem 2.21, Theorem 4.13, Theorem 5.16 and Theorem 5.23. We state the available result in that situation.

Theorem 8.36. *We assume that a, b, $\mathscr{K}$ and $\mathscr{R}$ satisfy the conditions* (8.35), (8.59), (8.60), (8.61), (8.62) *and* (8.63) *for some weight function $m : \bar{\mathcal{O}} \to [1, \infty)$ and some exponent $p \in [1, \infty)$. Consider the semigroup $S_{\mathcal{L}}$ associated to the transport equation* (8.1)-(8.3) *through Corollary 8.25. We assume further that $\mathscr{K}$ satisfies the strong positivity conditions* (8.94) *together with* (8.95) *and the first compactness property formulated in* (8.96). *We finally assume that* (8.102) *holds or* (8.103) *holds. In both cases, the conclusion* **(C3)** *holds as well as the ergodicity* **(E2)** *in $L^1_{\phi_1}$.*

We are not aware of any similar result for such a general transport equation, see however the next sections where more specific transport like equations are discussed. We do not try to improve the convergence result in the general case, but rather we aim to make one step further in the following particular situation where Doblin approach may be used.

Doblin condition. We suppose here that $\mathcal{O}$ is bounded, $K \in L^\infty(\mathcal{O})$, $\mathcal{R}^*_\Sigma \mathbf{1} = \mathcal{R}_\Sigma \mathbf{1} = \mathbf{1}$, and $k(y, y_*) \ge k_0 > 0$. We aim at establishing the Doblin condition

$$S_{\mathcal{L}}(T) f_0 \ge \kappa \langle f_0, \mathbf{1} \rangle,$$

which is (6.2) with $\psi_0 = \mathbf{1}$ and $g_0 = \kappa\mathbf{1}$. From (8.64) we have

$$\frac{d}{dt} \int_{\mathcal{O}} f dy = \int_{\mathcal{O}} f K dy \ge -\|K\|_\infty \int_{\mathcal{O}} f dy$$

and so

$$\int_{\mathcal{O}} f(t, y)\,dy \ge e^{-\|K\|_\infty t} \int_{\mathcal{O}} f_0(y)\,dy.$$

Now we define, for $\varphi_0 \in C^1_c(\mathcal{O})$, $\varphi_0 \ge 0$, $\int \varphi_0 = 1$, the solution φ to the equation

$$\begin{cases} \partial_t \varphi + \text{div}(a\varphi) = 0, \\ \gamma_+ \varphi = \mathcal{R}^*_\Sigma[\gamma_- \varphi]. \end{cases}$$

We have

$$\frac{d}{dt}\int_{\mathcal{O}} \varphi = 0, \qquad \text{and so} \quad \int_{\mathcal{O}} \varphi(t,y)\,dy = \int_{\mathcal{O}} \varphi_0(y)\,dy = 1,$$

and

$$\frac{d}{dt}\int_{\mathcal{O}} f\varphi = \int_{\mathcal{O}} \mathscr{K}[f]\varphi - \int_{\mathcal{O}} Kf\varphi \geq k_0 \int_{\mathcal{O}} f - \|K\|_\infty \int_{\mathcal{O}} f\varphi.$$

We deduce from Grönwall's inequality that, for any fixed $T > 0$,

$$\begin{aligned}\int_{\mathcal{O}} f(T,y)\varphi_0(y)\,dy &\geq e^{-\|K\|_\infty t}\int_{\mathcal{O}} f_0(y)\varphi_0(y)\,dy + k_0\int_0^T e^{-(T-t)\|K\|_\infty}\int_{\mathcal{O}} f(t,y)\,dydt \\ &\geq k_0 T e^{-T\|K\|_\infty}\int_{\mathcal{O}} f_0(y)\,dy =: \kappa\langle f_0, \mathbf{1}\rangle.\end{aligned}$$

This is nothing but the Doblin condition (6.2) since φ_0 is any non-negative function in $C^1_c(\mathcal{O})$ with $\int \varphi_0 = 1$.

In order to verify (6.3) in a quantitative way, we suppose that the conditions (8.35), (8.59), (8.60), (8.61), (8.62) and (8.63) are verified with the weight function $m = \mathbf{1}$ and the exponent $p = 1$. Note that in this case we have $\varpi = K \geq 0$. The first condition in (8.35) then imposes that $K \in L^\infty(\mathcal{O})$, and (8.102) reads

$$\eta_0 > \|\langle K\rangle\|_{L^\infty} C_0 + M_{\mathscr{R}} C_0 + \|a\|_{L^\infty(\mathcal{B}_0)} C_1 - \|b\|_{L^\infty(\mathcal{B}_0)} C_0.$$

We also assume that

$$\mathcal{R}^*_\Sigma \mathbf{1} = \mathcal{R}_\Sigma \mathbf{1} = \mathbf{1}, \tag{8.104}$$

and

$$\forall y, y_* \in \mathcal{O}, \quad k_0 \leq k(y,y_*) \leq k_1 \tag{8.105}$$

for some $k_1 > k_0 > 0$,

Theorem 8.37. *We assume that $\mathcal{O}$ is bounded and that the conditions* (8.35), (8.59), (8.60), (8.61), (8.62) *and* (8.63) *are satisfied by a, b, $\mathscr{K}$ and $\mathscr{R}$ for the weight function $m = \mathbf{1}$ and the exponent $p = 1$. We assume further that $\mathscr{K}$ satisfies the strong positivity conditions* (8.94) *together with* (8.95) *and the first compactness property formulated in* (8.96). *We finally assume that* (8.102), (8.104) *and* (8.105) *are satisfied. Then the exponential convergence in* **(E3$_1$)** *holds in L^1 with constructive constants C and ω.*

Proof of Theorem 8.37. We work in $X = L^1(\mathcal{O})$ and we normalize ϕ_1 by $\|\phi_1\|_{L^\infty} = 1$. We have proved above that (6.2) holds true with $\psi_0 = \mathbf{1}$ and $g_0 = \kappa\mathbf{1}$ for some explicit $\kappa > 0$, recalling that the assumption that $K \in L^\infty$ is nothing but the first condition in (8.35) when $m = \mathbf{1}$ and $p = 1$ since $b = K + \mathrm{div} a$. Due to the normalization $\|\phi_1\|_{L^\infty} = 1$, the condition (6.4) holds with $R_0 = 1$. It only remains to check the validity of (6.3) in order to be able to apply Theorem 6.2. Since we assume that the conditions (8.35), (8.59), (8.60), (8.61), (8.62) and (8.63) are satisfied for the weight function $m = \mathbf{1}$ and the exponent $p = 1$, we have that $\mathcal{R}_{\mathcal{B}}(\lambda_1) : L^1 \to L^1$ with $\|\mathcal{R}_{\mathcal{B}}(\lambda_1)\|_{\mathscr{B}(L^1)} \leq \frac{1}{\kappa_0 - \kappa_{\mathcal{B}}}$. This yields by duality that $\mathcal{R}^*_{\mathcal{B}}(\lambda_1) : L^\infty \to L^\infty$ with $\|\mathcal{R}^*_{\mathcal{B}}(\lambda_1)\|_{\mathscr{B}(L^\infty)} \leq \frac{1}{\kappa_0 - \kappa_{\mathcal{B}}}$. Since k is bounded by the constant k_1, we have on the other hand that $\mathcal{A}^* = \mathscr{K}^* : L^1 \to L^\infty$ with $\|\mathcal{A}^*\|_{\mathscr{B}(L^1, L^\infty)} \leq k_1$. We thus get

$$1 = \|\phi_1\|_{L^\infty} \leq \frac{k_1}{\kappa_0 - \kappa_{\mathcal{B}}}\|\phi_1\|_{L^1},$$

which yields (6.3) with $r_0 = \kappa(\kappa_0 - \kappa_{\mathcal{B}})/k_1$, and the proof is complete. ☐

8.6. **A word about the renewal equation.** We look at the case $\mathcal{O} = (0, +\infty)$ and $a(y) = 1$, which corresponds to the equation

$$\partial_t f + \partial_y f + Kf = 0 \tag{8.106}$$

with the boundary condition

$$(\gamma_- f)(t, 0) = \int_0^\infty r_{\mathcal{O}}(y_*) f(t, y_*) dy_*. \tag{8.107}$$

This renewal age structured model is standard in structured population dynamics, and the Krein-Rutman theorem is well-known for it, see for instance [34, 158, 187, 196, 338, 358]. The existence and uniqueness of (λ_1, f_1, ϕ_1) can even be obtained by explicit computations. However, it is not covered by the cases considered in Section 8.5 because $\mathscr{K} = 0$ here.

The singularity of this transport equation lies in the fact that **(H2)** is only guaranteed by the boundary condition. To fall into our splitting framework, we may replace the boundary condition by a singular source term $\mathcal{A}f = (\mathcal{R}_{\mathcal{O}} f)(0)\delta_0$, where δ_0 is the Dirac mass at the origin, and write $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{B}$ the generator of the free transport equation with zero flux boundary condition. This forces working in a space of measures, as in [277, 280]. We briefly present an alternative approach, which is more in the spirit of [34, 168] and which consists in working in the Lebesgue space L^1, first to solve the dual problem in $L^\infty = (L^1)'$ and next to use for instance Doblin's contraction to solve the primal problem.

We assume here that

$$0 \le K, r_{\mathcal{O}} \in L^\infty_{\rm loc}(0, \infty), \quad (r_{\mathcal{O}} - \alpha K)_+ \in L^\infty, \tag{8.108}$$

$$\lim_{y \to \infty} K(y) = +\infty, \quad \liminf_{y \to +\infty} r_{\mathcal{O}}(y) > 0, \tag{8.109}$$

for some $\alpha \in (0, 1)$, and we verify the usual conditions for the direct or the dual problem.

Condition (H1). Under assumption (8.108), the age structured equation (8.106)-(8.107) is well-posed in L^1 thanks to Proposition 8.23 and we may associate to it a positive semigroup $S_{\mathcal{L}}$ in L^1 with growth bound $\omega(S_{\mathcal{L}}) \le \kappa_1 := \|(r_{\mathcal{O}} - K)_+\|_{L^\infty}$ thanks to Corollary 8.25. We deduce that **(H1)** holds for the primal problem and thus also for the dual problem thanks to Lemma 2.2 and Lemma 2.3.

Condition (H2). The generator of the dual problem is

$$\mathcal{L}^* \phi = \partial_y \phi - K(y)\phi + \phi(0) r_{\mathcal{O}}(y)$$

with domain $D(\mathcal{L}^*) \subset W^{1,\infty}_{\rm loc}(\mathcal{O})$. From the second hypothesis in (8.109), there exist $y_0, \eta_0 \in (0, \infty)$ such that $r_{\mathcal{O}}(y) \ge \eta_0$ for any $y \ge y_0$. We then define

$$\phi_0(y) = \mathbf{1}_{[0, y_0)} + \eta_0 (y_1 - y) \mathbf{1}_{[y_0, y_1)}, \quad y_1 := y_0 + 1/\eta_0,$$

and we compute

$$\begin{aligned} \mathcal{L}^* \phi_0 &= r_{\mathcal{O}}(y) - K \ge -\|K - r_{\mathcal{O}}\|_{L^\infty(0, y_0)} \quad \text{on } (0, y_0), \\ \mathcal{L}^* \phi_0 &= r_{\mathcal{O}}(y) - \eta_0 - K\phi_0 \ge -\|K\|_{L^\infty(y_0, y_1)} \phi_0 \quad \text{on } (y_0, y_1), \\ \mathcal{L}^* \phi_0 &= 0 \quad \text{on } (y_1, \infty), \end{aligned}$$

so that in the three case $\mathcal{L}^* \phi_0 \ge \kappa_0 \phi_0$ with $\kappa_0 := -\max(\|K - r_{\mathcal{O}}\|_{L^\infty(0, y_0)}, \|K\|_{L^\infty(y_0, y_1)})$. Using Lemma 2.4-**(i)**, we have thus established that $\mathcal{L}$ satisfies **(H2)** with constructive constant κ_0.

Condition (H3) on the dual problem. We define the splitting $\mathcal{L}^* = \mathcal{A}^* + \mathcal{B}^*$ with $\mathcal{A}^* \phi := (\mathcal{R}^*_{\mathcal{O}} \phi)(y) = \phi(0) r_{\mathcal{O}}(y)$. From the first hypothesis in (8.109), for any $\kappa_* \le 0$ there exists $y_* \in [0, \infty)$ such that $K(y) \ge -\kappa_*$ for any $y \ge y_*$. Defining $m_* := e^{\kappa_* y} \mathbf{1}_{[0, y_*)} + e^{\kappa_* y_*} \mathbf{1}_{[y_*, \infty)}$, we compute

$$\mathcal{B}^* m_* = \kappa_* e^{\kappa_* y} \mathbf{1}_{[0, y_*)} - K m_* \le \kappa_* m_*.$$

Together with Proposition 8.23 and Corollary 8.25, we deduce that the operator $\mathcal{B} - \kappa_*$, with domain $D(\mathcal{B}) := \{f \in L^1(\mathcal{O});\ \partial_y f + Kf \in L^1(\mathcal{O}),\ f(0) = 0\}$, generates a contraction semigroup in $L^1_{m_*}(\mathcal{O})$, and thus a bounded semigroup in $L^1(\mathcal{O})$ because $m_*, m_*^{-1} \in L^\infty(\mathcal{O})$. In other words, we have established that $\omega(S_{\mathcal{B}}) = -\infty$. Now, we see that $\mathcal{R}_{\mathcal{B}^*}(\lambda) : L^\infty \to D(\mathcal{B}^*) \subset W^{1,\infty}_{\rm loc}([0, \infty))$ is bounded for any $\lambda \in \mathbb{R}$ and thus $\mathcal{A}^* \mathcal{R}_{\mathcal{B}^*}(\lambda) : L^\infty \to L^\infty$ is compact for any $\lambda \in \mathbb{R}$. We deduce from Lemma 2.8 and Remark 2.10, that $\mathcal{L}^*$ satisfies **(H3)**.

Using Lemma 2.8-*(1)*, we conclude to the existence of (λ_1, ϕ_1) solution to the dual eigenvalue problem. Now we turn to the existence, uniqueness, and exponential stability of $f_1 \in L^1$, by verifying that Doblin's condition (6.2) is satisfied.

Doblin condition. Denoting $S_t := S_{\mathcal{L}}(t)$, we have from the characteristics method

$$S_t f(y) = f(y-t) e^{-\int_0^t K(y-s)ds} \mathbf{1}_{t<y} + N(t-y) e^{-\int_0^y K(s)ds} \mathbf{1}_{t>y}$$

with $N(t) = \int_0^\infty r_{\mathcal{O}}(y_*) S_t f(y_*) dy_*$. Iterating this formula and using the positivity of S_t we get that for any $f \geq 0$

$$\begin{aligned} S_t f(y) &\geq \left(\int_0^{t-y} r_{\mathcal{O}}(y_*) N(t-y-y_*) e^{-\int_0^{y_*} K(s)ds} dy_* \right) e^{-\int_0^y K(s)ds} \mathbf{1}_{0<y<t} \\ &\geq \left(\int_0^{t-y} r_{\mathcal{O}}(t-y-\tau) N(\tau) e^{-\int_0^{t-y-\tau} K(s)ds} d\tau \right) e^{-\int_0^y K(s)ds} \mathbf{1}_{0<y<t}. \end{aligned}$$

Choosing $t_0 > 2y_0$ so that $r_{\mathcal{O}}(y) \geq \eta_0 > 0$ for all $y \geq t_0/2$, we obtain

$$S_{t_0} f(y) \geq \eta_0 \, e^{-\int_0^{t_0} K(s)ds} \left(\int_0^{t_0/4} N(\tau) d\tau \right) e^{-\int_0^y K(s)ds} \mathbf{1}_{0<y<t_0/4}.$$

From the expression of $N(t)$ we get by duality, using that $r_{\mathcal{O}} \geq \eta_0 \mathbf{1}_{(y_0,\infty)}$, that

$$S_{t_0} f(y) \geq \eta_0^2 \, e^{-2\int_0^{t_0} K(s)ds} \left(\int_0^\infty f(y_*) \left(\int_0^{t_0/4} S^*_\tau \mathbf{1}_{(y_0,\infty)}(y_*) d\tau \right) dy_* \right) \mathbf{1}_{0<y<t_0/4}. \tag{8.110}$$

Applying S_{t_1} to this inequality we deduce that for any $t_1 > 0$

$$\begin{aligned} S_{t_0+t_1} f(y) &\geq \eta_0^2 \, e^{-2\int_0^{t_0} K(s)ds} \left(\int_0^\infty f(y_*) \left(\int_0^{t_0/4} S^*_\tau \mathbf{1}_{(y_0,\infty)}(y_*) d\tau \right) dy_* \right) S_{t_1} \mathbf{1}_{0<y<t_0/4} \\ &\geq \eta_0^2 \, e^{-2\int_0^{t_0} K} e^{-\int_0^{t_1} K} \left(\int_0^\infty f(y_*) \left(\int_0^{t_0/4} S^*_\tau \mathbf{1}_{(y_0,\infty)}(y_*) d\tau \right) dy_* \right) \mathbf{1}_{t_1<y<t_0/4+t_1}. \end{aligned}$$

On the other hand, replacing f by $S_{t_1} f$ in (8.110) we obtain

$$S_{t_0+t_1} f(y) \geq \eta_0^2 \, e^{-2\int_0^{t_0} K(s)ds} \left(\int_0^\infty f(y_*) \left(\int_0^{t_0/4} S^*_{\tau+t_1} \mathbf{1}_{(y_0,\infty)}(y_*) d\tau \right) dy_* \right) \mathbf{1}_{0<y<t_0/4}. \tag{8.111}$$

The fact that $S^*_t \phi(y) \geq \phi(t+y) e^{-\int_0^t K(y+s)ds}$ ensures that for $t_1 > y_0$

$$S^*_{t_1} \mathbf{1}_{(y_0,\infty)} \geq e^{-\sup_{y\in[0,y_0]} \int_0^{t_1} K(y+s)ds} \mathbf{1}_{[0,y_0]}.$$

All together, we have proved that for any $t_0 > 4t_1 > 4y_0$ we have

$$S_{t_0+t_1} f(y) \geq c_0 \left(\int_0^\infty f(y_*) \left(\int_0^{t_0/4} S^*_\tau \mathbf{1}(y_*) d\tau \right) dy_* \right) \mathbf{1}_{t_1<y<t_0/4}$$

for some explicit constant c_0 and all $f \geq 0$. This is Doeblin's condition (6.2) with $T = t_0 + t_1$, and the functions $\psi_0 = \int_0^{t_0/4} S^*_\tau \mathbf{1} \, d\tau$ and $g_0 = c_0 \mathbf{1}_{(t_1, t_0/4)}$. We are now in position to prove the following result.

Theorem 8.38. *Under the assumptions* (8.108) *and* (8.109)*, the renewal equation* (8.106)-(8.107) *enjoys the conclusions* **(C3)** *and* **(E3$_1$)** *with quantitative rate in* L^1.

Despite the numerous results about the renewal age-structured model, we are not aware of any previous result with a constructive rate of convergence under such general assumptions.

Proof of Theorem 8.38. The conditions **(H1)**, **(H2)** and **(H3)** for $\mathcal{L}^*$ ensure the existence of $\lambda_1 \geq \kappa_0$ and $\phi_1 \in L^\infty$, $\phi_1 > 0$, that we normalize by $\|\phi_1\|_{L^\infty} = 1$. If we can prove that the conditions (6.2), (6.3) and (6.4) are verified, then the conclusions **(C1)** and **(E3$_1$)** follow by applying Theorem 6.2. Indeed, the contraction argument in the proof of Theorem 6.2 does not require the existence of f_1 and it can even be used for deriving the existence and uniqueness of f_1, see Remark 6.4. We have already proved (6.2) with the functions $\psi_0 = \int_0^{t_0/4} S^*_\tau \mathbf{1} \, d\tau$ and $g_0 = c_0 \mathbf{1}_{(t_1, t_0/4)}$. For proving (6.3), we start by recalling that $\phi_1 = \mathcal{R}^*_{\mathcal{B}}(\lambda_1) \mathcal{A}^* \phi_1 \in W^{1,\infty}_{loc}$ due to the informations derived on $\mathcal{R}^*_\mathcal{B}$ in **(H3)**. Consequently, there exists $y_1 > 0$ such that $\phi_1(y_1) > 1/2$, and we deduce from $\phi_1' \le (\lambda_1 + K)\phi_1$ that

$$\phi_1(y) \ge \frac{1}{2} e^{-\int_y^{y_1}(\lambda_1+K)}$$

for all $y \in (0, y_1)$. Choosing in the proof of the Doblin condition t_0 such that $y_1 < t_0/4$, we obtain that

$$\langle \phi_1, g_0 \rangle \ge \frac{c_0}{2} \int_{y_1/2}^{y_1} e^{-\int_y^{y_1}(\lambda_1+K)} dy,$$

which gives (6.3). For (6.4), we use that

$$\phi_1 = e^{-\lambda_1 \tau} S^*_\tau \phi_1 \le e^{-\lambda_1 \tau} S^*_\tau \mathbf{1}$$

for any $\tau > 0$ to deduce that

$$\phi_1 = \frac{4}{t_0} \int_0^{t_0} e^{-\lambda_1 \tau} S^*_\tau \phi_1 d\tau \le \frac{4 e^{|\lambda_1| t_0}}{t_0} \psi_0.$$

Finally, we check that the condition **(H5′)** is verified, so that **(C3)** is valid by virtue of Theorem 5.18 and Remark 5.19. The condition **(H5′)** is actually a direct consequence of the fact that (8.111) is verified for any $t_0 > 2y_0$ and $t_1 > 0$ together with the estimate

$$S^*_{\tau+t_1} \mathbf{1}_{(y_0,\infty)}(y) \ge e^{-\int_0^{t_1} K(y+s)ds} > 0$$

for any $t_1 > y_0$ and $\tau > 0$, and all $y > 0$. □

9. The growth-fragmentation equation

In this section, we are interested in the growth-fragmentation equation with equal mitosis kernel

$$\partial_t f(t,x) + \partial_x \big(a(x) f(t,x)\big) + K(x) f(t,x) = 4K(2x) f(t,2x) \tag{9.1}$$

and to its variant with an additional "growth speed" variable

$$\partial_t f(t,x,v) + v\partial_x \big(a(x) f(t,x,v)\big) + K(x) f(t,x,v) = 4 \int_1^2 K(2x) \wp(v,v_*) f(t,2x,v_*) dv_*, \tag{9.2}$$

with $x > 0$ and $v \in [1,2]$. For both equations, we assume that the total fragmentation rate K is a continuous function defined on $\mathbb{R}_+$ such that

$$\exists x_0 > 0, \quad K = 0 \text{ on } (0, 2x_0] \quad \text{and} \quad K > 0 \text{ on } (2x_0, \infty). \tag{9.3}$$

This condition ensures that no particle of size less than x_0 can be produced by division, and we thus consider the equations posed on the size space (x_0, ∞) with zero flux boundary condition $f(t, x_0) = 0$ or $f(t, x_0, v) = 0$. The growth rate a is supposed to be positive and globally Lipschitz on $[x_0, \infty)$, and we assume that

$$\lim_{x\to\infty} \frac{xK(x)}{a(x)} = +\infty. \tag{9.4}$$

For quantifying the positivity of the first eigenvalue, we also make the technical assumption that

$$\exists k > 0, \quad \lim_{x\to\infty} e^{x^k} K(x) = +\infty. \tag{9.5}$$

9.1. **The mitosis equation with mixing growth rate.** We are interested here in the growth-fragmentation equation (9.1) in the case where

$$\exists x_1 > x_0, \quad a(2x_1) \neq 2a(x_1). \tag{9.6}$$

As we will see below, this condition ensures some mixing property for the trajectories that guarantees the triviality of the boundary point spectrum.

We work in the space $X = L^1_m$ with a weight m that can be

$$\text{either} \quad m(x) = x^r, \quad r > 1, \qquad \text{or} \quad m(x) = \exp\Big(\eta \int_{x_0}^x \frac{K}{a}\Big), \quad 0 < \eta < 1. \tag{9.7}$$

Note that due to assumption (9.4), the weight $\exp\big(\eta\int_{x_0}^x K/a\big)$ is always stronger than x^r.

Theorem 9.1. *Suppose that* (9.3), (9.4), (9.5) *and* (9.6) *are satisfied. The first eigentriplet problem admits a unique solution* $(\lambda_1, f_1, \phi_1) \in \mathbb{R}\times X_+ \times X'_+$ *with the normalization* $\|\phi_1\| = \langle \phi_1, f_1\rangle = 1$, *and this triplet additionally satisfies* $\lambda_1 > 0$, $f_1 > 0$ *and* $\phi_1 > 0$. *Besides, there are some constructive constants* $C \geq 1$, $\omega > 0$ *such that*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t) f - \langle \phi_1, f\rangle f_1\|_X \leq C e^{-\omega t} \|f - \langle \phi_1, f\rangle f_1\|_X$$

for any $f \in X$ *and* $t \geq 0$.

This result is contained in the recent paper [354]. The novelty here is that all the constants are obtained constructively, which is not clear in [354]. We also provide what seems to us to be a more direct and comprehensive proof. We also refer to [35, 52, 80, 278] where the same result is obtained under stronger assumptions.

Before starting the proof of Theorem 9.1, let us briefly justify the relevance of the chosen weight functions m in (9.7). The dual operator associated to equation (9.1) is given by

$$\mathcal{L}^*\phi(x) = a(x)\phi'(x) - K(x)\phi(x) + 2K(x)\phi(x/2).$$

For $m(x) = x^r$, $r > 1$, we can compute

$$\mathcal{L}^* m(x) = \Big[r\frac{a(x)}{x} - (1 - 2^{1-r})K(x)\Big] m(x), \tag{9.8}$$

and for $m(x) = \exp\big(\eta\int_{x_0}^x K/a\big)$, $0 < \eta < 1$,

$$\mathcal{L}^* m(x) = \Big[2\exp\Big(-\eta\int_{x/2}^x \frac{K}{a}\Big) - (1-\eta)\Big] K(x) m(x). \tag{9.9}$$

Assumption (9.3) then ensures that $\mathcal{L}^* m \sim -\xi K m$ as $x \to +\infty$, with $\xi = 1 - 2^{1-r} > 0$ in the first case and $\xi = 1 - \eta > 0$ in the second case. In both cases, we deduce that

$$\mathcal{L}^* m \leq \kappa\, m + M \mathbf{1}_{(x_0,R)} m \tag{9.10}$$

for any $\kappa \geq 0$, by choosing $M > 0$ and $R > x_0$ large enough, and this type of Lyapunov inequality is pivotal in our analysis.

Condition (H1). Equation (9.1) is a particular case of equation (8.58) with $G = \mathfrak{g} = \mathscr{R} = 0$, $b = K + \mathrm{div} a$ and $\mathscr{K}[g](x) = 4K(2x)g(2x)$. We may then use Proposition 8.23-*(1)* to infer the well-posedness of equation (9.1) in $X = L^1_m(x_0, \infty)$, provided that the conditions (8.35) and (8.61) are met, with $0 < \alpha_{\mathscr{K}} < 1$, which is nothing but (8.63) when $\mathscr{R} \equiv 0$. To do so, we define the function

$$\varpi := K - a\frac{m'}{m},$$

which corresponds to ϖ_1 in (8.34). When $m(x) = x^r$ with $r > 1$, we have

$$\varpi(x) = K(x) - r\frac{a(x)}{x},$$

and for $m(x) = \exp\big(\eta\int_{x_0}^x K/a\big)$ with $0 < \eta < 1$, we have

$$\varpi(x) = (1-\eta)K(x).$$

In both cases, the fact that $a \in \text{Lip}$ ensures that $\varpi_q := \varpi + (1-1/q)a'$ enjoys $(\varpi_q)_- \in L^\infty$ for any $q \in [1,\infty)$. On the other hand, (9.4) guarantees that $K \lesssim \langle \varpi_+ \rangle$ and $a/x \lesssim \langle \varpi_+ \rangle$, and finally (8.35) is verified. The condition (8.61) is equivalent to the Lyapunov type condition

$$\mathscr{K}^*[m] \le (\alpha_{\mathscr{K}} \varpi_+ + M_{\mathscr{K}})m, \tag{9.11}$$

where $\mathscr{K}^*[m](x) = 2K(x)m(x/2)$. For $m(x) = x^r$ with $r > 1$, we compute

$$\mathscr{K}^*[m]/m = 2^{1-r}K,$$

and for $m(x) = \exp\big(\eta \int_{x_0}^x K/a\big)$ with $0 < \eta < 1$,

$$\frac{\mathscr{K}^*[m](x)}{m(x)} = 2\exp\Big(-\eta\int_{x/2}^x \frac{K}{a}\Big)K(x).$$

Using (9.4), we obtain that (9.11) is satisfied, for any $\alpha_{\mathscr{K}} \in (2^{1-r},1)$ in the first case, and for any $\alpha_{\mathscr{K}} \in (0,1)$ in the second case, by choosing $M_{\mathscr{K}}$ large enough.
We can then apply Proposition 8.23-*(1)* for associating to equation (9.1) a strongly continuous semigroup S in $X = L^1_m(x_0,\infty)$, and **(H1)** then follows from Lemma 2.2-(i). Moreover, we readily have that $\kappa_1 \le \kappa + M$ for any couple (κ, M) such that (9.10) is verified.

Condition (H2). We aim at verifying **(H2)** for some $\kappa_0 > 0$. Recalling assumption (9.5), we pick up $\ell > k$ and we consider the function

$$\phi_0(x) = xe^{-x^\ell/n}$$

with n large enough to be chosen later. We compute

$$\frac{\mathcal{L}^*\phi_0(x)}{\phi_0(x)} = \frac{a(x)}{x}\Big(1 - \frac{\ell}{n}x^\ell\Big) + K(x)\big(e^{\frac{1-2^{-\ell}}{n}x^\ell} - 1\big).$$

Choosing $R > x_0$ such that $xK(x)/a(x) \ge \frac{2\ell}{1-2^{-\ell}}$ and $K(x) \ge e^{-x^k}$ for all $x \ge R$, we get that

$$\begin{aligned}\frac{\mathcal{L}^*\phi_0(x)}{\phi_0(x)} &\ge \frac{a(x)}{x} + K(x)\Big(e^{\frac{1-2^{-\ell}}{n}x^\ell} - 1 - \frac{1-2^{-\ell}}{2n}x^\ell\Big)\\ &\ge e^{-x^k}\big(e^{\frac{1-2^{-\ell}}{n}x^\ell} - e^{\frac{1-2^{-\ell}}{2n}x^\ell}\big) \ge e^{\frac{1-2^{-\ell}}{2n}x^\ell - x^k}\big(e^{\frac{1-2^{-\ell}}{2n}R^\ell} - 1\big)\end{aligned}$$

on $[R,\infty)$. Choosing then $n \ge \frac{\ell}{2}R^\ell$, we have

$$\frac{\mathcal{L}^*\phi_0(x)}{\phi_0(x)} \ge \frac{a(x)}{2x}$$

on (x_0,R). Gathering the two above estimates, we deduce the existence of an explicit $\kappa_0 > 0$ such that $\mathcal{L}^*\phi_0 \ge \kappa_0\phi_0$. We conclude by invoking Lemma 2.4-(i).

Condition (H3). We consider the weight function $m(x) = x^r$ for some $r > 1$ or $m(x) = \exp\big(\eta\int_{x_0}^x K/a\big)$ with $0 < \eta < 1$ and we define the stronger weight function $m_1(x) = \exp\big(\eta_1\int_{x_0}^x K/a\big)$ for some $\eta_1 \in (\eta,1)$. We fix $\kappa_{\mathcal{B}} \in [0,\kappa_0)$, $M > 0$, and $R > x_0$ such that (9.10) is verified by m_1 with $\kappa = \kappa_{\mathcal{B}}$. Using the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A}f = M\mathbf{1}_{(x_0,R)}f$, the inequality (9.10) for m_1 reads $\mathcal{B}^*m_1 \le \kappa_{\mathcal{B}}m_1$ and this ensures (see the proof of Corollary 2.20) that $\kappa - \mathcal{B}$ is invertible in $L^1_{m_1}$ for any $\kappa > \kappa_{\mathcal{B}}$, with positive inverse, and

$$\|(\kappa - \mathcal{B})^{-1}\|_{\mathscr{B}(L^1_{m_1})} \le \frac{1}{\kappa - \kappa_{\mathcal{B}}}.$$

The operator $\mathcal{A}$ maps L^1_m into $L^1_{m_1}$ with

$$\|\mathcal{A}\|_{\mathscr{B}(L^1_m, L^1_{m_1})} \le M\frac{m_1(R)}{m(x_0)}.$$

Besides, due to the derivative part $\partial_x(a\,\cdot)$ in the operator $\mathcal{B}$, we also have that $\mathcal{R}_{\mathcal{B}}(\kappa)$ maps $L^1_{m_1}$ into $W^{1,1}_{\text{loc}}$. Finally, we have $\mathcal{R}_{\mathcal{B}}(\kappa)\mathcal{A} : L^1_m \to L^1_{m_1} \cap W^{1,1}_{\text{loc}}$, and thus $\mathcal{R}_{\mathcal{B}}(\kappa)\mathcal{A} \in \mathscr{K}(L^1_m)$, for any $\kappa \ge \kappa_0 > \kappa_{\mathcal{B}}$. We deduce from Lemma 2.8-*(2)* that the condition **(H3)** holds for both the primal and the dual problems.

Proof of the existence part of Theorem 9.1. We deduce from Theorem 2.21 that the conclusion **(C1)** about the existence of a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$ to the first eigentriplet problem holds true. □

Moreover, we have $\lambda_1 \geq \kappa_0 > 0$ and $f_1 \in W^{1,1}_{\mathrm{loc}} \cap L^1_m$ with $m(x) = \exp\big(\eta \int_{x_0}^x K/a\big)$ for any $\eta \in (0,1)$. For deriving similar additional estimates on ϕ_1, we can check directly that the condition **(H3)** holds for the dual operator $\mathcal{L}^*$.

Condition (H3) for $\mathcal{L}^*$. We consider the weight function $m(x) = x^r$ for some $r > 1$ or $m(x) = \exp\big(\eta \int_{x_0}^x K/a\big)$ with $0 < \eta < 1$ and we define the weaker weight function $m_0(x) = x^{r_0}$ for some $r_0 \in (1, r)$. We fix $\kappa_{\mathcal{B}} \in [0, \kappa_0)$, $M > 0$, and $R > x_0$ such that (9.10) is verified by m_0. Using again the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A}f = M\mathbf{1}_{(x_0,R)}f$, (9.10) means that $\mathcal{B}^* m_0 \leq \kappa_{\mathcal{B}} m_0$ and this ensures that for any $\kappa > \kappa_{\mathcal{B}}$ the operator $\kappa - \mathcal{B}^*$ is invertible in $L^\infty_{m_0}$, with positive inverse, and

$$\|(\kappa - \mathcal{B}^*)^{-1}\|_{\mathscr{B}(L^\infty_{m_0^{-1}})} \leq \frac{1}{\kappa - \kappa_{\mathcal{B}}}.$$

Because of the derivative part of $\mathcal{B}^*$, we also have that $\mathcal{R}_{\mathcal{B}^*}(\kappa) : L^\infty_{m_0^{-1}} \to W^{1,\infty}_{\mathrm{loc}}$. Besides, the operator $\mathcal{A}^* = \mathcal{A}$ maps $L^\infty_{m^{-1}}$ into $L^\infty_{m_0^{-1}}$ with

$$\|\mathcal{A}^*\|_{\mathscr{B}(L^\infty_{m^{-1}}, L^\infty_{m_0^{-1}})} \leq M \frac{m(R)}{m_0(x_0)}.$$

Finally we have that $\mathcal{R}_{\mathcal{B}^*}(\kappa)\mathcal{A} : L^\infty_{m^{-1}} \to L^\infty_{m_0^{-1}} \cap W^{1,\infty}_{\mathrm{loc}}$. Consequently $\phi_1 \in L^\infty_{m_0^{-1}} \cap W^{1,\infty}_{\mathrm{loc}}$ and

$$\|\phi_1\|_{L^\infty_{m_0^{-1}}} = \|(\lambda_1 - \mathcal{B}^*)^{-1}\mathcal{A}^*\phi_1\|_{L^\infty_{m_0^{-1}}} \leq \frac{m(R)}{m_0(x_0)} \frac{M}{\kappa_0 - \kappa_{\mathcal{B}}} \|\phi_1\|_{L^\infty_{m^{-1}}}. \tag{9.12}$$

We also easily deduce quantitative estimates of ϕ_1 in $W^{1,\infty}_{\mathrm{loc}}$ from the identity

$$\phi_1'(x) = \frac{1}{a(x)}\big[\lambda_1\phi_1(x) + K(x)\phi_1(x) - 2K(x)\phi_1(x/2)\big].$$

Condition (H4). The operator $\mathcal{L}$ satisfies the strong maximum principle. Let $\lambda \in \mathbb{R}$ and $f \in X_+ \cap D(\mathcal{L}) \setminus \{0\}$ such that $(\lambda - \mathcal{L})f \geq 0$, *i.e.*

$$\lambda f(x) + (af)'(x) + K(x)f(x) \geq 4K(2x)f(2x) \qquad \forall x > x_0.$$

Denoting by Λ_λ a function such that $\Lambda'_\lambda(x) = \frac{\lambda + K(x)}{a(x)}$, we get that

$$a(x)f(x) \geq 4\int_{x_0}^x e^{\Lambda_\lambda(y) - \Lambda_\lambda(x)} K(2y)f(2y)\,dy. \tag{9.13}$$

Since $K(2y) > 0$ for all $y > x_0$, $f \in X_+ \setminus \{0\}$, and $a(x) > 0$ for all $x > x_0$, we deduce from (9.13) that the set $\{x > x_0,\ f(x) > 0\}$ is an interval of the form $(\underline{x}, +\infty)$. Using again (9.13) we remark that we must have $\underline{x} = \max(x_0, \underline{x}/2)$, which enforces $\underline{x} = x_0$ and finally $f > 0$.

Proof of the uniqueness and positivity part of Theorem 9.1. We deduce from Theorem 4.13 the validity of the conclusion **(C2)** about existence, uniqueness and positivity of a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem. □

For deriving the exponential stability, we start by verifying a quantified irreducibility and aperiodicity condition on S, given in the next lemma, which then allows us to prove that the Harris condition (6.8) is met.

Lemma 9.2. *Assume that* (9.6) *is satisfied. Then for all $\varepsilon > 0$, $R_1 > x_0$, and $R_2 > x_0 + \varepsilon$, there exists $T_1 > 0$ such that for any $T > T_1$, there exists $c_T > 0$ such that*

$$S^*_T\phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_0+\varepsilon}^{R_2} \phi\,dx, \quad \forall\,\phi \geq 0.$$

Proof of Lemma 9.2. Throughout the proof we denote by c_t any positive constant that depend only on t. It is proved in [354, Prop. 5] the existence of $(x_2, x_3) \subset (x_0, \infty)$ such that for all $R_1 > x_0$ there exists $T_0 > 0$ such that for any $T > T_0$ and any $\phi \geq 0$

$$S_T^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2}^{x_3} \phi(x) dx. \tag{9.14}$$

We may now extend the integral to $[x_0 + \varepsilon, R_2]$. The Duhamel formula

$$S_{\mathcal{L}}^* = S_{\mathcal{B}_0}^* + S_{\mathcal{B}_0}^* \mathcal{A}_0 * S_{\mathcal{L}}^*$$

for the splitting $\mathcal{L}^* = \mathcal{A}_0^* + \mathcal{B}_0^*$ with $\mathcal{A}_0^* \phi = \mathscr{K}^*[\phi]$ and $\mathcal{B}_0^* \phi = b\phi' - K\phi$, also reads

$$S_t^* \phi(x) = \phi(X_t(x)) e^{-\int_0^t K(X_s(x)) ds} + 2 \int_0^t K(X_{t-s}(x)) S_s^* \phi(X_{t-s}(x)/2) e^{-\int_0^{t-s} K(X_{s'}(x)) ds'} ds, \tag{9.15}$$

where $X_t(x)$ is the solution to the characteristic equation

$$\dot{X}_t(x) = a(X_t(x)) \quad \text{with} \quad X_0(x) = x. \tag{9.16}$$

Applying (9.14) to $S_t^* \phi$, that we bound from below by the first in Duhamel's formula (9.15), we obtain

$$S_{T+t}^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2}^{x_3} \phi(X_t(x)) e^{-\int_0^t K(X_s(x)) ds} dx \geq c_T c_t \mathbf{1}_{(x_0,R_1)} \int_{X_t(x_2)}^{X_t(x_3)} \phi(y) dy.$$

Choosing t_0 such that $X_{t_0}(x_2) = x_3$, we get that for all $T > T_0 + t_0$

$$S_T^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2}^{X_{t_0}(x_3)} \phi(x) dx.$$

Iterating this argument and using the strict positivity of a we get for any $R_2 > x_2$ the existence of a time t_1 such that for all $T > T_0 + t_1$

$$S_T^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2}^{R_2} \phi(x) dx. \tag{9.17}$$

For decreasing the lower bound of the integral from x_2 to $x_0 + \varepsilon$, we iterate once Duhamel's formula (9.15) to get

$$S_t^* \phi(x) \geq 2 \int_0^t K(X_{t-s}(x)) \phi(X_s(X_{t-s}(x)/2)) e^{-\int_0^{t-s} K(X_{s'}(x)) ds' - \int_0^s K(X_{s'}(x)) ds'} ds$$

and then, using (9.17),

$$S_{T+t}^* \phi \geq c_t c_T \mathbf{1}_{(x_0,R_1)} \int_0^t \int_{x_2}^{R_2} K(X_{t-s}(x)) \phi(X_s(X_{t-s}(x)/2)) dx\, ds.$$

We can assume that $x_2 > 2x_0$ and $R_2 > 2x_2$. The fact that $x_2 > 2x_0$ ensures, due to assumption (9.3), that K is bounded from below by a positive constant on $[x_2, X_t(R_2)]$. We thus deduce, by using of a change of variables, that for any $t > 0$

$$S_{T+t}^* \phi \geq c_t c_T \mathbf{1}_{(x_0,R_1)} \int_{X_t(X_t(x_2)/2)}^{R_2/2} \phi(y) dy.$$

Since $X_t(x) \to x$ when $t \to 0$, we deduce for all $\zeta > 0$ the existence of $t > 0$ such that

$$S_{T+t}^* \phi \geq c_t c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2/2+\zeta}^{R_2/2} \phi(y) dy.$$

Since $R_2 > 2x_2$, we deduce by combining the above inequality with (9.17) that for all $T > T_0 + t_1 + t$

$$S_T^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{x_2/2+\zeta}^{R_2} \phi(x) dx.$$

Let us take $\zeta = x_0$. Since the sequence (u_n) defined by $u_0 = x_2$ and $u_{n+1} = u_n/2 + x_0$ converges to $2x_0$, we obtain by an iteration argument the existence of a time t_2 such that for all $T > T_0 + t_2$

$$S_T^* \phi \geq c_T \mathbf{1}_{(x_0,R_1)} \int_{2x_0+\varepsilon}^{R_2} \phi(x) dx.$$

Using a last time the argument with $\zeta = \varepsilon/2$ yields the desired result. □

We now prove another positivity result which allows making the time T independent of R_1 in Lemma 9.2.

Lemma 9.3. *Let $R_1 > 2x_0$. Then there exists $t_0 > 0$ such that for any $R > R_1$ we have*

$$S^*_{t_0}\mathbf{1}_{(x_0,R_1)} \geq c_R \mathbf{1}_{(x_0,R)}$$

for some $c_R > 0$.

Proof of Lemma 9.3. Since a is Lipschitz continuous, we can find $t_0 > 0$ small enough so that $X_{t_0}(x) \leq \alpha x$ for all $x > x_0$, with $\alpha > 1$ to be determined later. Then for any $t \in (0, t_0]$ and any $x \in (x_0, \frac{R_1}{\alpha})$, we have by using the first term in (9.15)

$$S^*_t \mathbf{1}_{(x_0,R_1)}(x) \geq c_{t_0}\mathbf{1}_{(x_0,R_1)}(X_t(x)) = c_{t_0} > 0.$$

Iterating once (9.15) and keeping only the second term, we get that for any $t \in (0, t_0]$ and any $x \in (2x_0, \frac{2R_1}{\alpha^2})$

$$S^*_t \mathbf{1}_{(x_0,R_1)}(x) \geq c_{t_0} \int_0^t \mathbf{1}_{(x_0,R_1)}(X_s(X_{t-s}(x)/2))\, ds = c_{t_0} t.$$

Choosing $\alpha > 1$ such that $\frac{R_1}{\alpha} > 2x_0$ and $\frac{2R_1}{\alpha^2} > R_1$, we deduce that for any $t \in (0, t_0]$ there exists $c_t > 0$ such that

$$S^*_t \mathbf{1}_{(x_0,R_1)} \geq c_t \mathbf{1}_{(x_0,2\alpha^{-2}R_1)}.$$

Dividing $[0, t_0]$ into n sub-intervals $[\frac{k}{n}t_0, \frac{k+1}{n}t_0]$, $0 \leq k \leq n-1$, and iterating the above inequality with $t = t_0/n$, we deduce for all integer $n \geq 1$ the existence of $c_n > 0$ such that

$$S^*_{t_0}\mathbf{1}_{(x_0,R_1)} \geq c_n \mathbf{1}_{(x_0,(2\alpha^{-2})^n R_1)}$$

and the proof is complete since $2\alpha^{-2} > 1$. □

With Lemmas 9.2 and 9.3, we are now in position to prove the convergence result in Theorem 9.1.

Proof of the exponential stability part of Theorem 9.1. We apply Theorem 6.3. We start by proving that (6.9) is verified, in a quantitative way, for the function $g_0 = \mathbf{1}_{(x_0+\varepsilon,R_2)}$ with a suitable choice of R_2 and ε. Choosing $r_0 \in (1, r)$ if $m(x) = x^r$ or any $r_0 > 1$ is $m(x) = \exp\big(\eta \int_{x_0}^x K/a\big)$ and defining $m_0(x) = x^{r_0}$ we have from (9.12), because of the normalization $\|\phi_1\|_{L^\infty_{m^{-1}}} = 1$,

$$\|\phi_1\|_{L^\infty_{m_0^{-1}}} \leq C_0$$

for some explicit constant $C_0 > 0$. Defining

$$R_2 := \inf\{R > 0;\ m_0(x)/m(x) \leq 1/2C_0,\ \forall\, x > R\},$$

we have

$$1 = \|\phi_1\|_{L^\infty_{m^{-1}}} = \sup_{(x_0,\infty)} \frac{\phi_1}{m} = \sup_{(x_0,R_2)} \frac{\phi_1}{m},$$

because

$$\sup_{(R_2,\infty)} \frac{\phi_1}{m} \leq \sup_{(R_2,\infty)} \frac{\phi_1}{m_0}\frac{m_0}{m} \leq C_0 \frac{1}{2C_0} < 1.$$

Together with the fact that $\phi_1' \in L^\infty_{\mathrm{loc}}$, with a quantitative estimate on $\|\phi_1'\|_{L^\infty(x_0,R_2)}$, we see that ϕ_1 has some quantifiable mass on $(x_0+\varepsilon, R_2)$ for $\varepsilon > 0$ small enough, which exactly means that $\langle g_0, \phi_1\rangle$ is quantified from below.

Now we prove that the Harris condition (6.8) is verified. Choosing $R_1 > 2x_0$ and combining Lemma 9.2 and Lemma 9.3, we have for any $\varepsilon > 0$ and $R_2 > x_0 + \varepsilon$ the existence of $T > 0$ such that for any $R > R_1$

$$S^*_T\phi \geq c_R \mathbf{1}_{(x_0,R)} \int_{x_0+\varepsilon}^{R_2} \phi\, dx, \quad \forall\, \phi \geq 0. \tag{9.18}$$

Defining $g_0 = \mathbf{1}_{(x_0+\varepsilon,R_2)}$, we deduce by duality that for all $f \geq 0$,

$$S_T f \geq c_R \langle f, \mathbf{1}_{(x_0,R)}\rangle g_0.$$

Let us now consider $A > 0$ and $f \in X_+$ such that $\|f\| \le A[f]_{\phi_1}$. Since $m_0(x)/m(x) \to 0$ as $x \to +\infty$ and $\|\phi_1/m_0\|_\infty \le C_0$, we have

$$\begin{aligned} [f]_{\phi_1} &= \int_{x_0}^R f\frac{\phi_1}{m} m + \int_R^\infty f m \frac{\phi_1}{m_0}\frac{m_0}{m} \\ &\le \langle f, \mathbf{1}_{(x_0,R)}\rangle \sup_{(x_0,R)} m + \|f\| C_0 \sup_{(R,\infty)} \frac{m_0}{m} \\ &\le \langle f, \mathbf{1}_{(x_0,R)}\rangle\, m(R) + \frac{1}{2}[f]_{\phi_1}, \end{aligned}$$

by choosing R large enough. We deduce that $S_T f \ge \frac{c_R}{2m(R)}[f]_{\phi_1} g_0$, which is equivalent to (6.8) where we recall the definition $\widetilde{S}_t := S_t e^{-\lambda_1 t}$.

Finally, we prove the Lyapunov condition (6.7). On the one hand, we get from (9.10) that

$$\frac{d}{dt}\widetilde{S}^*_t m = \widetilde{S}^*_t(\mathcal{L}^* - \lambda_1)m \le (\kappa_{\mathcal{B}} - \lambda_1)\widetilde{S}^*_t m + M\widetilde{S}^*_t(\mathbf{1}_{(x_0,R)} m).$$

On the other hand, arguing as in (6.6), we infer from (9.18) that

$$\phi_1 = e^{-\lambda_1 T} S^*_T \phi_1 \ge c_R e^{-\lambda_1 T}\langle g_0, \phi_1\rangle \mathbf{1}_{(x_0,R)}.$$

Combining both we deduce that

$$\frac{d}{dt}\widetilde{S}^*_t m \le (\kappa_{\mathcal{B}} - \lambda_1)\widetilde{S}^*_t m + \tilde{M}\phi_1$$

with $\tilde{M} = \frac{m(R)}{c_R}\frac{e^{\lambda_1 T}}{\langle g_0, \phi_1\rangle}$, and Grönwall's inequality then yields

$$\widetilde{S}^*_t m \le e^{(\kappa_{\mathcal{B}} - \lambda_1)t} m + \tilde{M} t e^{(\kappa_{\mathcal{B}} - \lambda_1)t}\phi_1.$$

This guarantees that (6.7) is verified with $\gamma_L = e^{(\kappa_{\mathcal{B}} - \lambda_1)T} \in (0,1)$ and $K = \tilde{M}T$.

We have proved that the conditions (6.13), (6.7) and (6.9) are verified. The conclusion of the proof then follows from Theorem 6.3. □

9.2. **The mitosis equation with non-mixing growth rate.** In this section, we investigate the case when the mixing condition (9.6) is not verified. In other words, we place ourselves under the singular condition that

$$\forall x > x_0, \quad a(2x) = 2a(x). \tag{9.19}$$

In this case, we still have the existence of a unique eigen-triplet (λ_1, f_1, ϕ_1) but the boundary point spectrum is not reduced to λ_1. As a consequence, the long time behavior of the semigroup does not stabilizes along f_1 but it exhibits periodic oscillations.

Theorem 9.4. *Suppose that* (9.3), (9.4), (9.5) *and* (9.19) *are satisfied. The first eigentriplet problem admits a unique solution* $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$ *with the normalization* $\|\phi_1\| = \langle \phi_1, f_1\rangle = 1$, *and this triplet additionally satisfies* $\lambda_1 > 0$, $f_1 > 0$ *and* $\phi_1 > 0$.
Besides, $\Sigma^+_P(\mathcal{L}) = \{\lambda_1 + ik\alpha,\ k \in \mathbb{Z}\}$ *for some quantifiable* $\alpha > 0$, *there exists a family* $(g_k, \psi_k)_{k\in\mathbb{Z}}$ *of corresponding primal and dual eigenvectors that verifies* $\langle \psi_k, g_\ell\rangle = \delta_{k\ell}$, *and for all* $f \in L^1_{\phi_1}$, *we have the convergence*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t)(f - \Pi f)\|_{L^1_{\phi_1}} \to 0 \qquad \text{as } t \to +\infty,$$

where $\Pi f = \lim_{n\to\infty} \frac{1}{n}\sum_{\ell=0}^{n}\sum_{k=-\ell}^{\ell} \langle \psi_k, f\rangle g_k$.

This new result complements the scarce literature on the long time behavior of equation (9.1) in the singular case (9.19) which, to the best of our knowledge, is limited to the references [50, 169, 188]. We will actually prove that the convergence in Theorem 9.4 also holds in other spaces, and this will be the occasion to compare our method and results to the three above mentioned papers.

The proof of the conclusion **(C2)** in Section 9.1 does not use the mixing assumption (9.6). It thus also proves the existence, uniqueness and strict positivity of eigentriplet (λ_1, f_1, ϕ_1) under the assumptions of Theorem 9.4, as well as the fact that equation (9.1) is associated with a semigroup

S in X. For proving the long time convergence result, we start by verifying that this semigroup extends to other relevant Banach spaces.

Well-posedness in entropic L^p and M^1 spaces. The dual eigenfunction ϕ_1 satisfies by definition $\mathcal{L}^*\phi_1 = \lambda_1\phi_1$ and the rescaled semigroup $\widetilde{S}_t = S_t e^{-\lambda_1 t}$ is thus a contraction for the norm of $L^1_{\phi_1}$. In particular S_t is a bounded operator for this norm and, since L^1_m is dense in $L^1_{\phi_1}$, we can uniquely extend the semigroup S into a strongly continuous semigroup in $L^1_{\phi_1}$. Similarly, due to the weak-$*$ density of $L^1_{\phi_1}$ into $M^1_{\phi_1}$, this semigroup extends uniquely into a weakly-$*$ continuous semigroup in $M^1_{\phi_1}$. We still denote by S these extensions.
The General Relative Entropy principle, see [269, 49], ensures that the weighted L^p sub-spaces of $L^1_{\phi_1}$ defined by

$$X_p := L^p_{f_1^{1-p}\phi_1}(x_0,\infty) \quad \text{for } p\in[1,\infty) \quad \text{and} \quad X_\infty := L^\infty_{f_1^{-1}}(x_0,\infty)$$

are invariant under the semigroup S and the restriction to these spaces is a contraction. Besides, Jensen's inequality yields that it is a decreasing sequence for the inclusion

$$p>q \qquad \Longrightarrow \qquad X_p \supset X_q.$$

Since $X_\infty \subset X_p$ is dense, we can infer the strong continuity of S in X_p from the strong continuity in X_1 by writing for any $f\in X_\infty$

$$\|\widetilde{S}_t f - f\|^p_{X_p} \le \|\widetilde{S}_t f - f\|^{p-1}_{X_\infty}\|\widetilde{S}_t f - f\|_{X_1} \le 2^{p-1}\|f\|^{p-1}_{X_\infty}\|\widetilde{S}_t f - f\|_{X_1} \to 0,$$

as $t\to 0$.

Long-time convergence in $M^1_{\phi_1}$. We start by giving some useful properties of the dual semigroup S^* in $X' = L^\infty_{m^{-1}}$. Splitting $\mathcal{L}^*$ as $\mathcal{L}^* = \mathcal{A}_0^* + \mathcal{B}_0^*$ with $\mathcal{A}_0^*\phi = \mathscr{K}^*[\phi]$, so that $\mathcal{B}_0^*\phi = a\phi' - K\phi$, the Duhamel formula

$$S^*_{\mathcal{L}} = S^*_{\mathcal{B}_0} + S^*_{\mathcal{B}_0}\mathcal{A}_0 * S^*_{\mathcal{L}}$$

ensures that $\bar\varphi(t,x) := S^*_t\phi(x)$ is a fixed point of the operator Γ defined by

$$\begin{aligned}(9.20)\quad \Gamma\varphi(t,x) &:= S^*_{\mathcal{B}_0}(t)\phi(x) + \big[S^*_{\mathcal{B}_0}\mathcal{A}_0 * \varphi(\cdot,x)\big](t)\\ &= \phi(X_t(x))e^{-\int_0^t K(X_s(x))ds} + 2\int_0^t K(X_{t-s}(x))\varphi(s,X_{t-s}(x)/2)e^{-\int_0^{t-s}K(X_{s'}(x))ds'}ds,\end{aligned}$$

where we recall that $X_t(x)$ is the solution to the characteristic equation (9.16). It turns out that Γ has a unique fixed point in $L^\infty_{\rm loc}([0,\infty)\times(x_0,\infty))$, and that this fixed point also lies in any closed subset of $L^\infty_{\rm loc}([0,\infty)\times(x_0,\infty))$ which is left invariant by Γ. This property is proved in [169] or in [35, Sec. 6.3], by building $\bar\varphi$ thanks to the Banach-Picard fixed point theorem. It has very useful consequences, as for instance the fact that if $\phi\in C(x_0,\infty)$, then $\bar\varphi\in C([0,\infty)\times(x_0,\infty))$. In particular, this implies that $C(x_0,\infty)\cap L^\infty_{m^{-1}}$ is invariant under the semigroup S^*. Since $C(x_0,\infty)\cap L^\infty_{m^{-1}}$ is a dense subspace of $C_{0,\phi_1}(x_0,\infty)$, this ensures that C_{0,ϕ_1} is invariant under S^* and that the duality relation

$$\langle S_t f,\phi\rangle = \langle f, S^*_t\phi\rangle$$

is valid for any $f\in M^1_{\phi_1}$ and $\phi\in C_{0,\phi_1}$. The proof of the next result crucially relies on another application of the fact that the fixed point of Γ belongs to any closed invariant subset.

Proposition 9.5. *Suppose that* (9.3), (9.4), (9.5) *and* (9.19) *are satisfied. Then* $\Sigma_P^+(\mathcal{L}) = \{\lambda_1 + ik\alpha,\ k\in\mathbb{Z}\}$ *for some* $\alpha>0$, *there exists a family* $(g_k,\psi_k)_{k\in\mathbb{Z}}$ *of corresponding primal and dual eigenvectors that verifies* $\langle\psi_k,g_\ell\rangle = \delta_{k\ell}$, *and for all* $f\in M^1_{\phi_1}$ *we have the convergence*

$$(9.21)\qquad \widetilde{S}_t f - \widetilde{S}_t\Pi f \to 0, \quad \textit{as}\quad t\to\infty, \quad \Pi f := \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n}\sum_{k=-\ell}^{\ell}\langle\psi_k,f\rangle g_k,$$

both convergences having to be understood in the sense of the weak-$*$ topology.

Note that we did not specify the space in which we define the boundary point spectrum $\Sigma_P^+(\mathcal{L})$ in Proposition 9.5. It is because this set is the same in all the Banach lattices we consider. Indeed, any $g\in M^1_{\phi_1}$ such that $\mathcal{L}g = \lambda g$ for some $\lambda\in\mathbb{C}$ with $\Re e(\lambda) = \lambda_1$ satisfies $|g|\in \mathrm{Span}(f_1)$, so that $g\in X = L^1_m$ for any weight m as in (9.7).

Proof of Proposition 9.5. Step 1. The rescaled semigroup $\widetilde{S}$ is a contraction semigroup in $M^1_{\phi_1} = (C_{0,\phi_1})'$. This ensures in particular that for all $f \in M^1_{\phi_1}$ the trajectory $(\widetilde{S}_t f)_{t\geq 0}$ is bounded in $M^1_{\phi_1}$. We can thus use Theorem 5.23-*(2)* to infer the non-triviality of the boundary spectrum, by proving that the conclusion cannot hold, see Remark 5.26-*(ii)*. We start from the fact that for any $\phi \in C_{0,\phi_1}(x_0,\infty)$, the solution $S^*\phi$ to the dual mitosis equation is the unique fixed point of Γ defined in (9.20), and that it belongs to any closed invariant subset of $C([0,\infty)\times(x_0,\infty))$. For $y > x_0$ we define the set

$$\mathcal{E}_y = \{x > x_0,\ x = 2^k y \text{ for some } k \in \mathbb{Z}\}$$

and we consider a function ϕ such that $\phi(x) = 0$ if $x \in \mathcal{E}_y$ and $\phi(x) > 0$ if $x \notin \mathcal{E}_y$. Then, since (9.19) ensures that $X_t(2x) = 2X_t(x)$ for all $t \geq 0$, the set

$$\big\{\varphi \in C([0,\infty)\times(x_0,\infty)),\ \varphi(t,x) = 0 \text{ if } X_t(x) \in \mathcal{E}_y \text{ and } \varphi(t,x) > 0 \text{ if } X_t(x) \notin \mathcal{E}_y\big\}$$

is invariant under Γ. Consequently, the unique fixed point $S^*_t\phi$ belongs to this set, and we deduce that $S^*_t\phi(x) = 0$ if and only if $X_t(x) \in \mathcal{E}_y$. In other words, by duality, $\operatorname{supp}(S_t\delta_x) \subset \mathcal{E}_y$ if and only if $X_t(x) \in \mathcal{E}_y$, and in particular $\operatorname{supp}(S_t\delta_x) \subset \mathcal{E}_{X_t(x)}$ for all $x > x_0$ and $t \geq 0$. This prevents the convergence of $\widetilde{S}_t\delta_x$ toward $\langle \delta_x, \phi_1\rangle f_1$ and we infer from (the negation of) Theorem 5.23-*(2)* that the boundary point spectrum cannot be trivial.

Step 2. We next formulate the following simple but fundamental observation. If (λ, f) is a solution to the eigenvalue problem in $X = M^1_m$, then $f \in D(\mathcal{L}) \subset BV_{\rm loc} \subset L^1_{\rm loc}$ and it is a solution to the eigenvalue problem in $X = L^1_m$. Symmetrically, if (λ, ϕ) is a solution to the eigenvalue problem in $Y = L^\infty_{m_1^{-1}}$, then $\phi \in D(\mathcal{L}^*) \subset \mathrm{Lip}_{\rm loc} \cap L^\infty_{m_2^{-1}}$, $m_1(x)/m_2(x) \to 0$ as $x \to \infty$, and it is a solution to the eigenvalue problem in $Y = C_{0,m_1}$. In other words, the point spectrum and the associated eigenelements are the same in the two frameworks $(L^1_m, L^\infty_{m^{-1}})$ and $(M^1_m, C_{0,m})$. Now, as a consequence of this observation and Step 1, we know that the boundary point spectrum $\Sigma^+_P(\mathcal{L})$ is not trivial. Because we have proved that $(\kappa - \mathcal{B})^{-1}\mathcal{A}$ is compact in L^1_m, $\kappa < \lambda_1$, we may apply Theorem 5.7 and we obtain that $\Sigma^+_P(\mathcal{L}) = \{\lambda_1\} + i\alpha\mathbb{Z}$ for some $\alpha > 0$, and each eigenvalue is algebraically simple. Using finally Theorem 5.25 in the situation *(2)*, we get the weak-$*$ convergence (9.21). ☐

This result is proved by means of entropy techniques in [169] for a linear growth rate $a(x) = x$, by taking advantage of the explicit formulation of the eigenvectors g_k and ψ_k in terms of f_1 and ϕ_1 in that case. Here we extend it to any a satisfying $a(2x) = 2a(x)$. Note that arguing similarly as in [169], the convergence (9.21) may be strengthened into an exponential strong convergence in M^1_m for $m(x) = x^r$, $r > 1$, or $m(x) = \exp\big(\eta\int_{x_0}^x K/a\big)$, $0 < \eta < 1$, meaning that there is a spectral gap between $\Sigma_+(\mathcal{L})$ and the rest of the spectrum in these spaces.

Long-time convergence in X_p. We prove the following result, the case $p = 1$ of which corresponds to the convergence result of Theorem 9.4.

Proposition 9.6. *Under the same assumptions as in Proposition 9.5, the convergence* (9.21) *holds for the strong topology in X_p, $1 \leq p < \infty$ for all $f \in X_p$, and the convergence of the Fejér sum in the definition of the projector Π is also for this topology.*

Proof. The case $p = 1$ is an immediate consequence of Theorem 5.25, case *(4)*. The proof in the case $p > 1$ is a direct adaptation of the case $p = 1$. We aim at verifying that the trajectories $(\widetilde{S}_t f)_{t\geq 0}$ are relatively compact in X_p. We have already seen that $X_\infty \subset X_p$ is dense. Besides, the domain $D(\mathcal{L})$ of the generator $\mathcal{L} - \lambda_1$ of $\widetilde{S}$ in X_p is also dense in X_p, so that it suffices to check the relative compactness of $(\widetilde{S}_t f)_{t\geq 0}$ for $f \in X_\infty \cap D(\mathcal{L})$. For f in $X_\infty \cap D(\mathcal{L})$ the bounds

$$\|\widetilde{S}_t f\|_{X_p} \leq \|f\|_{X_p}, \quad \|\mathcal{L}\widetilde{S}_t f\|_{X_p} = \|\widetilde{S}_t\mathcal{L} f\|_{X_p} \leq \|\mathcal{L} f\|_{X_p} \quad \text{and} \quad \|\widetilde{S}_t f\|_{X_\infty} \leq \|f\|_{X_\infty}$$

yield the relative compactness of $(\widetilde{S}_t f)_{t\geq 0}$, the second bound guaranteeing uniform $W^{1,1}_{\rm loc}$ estimates. We can thus apply the case *(1)* of Theorem 5.25 to deduce the convergence (9.21) in X_p for the strong topology. ☐

Propositon 9.6 extends the result of [50] where it is proved in the case $p = 2$ for $a(x) = x$ by taking advantage of the Hilbert structure of X_2 and of the explicit formulation of the eigenvectors g_k and

ψ_k in terms of f_1 and ϕ_1. In this Hilbert setting it is proved that the Fourier series $\sum_{k=-n}^{n}\langle f,\psi_k\rangle g_k$ converges as n goes to infinity, and Πf is then given by the limit.

About the value of α. For ensuring that the boundary spectrum is discrete, we have used a compactness argument. The period $2\pi/\alpha$ of the periodic semigroup $\widetilde{S}\Pi$ is thus not quantified. It is expected to be equal to the time needed for a particle to double its size by following the flow of a, namely

$$\frac{2\pi}{\alpha} = \int_x^{2x} \frac{dt}{a(t)}, \tag{9.22}$$

which is independent of the choice of $x \geq x_0$ due to the condition $a(2t) = 2a(t)$. This is known to be true in the case of a linear growth rate $a(x) = x$, see [134] or [50], and also for a general when the size domain is $(x_0, 4x_0)$, see [188], where explicit computations can be carried out. In the general case, we have not been able to prove (9.22). Yet the fact that for any $x > x_0$ the support of $S_t\delta_x$ is a subset of $\mathcal{E}_{X_t(x)}$ guarantees that the period cannot be too small as shown now.

Proposition 9.7. *We have the estimate*

$$\frac{2\pi}{\alpha} \geq \ell_a := \int_x^{2x} \frac{dt}{a(t)}. \tag{9.23}$$

Proof of Proposition 9.7. Let $x > x_0$ such that $\psi_1(x) \neq 0$ (actually any $x > x_0$ is suitable). We have

$$\widetilde{S}_t\delta_x - \widetilde{S}_t\Pi\delta_x \to 0,$$

as $t \to +\infty$, and $\operatorname{supp}\widetilde{S}_t\delta_x \subset \mathcal{E}_{X_t(x)}$, so $\operatorname{supp}\widetilde{S}_t\Pi\delta_x \subset \mathcal{E}_{X_t(x)}$ since $\widetilde{S}_t\Pi\delta_x$ is periodic. Besides,

$$\widetilde{S}_t\Pi\delta_x = \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=0}^{n}\sum_{k=-\ell}^{\ell}\psi_k(x)e^{i\alpha kt}g_k$$

and, since $\psi_1(x) \neq 0$, the period of this periodic function of time is $\frac{2\pi}{\alpha}$. But since $\operatorname{supp}\widetilde{S}_t\Pi\delta_x \subset \mathcal{E}_{X_t(x)}$ and the period of the set $\mathcal{E}_{X_t(x)}$ is ℓ_a, we have proved that (9.23) holds. □

On the other hand, we can use Theorem 6.5 for deriving a quantified lower bound on α, and thus an upper bound on the period. We work in the space $X = L^1_m$, recalling that $\Sigma_P^+(\mathcal{L})$ is the same in this space and in $M^1_{\phi_1}$.

Proposition 9.8. *There exists a constructive constant $\alpha_1 > 0$ such that $\Sigma(\mathcal{L}) \cap B(\lambda_1, \alpha_1) = \{\lambda_1\}$. In particular, $\alpha \geq \alpha_1$.*

To prove this result, we check that the conditions (6.7), (6.9) and (6.13) are verified, and we invoke Theorem 6.5. We start with a lemma which, together with Lemma 9.3, will guarantee the validity of (6.13).

Lemma 9.9. *For all $\varepsilon > 0$, $R_1 > x_0$, and $R_2 > x_0 + \varepsilon$, there exist $T > 0$ and $c_T > 0$ such that*

$$\int_0^T S_t^*\phi\,dt \geq c_T\mathbf{1}_{(x_0,R_1)}\int_{x_0+\varepsilon}^{R_2}\phi\,dx, \quad \forall\,\phi \geq 0.$$

Proof of Lemma 9.9. Throughout the proof we denote by c_t any positive constant that depend only on t. From the Duhamel formula (9.15), we get by positivity that for any $\phi \geq 0$

$$\int_0^T S_t^*\phi(x)dt \geq \int_0^T \phi(X_t(x))e^{-\int_0^t K(X_s(x))ds}dt.$$

We deduce that for all $x \in (x_0, R_1)$ and for T_0 large enough so that $X_{T_0}(x_0) > R_2$, we have

$$\int_0^{T_0} S_t^*\phi(x)dt \geq c_{T_0}\int_x^{X_T(x)}\phi(y)dy \geq c_{T_0}\int_{R_1}^{R_2}\phi(y)dy,$$

and the conclusion follows if $R_1 \leq x_0 + \varepsilon$. If not, we have with the same argument the existence of T such that for all $x \in (x_0, R_1)$

$$\int_0^T S_t^*\phi(x)dt \geq c_T\int_{\max(R_1,2x_0+\varepsilon)}^{2R_2}\phi(y)dy.$$

Iterating once Duhamel's formula and using that $X_t(x/2) = X_t(x)/2$ and (9.3) we get that for all $t \geq 0$ and all $y \in (2x_0 + \varepsilon, R_2)$

$$S_t^* \phi(y) \geq c_t \int_0^t K(X_{t-s}(y))\phi(X_s(X_{t-s}(y)/2))ds \geq c_t \phi(X_t(y/2)).$$

This yields for all $x \in (x_0, R_1)$

$$\begin{aligned}\int_0^{T+t} S_s^* \phi(x)ds &\geq \int_t^{T+t} S_s^* \phi(x)ds = \int_0^T S_s^* S_t^* \phi(x)ds \\ &\geq c_T \int_{\max(R_1, 2x_0+\varepsilon)}^{2R_2} S_t^* \phi(y)dy \\ &\geq c_T c_t \int_{\max(R_1, 2x_0+\varepsilon)}^{2R_2} \phi(X_t(y/2))dy = c_T c_t \int_{\max(X_t(\frac{R_1}{2}), X_t(x_0+\frac{\varepsilon}{2}))}^{X_t(R_2)} \phi(z)dz.\end{aligned}$$

Choosing $t > 0$ small enough so that $\max(X_t(\frac{R_1}{2}), X_t(x_0 + \frac{\varepsilon}{2})) \leq \max(\frac{R_1+\varepsilon}{2}, x_0 + \varepsilon)$, we get

$$\int_0^{T_1} S_s^* \phi(x)ds \geq c_{T_1} \int_{\max(\frac{R_1+\varepsilon}{2}, x_0+\varepsilon)}^{R_2} \phi(z)dz, \quad \text{for} \quad T_1 := T + t.$$

Iterating the argument we can build an increasing sequence of times T_n such that

$$\int_0^{T_n} S_t^* \phi(x)dt \geq c_{T_n} \int_{\max(u_n, x_0+\varepsilon)}^{R_2} \phi(z)dz, \ \forall n \geq 0,$$

where (u_n) is defined by $u_0 = R_1$ and $u_{n+1} = \frac{u_n+\varepsilon}{2}$. Since this sequence (u_n) converges to $\varepsilon < x_0 + \varepsilon$, we get the conclusion by taking n large enough. □

We are now in position to prove Proposition 9.8.

Proof of Proposition 9.8. Arguing similarly as in the proof of Theorem 9.1 and using Lemma 9.9 instead of Lemma 9.2, we can prove that the conditions (6.7), (6.9) and (6.13) are verified. Applying Theorem 6.5 then gives the result. □

9.3. **The model with variability.** In this last part, we consider the model with variability given by the equation (9.2). Compared to equation (9.1), the main consequence of introducing a variability in terms of the spectrum and the asymptotic behavior is that for equation (9.2) the boundary spectrum is trivial and the first eigenfunction is exponentially stable, no matter if a satisfies (9.6) or (9.19).

Additionally to the assumptions (9.3), (9.4) and (9.5), we ask that

$$K(x) = O\Big(\exp\Big(\delta \int_{x_0}^{x/2} K/a\Big)\Big) \quad \text{as } x \to +\infty, \tag{9.24}$$

for some $\delta > 0$. About the variability kernel $\wp$ we suppose that

$$\int_1^2 \wp(v, v_*)dv = 1, \ \forall v_* \in [1, 2], \quad \wp \in W^{1,\infty}([1, 2]^2) \quad \text{and} \quad \wp \geq \wp_* \tag{9.25}$$

for some $\wp_* > 0$. We still work in the space $X = L^1_m$ by considering the weight $m = m(x)$ as function of (x, v) constant in v.

Theorem 9.10. *Suppose that* (9.3)*,* (9.4)*,* (9.5)*,* (9.24) *and* (9.25) *are satisfied. The first eigentriplet problem for equation* (9.2) *admits a unique solution* $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$ *with the normalization* $\|\phi_1\| = \langle \phi_1, f_1 \rangle = 1$*, and this triplet additionally satisfies* $\lambda_1 > 0$*,* $f_1 > 0$ *and* $\phi_1 > 0$*. Besides, there are some constructive constants* $C \geq 1$*,* $\omega > 0$ *such that*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t)f - \langle \phi_1, f \rangle f_1\|_X \leq Ce^{-\omega t}\|f - \langle \phi_1, f \rangle f_1\|_X$$

for any $f \in X$ *and* $t \geq 0$.

Yet expected, this result was known only in the case of a discrete set of variabilities [108, 328]. Theorem 9.10 is thus new in the literature.

Because of the assumption (9.25), we easily see that the construction of the semigroup and the proof of the conditions **(H1)**, **(H2)** and **(H4)** given in Section 9.1 for the model without variability readily extend to the model (9.2). We thus only have to verify **(H3)** and some Harris type condition.

Condition (H3). Let $\delta \in (0,1)$ such that (9.24) is verified, and consider the weight function $m(x) = x^r$ with $r > 1$ or $m(x) = \exp\big(\eta \int_{x_0}^x K/a\big)$ with $\eta \in (0, 1-\delta)$. We also use the two other weights

$$m_1(x) = \exp\big(\eta_1 \int_{x_0}^x K/a\big), \quad m_2(x) = \exp\big(\eta_2 \int_{x_0}^x K/a\big)$$

for some $\eta_1 \in (\eta, 1-\delta)$ and $\eta_2 = \eta_1 + \delta$. We combine the two different splittings $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and $\mathcal{L} = \mathcal{A}_0 + \mathcal{B}_0$, where

$$\mathcal{A}f(x,v) = M\mathbf{1}_{(x_0,R)}(x) f(x,v), \quad \mathcal{A}_0 f(x,v) = 4\int_1^2 K(2x)\wp(v,v_*) f(x,v_*)\, dv_*.$$

We prove that for any $\kappa > \kappa_{\mathcal{B}}$ the operator

$$\mathcal{C} := (\kappa - \mathcal{B}_0)^{-1}\mathcal{A}_0(\kappa - \mathcal{B})^{-1}\mathcal{A}$$

is well defined and maps continuously L^1_m into $L^1_{m_1} \cap W^{1,1}_{\mathrm{loc}}$, in the sense that if (f_n) is bounded in L^1_m then the image is bounded in $L^1_{m_1} \cap W^{1,1}((x_0,R)\times[1,2])$ for all $R > 0$. In particular, $\mathcal{C} \in \mathcal{K}(L^1_m)$. More precisely, for any $\kappa > 0$, we prove

$$L^1_m \xrightarrow{\ \mathcal{A}\ } L^1_{m_2} \xrightarrow{(\kappa-\mathcal{B})^{-1}} L^1_{m_2} \xrightarrow{\ \mathcal{A}_0\ } L^1_{m_1} \cap W^{1,1}_{v,\mathrm{loc}} \xrightarrow{(\kappa-\mathcal{B}_0)^{-1}} L^1_{m_1} \cap W^{1,1}_{\mathrm{loc}}$$

where $W^{1,1}_{v,loc} := \{f \in L^1_{\mathrm{loc}}((x_0,\infty)\times[1,2]),\ \partial_v f \in L^1_{\mathrm{loc}}((x_0,\infty)\times[1,2])\}$.

The results for $\mathcal{A}$ and $(\kappa-\mathcal{B})^{-1}$ are proved as in the case without variability. For the third one, the fact that $\mathcal{A}_0$ maps $L^1_{m_2}$ in $L^1_{m_1}$ follows from assumption (9.24), and the fact that the range is in $W^{1,1}_{v,loc}$ is a direct consequence of the assumption that $\wp \in W^{1,\infty}([1,2]^2)$.

Finally we consider $\kappa - \mathcal{B}_0$ and we first verify that it is invertible in $L^1_{m_1}$ for any $\kappa > 0$. If $(\kappa - \mathcal{B}_0)g = f$, then necessarily

$$g(x,v) = \frac{1}{va(x)}\int_{x_0}^x e^{(\Lambda_\kappa(y)-\Lambda_\kappa(x))/v} f(y,v)\, dy, \tag{9.26}$$

where $\Lambda_\kappa(x) = \int_{x_0}^x \frac{\kappa+K}{a}$, and consequently

$$g(x,v)m_1(x) = \frac{e^{(\eta_1\Lambda(x)-\Lambda_\kappa(x))/v}}{va(x)}\int_{x_0}^x e^{(\Lambda_\kappa(y)-\eta_1\Lambda_0(y))/v} f(y,v)m_1(y)\, dy.$$

Since

$$\Lambda_\kappa(x) - \eta_1\Lambda_0(x) = (1-\eta_1)\Lambda_{\kappa/(1-\eta_1)}(x)$$

we have for all $v \in [1,2]$

$$\begin{aligned}\int_{x_0}^\infty \Big(\frac{\kappa}{1-\eta_1} + K(x)\Big) g(x,v)m_1(x)\, dx \\ &= \int_{x_0}^\infty \frac{1}{v}\Lambda'_{\kappa/(1-\eta_1)}(x) e^{-\frac{1-\eta_1}{v}\Lambda_{\kappa/(1-\eta_1)}(x)} \int_{x_0}^x e^{(1-\eta_1)\Lambda_{\kappa/(1-\eta_1)}(y,v)} f(y,v)m_1(y)\, dydx \\ &= \int_{x_0}^\infty e^{\frac{1-\eta_1}{v}\Lambda_{\kappa/(1-\eta_1)}(y)} f(y,v)m_1(y) \int_y^\infty \frac{1}{v}\Lambda'_{\kappa/(1-\eta_1)}(x) e^{-\frac{1-\eta_1}{v}\Lambda_{\kappa/(1-\eta_1)}(x)} dx\, dy \\ &= \frac{1}{1-\eta_1}\int_{x_0}^\infty f(y,v)m_1(y)\, dy.\end{aligned}$$

We deduce that the operator $\kappa - \mathcal{B}_0$ is invertible in $L^1_{m_1}$ with $\|(\kappa-\mathcal{B})^{-1}\| \le 1/\kappa$. We have also proved that $(\kappa-\mathcal{B}_0)^{-1}$ maps $L^1_{m_1}$ into $L^1_{Km_1}$ with $\|(\kappa-\mathcal{B}_0)^{-1}\|_{\mathscr{B}(L^1_{m_1},L^1_{Km_1})} \le \frac{1}{1-\eta_1}$. The fact that it maps $W^{1,1}_{v,loc}$ into $W^{1,1}_{\rm loc}$ readily follows from the formula (9.26). We conclude to the compactness of $\mathcal{C}$ and then to the validity of **(H3)**. Indeed, we can write (2.20) as

$$\hat f_n = \mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A}\hat f_n + \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n,$$

but also as

$$\hat f_n = \mathcal{R}_{\mathcal{B}_0}(\lambda_n)\mathcal{A}_0\hat f_n + \mathcal{R}_{\mathcal{B}_0}(\lambda_n)\varepsilon_n.$$

Combining both, we get

$$\hat f_n = \mathcal{C}\hat f_n + \big[\mathcal{R}_{\mathcal{B}_0}(\lambda_n)\mathcal{A}_0\mathcal{R}_{\mathcal{B}}(\lambda_n) + \mathcal{R}_{\mathcal{B}_0}(\lambda_n)\big]\varepsilon_n.$$

Since $\mathcal{C}$ is compact, we conclude to **(H3)** with the same argument as in the proof of Lemma 2.8.

From **(H1)**, **(H2)**, **(H3)** and **(H4)** we infer the conclusion **(C2)** about existence and uniqueness of (λ_1, f_1, ϕ_1), which gives a part of Theorem 9.10. For the quantitative exponential stability, we start with a lemma.

Lemma 9.11. *For all $\varepsilon > 0$, $R_1 > x_0$, and $R_2 > x_0 + \varepsilon$, there exist $T > 0$ and $c_T > 0$ such that*

$$S^*_T\phi \ge c_T \mathbf{1}_{(x_0,R_1)\times[1,2]} \int_1^2\int_{x_0+\varepsilon}^{R_2} \phi\, dxdv, \quad \forall\, \phi \ge 0. \tag{9.27}$$

Proof of Lemma 9.11. Let us fix $\varepsilon > 0$, $R_1 > x_0$, and $R_2 > x_0 + \varepsilon$. Throughout the proof we denote by c_t any positive constant that depend on t, and also possibly on the ingredients of the model g, K, $\wp$ and on ε, R_1, R_2, but is independent of $(x, v) \in (x_0, R_1) \times [1, 2]$.

First step. We prove first that there exists $T_1 > 0$ and $x_3 > x_2 > x_0$ such that $x_3 > \max(R_2, 2x_2)$ and

$$S^*_{T_1}\phi \ge c_{T_1} \mathbf{1}_{(x_0,R_1)\times[1,2]} \int_1^2\int_{x_2}^{x_3} \phi\, dxdv, \tag{9.28}$$

for all $\phi \ge 0$. We start from the Duhamel formula

$$\begin{aligned} S^*_t\phi(x,v) &= \phi(X^v_t(x), v)e^{-\int_0^t K(X^v_s(x))ds} \\ &\quad + 2\int_0^t\int_1^2 K(X^v_s(x))S^*_{t-s}\phi(X^v_s(x)/2, v_*)e^{-\int_0^s K(X^v_{s'}(x))ds'}\wp(v_*, v)dv_*ds, \end{aligned} \tag{9.29}$$

where $X^v_t(x)$ is the solution to the characteristic equation

$$\dot X^v_t(x) = v\, a(X^v_t(x)) \quad \text{with} \quad X^v_0(x) = x.$$

Iterating twice (9.29), using positivity and the fact that K and a are locally bounded and $\wp$ is bounded from below, we deduce that

$$\begin{aligned} &S^*_t\phi(x,v) \ge \\ &\quad c_t\int_0^t\!\!\int_1^2\!\!\int_0^s\!\!\int_1^2 K(X^v_{s'}(x))K(X^{v_*}_{s-s'}(X^v_{s'}(x)/2))\phi\big(X^{v_{**}}_{t-s}\big(X^{v_*}_{s-s'}\big(X^v_{s'}(x)/2\big)/2\big), v_{**}\big)\, dv_{**}ds'dv_*ds, \end{aligned}$$

on $(x_0, R_1) \times [1, 2]$, for all $\phi \ge 0$. Let t_0 be such that $X^1_{t_0}(x_0) = 2x_0 + 1$. Then, for $t > 2t_0$, we deduce, from the fact that K is locally bounded from below on $(2x_0, \infty)$, that

$$S^*_t\phi(x,v) \ge c_t\int_{2t_0}^t\int_1^2\int_{t_0}^s\int_1^2 \phi\big(X^{v_{**}}_{t-s}\big(X^{v_*}_{s-s'}\big(X^v_{s'}(x)/2\big)/2\big), v_{**}\big)\, dv_{**}ds'dv_*ds.$$

For $t > 2t_0 + 2$, by using the Fubini-Tonelli theorem, we thus have

$$S^*_t\phi(x,v) \ge c_t\int_1^2\int_{t-1}^t\int_{t_0}^{t_0+1}\left(\int_1^2 \phi\big(X^{v_{**}}_{t-s}\big(X^{v_*}_{s-s'}\big(X^v_{s'}(x)/2\big)/2\big), v_{**}\big)dv_*\right)ds'dsdv_{**}.$$

Using now a change of variables, we get

$$\begin{aligned} S^*_t\phi(x,v) &\ge c_t\int_1^2\int_{t-1}^t\int_{t_0}^{t_0+1}\left(\int_{X^{v_{**}}_{t-s}(X^1_{s-s'}(X^v_{s'}(x)/2)/2)}^{X^{v_{**}}_{t-s}(X^2_{s-s'}(X^v_{s'}(x)/2)/2)} \phi(y, v_{**})dy\right)ds'dsdv_{**} \\ &\ge c_t\int_1^2\int_{X^2_1(X^1_{t-t_0}(X^2_{t_0+1}(R_1)/2)/2)}^{X^2_{t-t_0-2}(x_0/2)/2} \phi(y, v_{**})\, dydv_{**}. \end{aligned}$$

Due to the strict positivity of a, we can choose $t = T_1$ large enough so that

$$X^2_{T_1-t_0-2}(x_0/2)/2 > \max\big(R_2, 2X^2_1(X^1_{T_1-t_0}(X^2_{t_0+1}(R_1)/2)/2)\big),$$

and we obtain (9.28) by setting $x_3 = X^2_{T_1-t_0-2}(x_0/2)/2$ and $x_2 = X^2_1(X^1_{T_1-t_0}(X^2_{t_0+1}(R_1)/2)/2)$, which concludes the first step of the proof.

Second step. We deduce (9.27) from (9.28) as follows. On the one hand, applying (9.28) to the function $S^*_t\phi$, we obtain

$$S^*_{T_1+t}\phi \geq c_{T_1}\mathbf{1}_{(x_0,R_1)\times[1,2]}\int_1^2\int_{x_2}^{x_3} S^*_t\phi\, dxdv.$$

On the other hand, iterating once the Duhamel formula (9.29), we get by positivity that

$$S^*_t\phi(x,v) \geq c_t\bigg[\phi(X^v_t(x),v) + \int_0^t\int_1^2 K(X^v_s(x))\phi(X^{v_*}_{t-s}(X^v_s(x)/2),v_*)dv_*ds\bigg].$$

We first assume $x_2 > 2x_0$. In that case, the term $K(X^v_s(x))$ is bounded from below uniformly in $s\in[0,t]$, $v\in[1,2]$ and $x\in[x_2,x_3]$, so that we infer from the two above inequalities that

$$S^*_{T_1+t}\phi \geq c_{T_1}c_t\mathbf{1}_{(x_0,R_1)\times[1,2]}\int_1^2\int_{x_2}^{x_3}\bigg[\phi(X^v_t(x),v) + \int_0^t\int_1^2 \phi(X^{v_*}_{t-s}(X^v_s(x)/2),v_*)dv_*ds\bigg]dxdv.$$

By a change of variable, we have

$$\int_{x_2}^{x_3}\phi(X^v_t(x),v)dx \geq c_t\int_{X^2_t(x_2)}^{x_3}\phi(y,v)dy$$

and

$$\int_0^t\int_{x_2}^{x_3}\phi(X^{v_*}_{t-s}(X^v_s(x)/2),v_*)dxds \geq c_t\int_{X^2_t(X^2_t(x)/2)}^{x_3/2}\phi(y,v_*)dy.$$

Since $X^2_t(x)\to x$ as $t\to 0$, we deduce that we can find, for any $\zeta>0$, a time $t>0$ such that

$$S^*_{T_1+t}\phi \geq c_{T_1}c_t\mathbf{1}_{(x_0,R_1)\times[1,2]}\bigg[\int_1^2\int_{x_2+\zeta}^{x_3}\phi(y,v)dydv + \int_1^2\int_{x_2/2+\zeta}^{x_3/2}\phi(y,v_*)dydv_*\bigg].$$

As $x_3 > 2x_2$, we can choose ζ small enough so that $x_3/2 > x_2+\zeta$, and we get

$$S^*_{T_1+t}\phi \geq c_{T_1}c_t\mathbf{1}_{(x_0,R_1)\times[1,2]}\int_1^2\int_{x_2/2+\zeta}^{x_3}\phi(y,v)dydv.$$

Impose additionally that $\zeta\leq x_0$. Since the sequence (u_n) defined by $u_0 = x_2$ and $u_{n+1} = u_n/2+\zeta$ converges to $2\zeta\leq 2x_0$, we deduce by an iteration argument the existence of a time T_2 such that

$$S^*_{T_2}\phi(x) \geq c_{T_2}\mathbf{1}_{(x_0,R_1)}(x)\int_1^2\int_{2x_0+\varepsilon}^{x_3}\phi(y,v)dydv. \tag{9.30}$$

Using a last time the argument with $\zeta\leq\varepsilon/2$ yields (9.27), since $x_3 > R_2$.
In the case where $x_2\leq 2x_0$, (9.28) directly implies (9.30) with $T_2 = T_1$, and only one iteration of the extension argument is enough for concluding. □

We are now in position to finish the proof of Theorem 9.10.

Proof of Theorem 9.10. The proof is exactly the same as for Theorem 9.1, Lemma 9.11 replacing Lemma 9.2. The only missing information is a quantitative $L^\infty_{\rm loc}$ estimate on the derivatives $\partial_v\phi_1$ and $\partial_x\phi_1$, in order to use the same argument as in the proof of Proposition 9.8 for verifying (6.9). The estimate on $\partial_x\phi_1$ follows directly from the equation $\mathcal{L}^*\phi_1 = \lambda_1\phi_1$, which also reads

$$\partial_x\phi_1 = \frac{1}{va(x)}\bigg[\lambda_1\phi_1 + K\phi_1 - 2K(x)\int_1^2\phi_1(x,v_*)\wp(v_*,v)dv_*\bigg].$$

For $\partial_v\phi_1$ we argue by duality, using that

$$\|\phi\|_{L^\infty} = \sup_{\|f\|_{L^1}=1}\langle\phi,f\rangle.$$

We start from

$$\phi_1 = (\lambda_1-\mathcal{B}^*_0)^{-1}\mathcal{A}^*_0\phi_1,$$

which yields

$$\begin{aligned}\|\partial_v\phi_1\|_{L^\infty((x_0,R)\times[1,2]} &= \sup_{\|f\|_{L^1}=1} \langle \partial_v(\lambda_1-\mathcal{B}_0^*)^{-1}\mathcal{A}_0^*\,\phi_1, f\rangle \\ &= \sup_{\|f\|_{L^1}=1} \langle \phi_1, \mathcal{A}_0(\lambda_1-\mathcal{B}_0)^{-1}\partial_v f\rangle\end{aligned}$$

where the supremum can be taken over the functions $f \in C^1_c((x_0,R)\times(1,2))$. Using an integration by parts in v_*, we have

$$\begin{aligned}\mathcal{A}_0(\lambda_1-\mathcal{B}_0)^{-1}\partial_v f(x,v) &= 4\int_1^2 \frac{K(2x)}{v_* a(x)} \int_{x_0}^x e^{(\Lambda_{\lambda_1}(y)-\Lambda_{\lambda_1}(x))/v_*}\partial_v f(y,v_*)dy\,\wp(v,v_*)dv_* \\ &= -4\int_1^2\int_{x_0}^x \partial_{v_*}\Big(\frac{K(2x)}{v_* a(x)} e^{(\Lambda_{\lambda_1}(y)-\Lambda_{\lambda_1}(x))/v_*}\wp(v,v_*)\Big) f(y,v_*)dydv_*.\end{aligned}$$

Since $\|\phi_1\|_{L^\infty((x_0,R)\times[1,2])} \le m(R)$, we deduce

$$\begin{aligned}&\|\partial_v\phi_1\|_{L^\infty((x_0,R)\times[1,2])} \\ &\quad\le 4m(R) \sup_{(y,v_*)\in(x_0,R)\times[1,2]} \int_{x_0}^R\int_1^2 \Big|\partial_{v_*}\Big(\frac{K(2x)}{v_* a(x)} e^{(\Lambda_{\lambda_1}(y)-\Lambda_{\lambda_1}(x))/v_*}\wp(v,v_*)\Big)\Big| dvdx,\end{aligned}$$

and this last quantity is finite due to the assumptions made on the functions a, K and $\wp$. □

10. The kinetic linear Boltzmann equation

In this section, we consider the kinetic linear Boltzmann type equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\Phi(x)\cdot\nabla_v f = \mathscr{K}[f] - Kf, \quad \text{in} \quad (0,\infty)\times\mathcal{O} \tag{10.1}$$

on the function $f = f(t,x,v)$, $t \ge 0$, $(x,v)\in\mathcal{O} := \Omega\times\mathbb{R}^d$. We assume that $K = K(x,v) \ge 0$ and that the collision operator $\mathscr{K}$ is linear and defined by

$$\mathscr{K} = r\mathscr{K}_1, \quad (\mathscr{K}_1 g)(x,v) := \int_{\mathbb{R}^d} k\, g_*\, dv_*, \tag{10.2}$$

for a real number $r > 0$ and some collision kernel $k : \Omega\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}_+$. Here and below, we use the common shorthands

$$g_* := g(v_*), \quad k := k(x,v,v_*), \quad k_* := k(x,v_*,v).$$

The most classical example for the collisional operator $\mathcal{C} = \mathscr{K} - K$ is the mass conservative operator

$$(\mathcal{C}g)(v) := \int_{\mathbb{R}^d} |v-v_*|^\gamma\{\mathscr{M} g_* - \mathscr{M}_* g\}\, dv_*, \tag{10.3}$$

for some function $\mathscr{M}\in L^1_+(\mathbb{R}^d)$ and some exponent $\gamma\in\mathbb{R}$, which includes the relaxation operator

$$(\mathcal{C}g)(v) := \sigma(\mathscr{M}\rho_g - g), \quad \rho_g := \int_{\mathbb{R}^d} g_*\, dv_*. \tag{10.4}$$

We make the follwing strong positivity and boundedness assumption on the collision kernel k and the function K. There exist $\gamma \ge 0$ and $K_i > 0$ such that

$$\forall\,(x,v)\in\Omega\times\mathbb{R}^d, \quad K_0 \le K(x,v)\langle v\rangle^{-\gamma} \le K_1. \tag{10.5}$$

There exists a weight function $m : \mathbb{R}^d \to [1,\infty)$ such that

$$\forall\, p\in[1,\infty], \quad k\, m_*^{-1} m \in L^\infty_x L^p_v L^{p'}_{v_*}. \tag{10.6}$$

For all $R > 0$, there exists $k_R > 0$ such that

$$\forall\,(x,v,v_*)\in\Omega\times B_R\times B_R, \quad k(x,v,v_*) \ge k_R. \tag{10.7}$$

It is worth emphasizing that for $\mathscr{K}$ and K defined in (10.3), the above assumptions are met when $m := \mathscr{M}^{-1/2} : \mathbb{R}^d \to [1,\infty)$ (so that in particular $\mathscr{M} > 0$ a.e.) and $\mathscr{M}^{1/2}\langle v\rangle^\gamma \in L^1\cap L^\infty$. We finally assume that for some weight function $m_1 : \mathbb{R}^d \to [1,\infty)$ such that $m_1/m \to \infty$ at infinity, we have

$$k\, m_*^{-1} m_1 \in L^\infty_x L^2_{vv_*}, \tag{10.8}$$

what holds true for the relaxation operator when $\mathscr{M} m_1 \in L^2(\mathbb{R}^d)$ and $m^{-1} \in L^2(\mathbb{R}^d)$, and that for some weight function $m_0 : \mathbb{R}^d \to [1,\infty)$ such that $m_0/m \to 0$ at infinity, we have

$$k\, m_{0*}^{-1} m \in L^\infty_x L^1_{vv_*}, \tag{10.9}$$

what holds true for the relaxation operator when $\mathscr{M} m \in L^1(\mathbb{R}^d)$ and $m_0^{-1} \in L^1(\mathbb{R}^d)$.

For the space domain Ω, we consider the two following cases:

(1) $\Omega := \mathbb{T}^d$, the torus;
(2) $\Omega := \mathbb{R}^d$, the whole space.

In case (1), and for the sake of simplicity, we will always assume that $\Phi = 0$. In case (2), we will need a confinement mechanism which will be provided by the mean of the confinement force associated to the confinement potential Φ. We do not consider here the case of a bounded domain with zero influx boundary condition because (1) our approach applies exactly as for the torus case and (2) this case has already been considered in the pioneering work by Vidav [351], where existence, uniqueness and exponential stability (with non constructive constants) have been established. We do not consider either the case of a bounded domain complemented with a reflection as we will consider in Section 11 for the kinetic Fokker-Planck evolution equation, because we have not been able to establish some crucial regularity estimates which seem to be necessary in our approach. We let this issue for a future work.

10.1. **The torus.**

In this section, we are first concerned with the kinetic linear Boltzmann equation in the torus

$$\partial_t f + v\cdot\nabla_x f = \mathscr{K}[f] - Kf, \quad \text{in} \quad (0,\infty)\times\mathbb{T}^d\times\mathbb{R}^d. \tag{10.10}$$

We make the boundedness and strong positivity assumptions listed above together with the additional assumption

$$k\, m_*^{-1} m_1, k\, m_{1*}^{-1/2} m \in L^\infty_{xvv_*}, \quad \sum_{u\in\mathbb{Z}^d} \|m_1^{-1/2}(u+\cdot)\|_{L^\infty(\mathbb{T}^d)} < \infty, \tag{10.11}$$

for some m_1 such that $m/m_1 \in L^1 \cap L^2$.

Theorem 10.1. *For the kinetic equation* (10.10) *in the torus and under conditions* (10.5)-(10.6)-(10.7)-(10.8)-(10.9) *and* (10.11) *for some weight functions m, m_0, m_1, there exists $r^* > 0$ such that for any $r \ge r^*$, the conclusions* **(C3)** *holds in L^2_m and the conclusion* **(E3$_1$)** *holds in L^1_m.*

Our result may be compared to [351] which establishes the same result without constructive rate and to [103] which establishes the same result using a probabilistic approach, both in the case of a bounded domain with zero influx boundary condition. It also extends to a non mass conservative situation the many results devoted to the conservative framework, see for instance the recent papers [285, 200, 79] and the references therein. When $\gamma > 0$, we may probably establish the same above result under the sole condition $r > 0$ (no need for r to be large enough) by using some arguments developed in the next section.

We present now the proof of Theorem 10.1 by establishing that the conditions presented in the abstract part are satisfied.

Condition (H1). For an exponent $p \in [1,\infty)$ and a weight function m satisfying (10.6), we set

$$k_\infty := \|k m_*^{-1} m\|_{L^\infty_x L^p_v L^{p'}_{v_*}} < \infty.$$

Considering then a solution f to the evolution equation (10.10), we compute

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int f^p m^p &= \int \mathscr{K}[f] f^{p-1} m^p - K(x,v) f^p m^p \\ &\le \|\mathscr{K}[f] m\|_{L^p} \Big(\int f^p m^p\Big)^{1-1/p} - \int K(x,v) f^p m^p \\ &\le r k_\infty \int f^p m^p - K_0 \int \langle v\rangle^\gamma f^p m^p,\end{aligned}$$

where we have used twice the Holder inequality. This differential inequality together with the Gronwall lemma provides an apriori estimates about the growth of the L^p_m norm. As a consequence, the same arguments as in section 8.3 imply that $S_\mathcal{L}$ is a positive semigroup in L^p_m with growth bound $\omega(S_\mathcal{L}) = r k_\infty - K_0$. In particular, condition **(H1)** holds thanks to Lemma 2.2.

Condition (H2). For $f_0 := \mathbf{1}_{\mathbb{T}^d \times B_1}$, where B_1 denotes the unit ball in $\mathbb{R}^d_v$, we compute

$$\mathcal{L} f_0 = \mathscr{K}[f_0] - K f_0 \ge \inf_{v\in B_1} \big\{ r \mathscr{K}_1[f_0] - K \big\} f_0.$$

Using (10.5) and the strong positivity condition (10.7), we get

$$\inf_{v\in B_1} \big\{ \mathscr{K}[f_0] - K \big\} \ge r k_1 - 2^{\gamma/2} K_0 =: \kappa_0, \tag{10.12}$$

which provides a constructive lower bound of the set $\mathcal{I}$ defined in (2.15) thanks to Lemma 2.4-**(ii)**. We have thus established that $\mathcal{L}$ satisfies **(H2)**.

Condition (H3). We define the operator

$$\mathcal{B} f := -v\cdot\nabla_x f - K(x,v) f,$$

and we assume $\kappa_\mathcal{B} := -\inf K \le -K_0 < \kappa_0$, what holds whenever $r \ge r^*$, with $r^* > 0$ large enough thanks to (10.12). In the sequel, we assume $p = 2$ and we work in $X = L^2_m$. We immediately deduce that $\mathcal{B} - \kappa$ is dissipative for any $\kappa > \kappa_\mathcal{B}$, and thus $\mathcal{R}_\mathcal{B}(z)$ is bounded in $\mathscr{B}(L^2_m)$, uniformly in $z \in \Delta_\kappa$. For $\kappa > \kappa_\mathcal{B}$ and $g \in L^2_m$, the function $f = \mathcal{R}_\mathcal{B}(\kappa) g$ satisfies

$$v\cdot\nabla_x f + (\kappa + K) f = g \quad \text{in} \quad \mathcal{O},$$

from what we deduce

$$(\kappa - \kappa_\mathcal{B}) \int_\mathcal{O} f^2 m^2 \le \int_\mathcal{O} (\kappa + K) f^2 m^2 = \int_\mathcal{O} f g m^2,$$

and finally

$$\|f\|^2_{L^2_m} \le \frac{1}{\kappa - \kappa_\mathcal{B}} \|g\|^2_{L^2_m}.$$

Because of assumption (10.8), and defining $\mathcal{A} := \mathscr{K}$, we immediately deduce that

$$\mathcal{A}\mathcal{R}_\mathcal{B}(\kappa) : L^2_m \to L^2_{m_1}. \tag{10.13}$$

On the other hand, from the classical averaging lemma [182], we know that

$$\mathcal{A}_\varphi \mathcal{R}_\mathcal{B}(\kappa) : L^2(\mathcal{O}) \to H^{1/2}(\mathcal{O}), \tag{10.14}$$

where for $\varphi = \varphi_1 \otimes \varphi_2 \in C^1_c(\mathcal{O}) \otimes C^1_c(\mathbb{R}^d)$, we have defined the mapping $\mathcal{A}_\varphi : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ by

$$\mathcal{A}_\varphi(f)(x) := \varphi_1(x,v) \int_{\mathbb{R}^d} f(x,v_*) \varphi_2(v_*)\, dv_*.$$

By classical approximation arguments, there exists a sequence (φ_n) such that $\varphi_n \to k$ in the space $L^\infty(\mathbb{T}^d; L^2_{m_1 \otimes m^{-1}}(\mathbb{R}^d \times \mathbb{R}^d))$ and such that φ_n is a linear combination of functions of $C^1_c(\mathcal{O}) \otimes C^1_c(\mathbb{R}^d)$. As a consequence of (10.13) and (10.14), we deduce that $\mathcal{A}\mathcal{R}_\mathcal{B}(\kappa) \in \mathscr{K}(L^2_m)$ and next $(\mathcal{R}_\mathcal{B}(\kappa)\mathcal{A})^2 \in \mathscr{K}(L^2_m)$ for any $\kappa > \kappa_\mathcal{B}$. We may use Lemma 2.8 (and Remark 2.9-(2)) with $N = 2$ in $X = L^2_m$, and deduce that **(H3)** holds.

Condition (H4). We start with a result of independent interest about strict positivity. Such an argument is reminiscent from [88, 327] in the study of the Boltzmann equation and has been used for instance in [293, 79].

Lemma 10.2. *For any $\varrho, \varrho_*, t > 0$, there exists $c > 0$ such that*

$$(S_{\mathcal{L}}(t)f_0)(x,v) \ge c\,\mathbf{1}_{B_\varrho}(v) \int_{\mathbb{T}^d \times B_{\varrho_*}} f_0\, dv_* dx_*, \tag{10.15}$$

for all $f_0 \ge 0$ and $(x,v) \in \mathbb{T}^d \times \mathbb{R}^d$.

Proof of Lemma 10.2. We observe that the semigroup $S_{\mathcal{B}}$ has explicit representation

$$(S_{\mathcal{B}}(t)f_0)(x,v) = f_0(x - vt, v) e^{-\int_0^t K(x - \tau v, v) d\tau}.$$

We next write the associated iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{B}} + S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{B}} + S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{B}}\mathscr{K} * S_{\mathcal{L}}.$$

Since all the terms are nonnegative, we may through away the first terms and the last one, and we get

$$S_{\mathcal{L}} \ge S_{\mathcal{B}} * \mathscr{K} S_{\mathcal{B}} * \mathscr{K} S_{\mathcal{B}}.$$

On the one hand, using the explicit expression of $S_{\mathcal{B}}$ and (10.7), we have

$$(\mathscr{K}_1 S_{\mathcal{B}}(s) f_0)(y,w) \ge k_{\varrho'} e^{-sK_{\varrho_*}} \mathbf{1}_{B_{\varrho'}}(w) \int_{B_{\varrho_*}} f_0(y - w_* s, w_*) dw_* =: g(s,y,w),$$

for any $s > 0$ and any $\varrho' \ge \varrho_* > 0$, with $K_\varrho := \sup_{z \in \mathbb{T}^d, |v| \le \varrho} K(z,v)$. On the other hand, for the same reasons, we have

$$\begin{aligned}
(\mathscr{K}_1 S_{\mathcal{B}} * g(t))(x,v) &= \int_0^t \int_{\mathbb{R}^d} k(x,v,v_*) g(s, x - v_*(t-s), v_*) e^{-\int_0^{t-s} K(x - \tau v_*, v_*) d\tau} dv_* ds \\
&\ge k_{\varrho'} \mathbf{1}_{B_\varrho}(v) \int_0^t \int_{B_{\varrho'}} g(s, x - v_*(t-s), v_*) e^{-(t-s)K_{\varrho'}} dv_* ds \\
&\ge k_{\varrho'}^2 e^{-tK_{\varrho'}} \mathbf{1}_{B_\varrho}(v) \int_0^t \int_{B_{\varrho'}} \int_{B_{\varrho_*}} f_0(x - v_*(t-s) - w_* s, w_*) dw_* dv_* ds \\
&\ge k_{\varrho'}^2 e^{-tK_{\varrho'}} \mathbf{1}_{B_\varrho}(v) \int_0^{t/2} \int_{B_{\varrho_*}} \int_{B(x + w_* s, (t-s)\varrho')} f_0(y_*, w_*) \frac{dy_*}{(t-s)^d} dw_* ds \\
&\ge k_{\varrho'}^2 \frac{e^{-tK_{\varrho'}}}{(t/2)^d} \mathbf{1}_{B_\varrho}(v) \int_0^{t/2} \int_{B_{\varrho_*}} \int_{\mathbb{T}^d} f_0(y_*, w_*) dy_* dw_* ds \\
&\ge k_{\varrho'}^2 \frac{e^{-tK_{\varrho'}}}{(t/2)^d} \mathbf{1}_{B_\varrho}(v) \frac{t}{2} \int_{B_{\varrho_*}} \int_{\mathbb{T}^d} f_0(y_*, w_*) dy_* dw_*,
\end{aligned}$$

for any $t > 0$ and $\varrho' \ge \max(\varrho, \varrho_*)$ such that $t\varrho' \ge 2$, in such a way that $B(z, (t/2)\varrho') \supset \mathbb{T}^d$. We then have

$$(\mathscr{K}_1 S_{\mathcal{B}} * \mathscr{K}_1 S_{\mathcal{B}}(t))(x,v) \quad \ge \quad k_{\varrho'}^2 \frac{e^{-tK_{\varrho'}}}{(t/2)^{d-1}} \mathbf{1}_{B_\varrho}(v) \int_{\mathbb{T}^d} \int_{B_{\varrho_*}} f_{0*}\, dw_* dx_*$$

for any $t \ge 2/\varrho'$. We finally conclude

$$S_{\mathcal{L}}(t) f_0(x,v) \ge r^2 k_{\varrho'}^2 e^{-tK_{\varrho'}} \int_{2/\varrho'}^t \frac{ds}{(s/2)^{d-1}}\, \mathbf{1}_{B_\varrho}(v) \int_{\mathbb{T}^d} \int_{B_{\varrho_*}} f_{0*}\, dw_* dx_*,$$

from what we deduce (10.15) by choosing $\varrho' = 8/t$. $\square$

We now consider $\lambda \ge \lambda_1$ and $0 \le f \in L^2_m$, $f \not\equiv 0$, such that

$$\lambda f + v \cdot \nabla_x f + Kf - \mathscr{K}[f] \ge 0 \quad \text{in} \quad \mathbb{T}^d \times \mathbb{R}^d.$$

We fix $\varrho_* > 0$ such that $f \not\equiv 0$ on B_{ϱ_*}. From (2.13), we have

$$f \ge \int_0^\infty e^{-(1+\lambda)t} S_{\mathcal{L}}(t) f dt,$$

and we conclude that $f > 0$ a.e. on any set $\mathbb{T}^d \times B_\varrho$ thanks to Lemma 10.2. We have established that the strong maximum holds true, and thus **(H4)**.

Condition (H5). Assume that $(\lambda, f) \in \mathbb{C} \times X \backslash \{0\}$ satisfies

$$\mathcal{L}f = \lambda f, \quad \mathcal{L}|f| = (\Re e\lambda)|f| = \Re e(\text{sign} f)\mathcal{L}f.$$

From **(H4)** and the first identity satisfied by $|f|$, we know that $|f| > 0$ a.e. on $\mathbb{T}^d \times \mathbb{R}^d$. Using the second identity, we get

$$\mathscr{K}[|f|] = \Re e(\text{sign} f)\mathscr{K}[f].$$

Writing $f = e^{i\alpha}|f|$, we deduce

$$\int_{\mathbb{R}^d} k(x, v, v_*)|f(x, v_*)|(1 - \cos(\alpha - \alpha_*))dv_* = 0 \quad \text{a.e. on } \mathbb{T}^d \times \mathbb{R}^d,$$

and thus $\alpha = \alpha(x)$ thanks to (10.7). Next, coming back to the first equation, we have

$$\begin{aligned} \lambda|f|e^{i\alpha} &= \mathcal{L}(|f|e^{i\alpha}) \\ &= e^{i\alpha}\mathcal{L}|f| - |f|e^{i\alpha} iv \cdot \nabla_x \alpha \\ &= e^{i\alpha}(\Re e\lambda)|f| - |f|e^{i\alpha} iv \cdot \nabla_x \alpha. \end{aligned}$$

The equation simplifies into

$$v \cdot \nabla_x \alpha = \Im m\lambda,$$

so that $\alpha(x) = \alpha$ is a constant and the reverse Kato's inequality holds.

Alternatively to **(H5)**, we readily infer from Lemma 10.2 that the variant condition **(H5′)** is verified.

At this stage, because of Theorem 2.21, Theorem 4.13 and Theorem 5.16 (or Theorem 5.18), we deduce the conclusions **(C1)**, **(C2)** and **(C3)** about the existence and uniqueness of the eigentriplet (λ_1, f_1, ϕ_1) which satisfies $f_1 > 0$, $\phi_1 > 0$, λ_1 is algebraically simple and on the triviality of the boundary punctual spectrum. We now establish the exponential asymptotic stability with constructive constants.

We start with a gain of unifom boundness estimate.

Lemma 10.3. *There exists $N \geq 1$ such that $(\mathcal{A}\mathcal{R}_\mathcal{B}(\kappa))^N : L^1_m \to L^\infty_m$, for any $\kappa > \kappa_\mathcal{B}$. As a consequence, $\phi_1 \in L^\infty_{m^{-1}}$.*

Proof of Lemma 10.3. Step 1. We argue silmilarly as in [279, Sec. 3.1]. On the one hand, denoting $\mathcal{A}_1 = \mathscr{K}_1$, so that $\mathcal{A} = r\mathcal{A}_1$, we have for any $f_0 \in L^1_m$

$$(\mathcal{A}_1 S_\mathcal{B}(t) f_0)(x, v) = \int_{\mathbb{R}^d} k(x, v, v_*) f_0(x - v_* t, v_*)\, e^{-\int_0^t K(x - v_*\tau, v_*)d\tau}\, dv_*$$

and, using estimate (10.11), we deduce that

$$\begin{aligned} \|m_1 \mathcal{A}_1 S_\mathcal{B}(t) f_0\|_{L^1_x L^\infty_v} &\leq \|k m_*^{-1} m_1\|_{L^\infty_{xvv_*}} \int_\mathcal{O} |f_0(x - v_* t, v_*)| m_* dv_* dx\, e^{t\kappa_\mathcal{B}} \\ &\lesssim \|f_0\|_{L^1_m}\, e^{t\kappa_\mathcal{B}}, \end{aligned}$$

for any $t \geq 0$. Now, we consider $f_0 \in L^1_x L^\infty_{vm}$, we write

$$(\mathcal{A}_1 S_\mathcal{B}(t) f_0)(x, v) = \sum_{u \in \mathbb{Z}^d} \int_{\mathbb{T}^d} k(x, v, v_*) f_0(x - ut - v_* t, u + v_*)\, e^{-\int_0^t K(x - (u + v_*)\tau, u + v_*)d\tau}\, dv_*$$

and using estimate (10.11) again, we compute

$$\begin{aligned} (\mathcal{A}_1 S_\mathcal{B}(t) f_0)(x, v) m(v) &\leq \|k m_{1*}^{-1/2} m\|_{L^\infty_{xvv_*}} \sum_{u \in \mathbb{Z}^d} \int_{\mathbb{T}^d} m_1^{1/2}(u + v_*) f_0(x - ut - v_* t, u + v_*)\, e^{t\kappa_\mathcal{B}}\, dv_* \\ &\lesssim e^{t\kappa_\mathcal{B}} \Big(\sum_{u \in \mathbb{Z}^d} \|m_1^{-1/2}(u + \cdot)\|_{L^\infty(\mathbb{T}^d)} \Big) \int_{t\mathbb{T}^d} \|f_0(y, \cdot)\|_{L^\infty_{m_1}} \frac{dy}{t^d} \\ &\lesssim e^{t\kappa_\mathcal{B}} \Big(1 + \frac{1}{t^d}\Big) \|f_0\|_{L^1_x L^\infty_{vm_1}}. \end{aligned}$$

Defining $\widetilde{u}(t) := e^{-\kappa t}\mathcal{A}_1 S_\mathcal{B}(t)$, $\kappa > \kappa_\mathcal{B}$, we have first established $\widetilde{u} : L^1_m \to L^1_x L^\infty_{vm_1}$ unifomly in time, and thus $\widetilde{u} : L^1_x L^\infty_{vm_1} \to L^1_x L^\infty_{vm_1}$ unifomly in time because $L^1_x L^\infty_{vm_1} \subset L^1_m$ (we use here the fact that $m/m_1 \in L^1$). On the other hand, we have establised that $t^d \widetilde{u} : L^1_x L^\infty_{vm_1} \to L^\infty_m$ uniformly in time. Using [276, Prop. 2.5] with $X := L^1_x L^\infty_{vm_1}$ and $Y := L^\infty_m$, we deduce $\widetilde{u}^{*(d+1)} : L^1_x L^\infty_{vm_1} \to L^\infty_m$ uniformly in time, and we thus conclude that $\widetilde{u}^{*N} : L^1_m \to L^\infty_m$ uniformly in time, for $N := d+2$. Using formula (2.13), we deduce that $(\mathcal{A}\mathcal{R}_\mathcal{B})^N(z) : L^1_m \to L^\infty_m$, uniformly for any $z \in \Delta_\kappa$.

Step 2. In particular, $(\mathcal{A}\mathcal{R}_\mathcal{B})^N(z) : L^1_m \to L^2_m$ because $L^\infty_{m_1} \subset L^2_m$ (we use here the fact that $m/m_1 \in L^2$). By duality, we deduce that $(\mathcal{R}_{\mathcal{B}^*}\mathcal{A}^*)^N(z) : L^2_{m^{-1}} \to L^\infty_{m^{-1}}$. Coming back to the eigenvalue equation

$$\mathcal{A}^*\phi_1 + \mathcal{B}^*\phi_1 = \lambda_1\phi_1,$$

we deduce

$$\phi_1 = \mathcal{R}_{\mathcal{B}^*}(\lambda_1)\mathcal{A}^*\phi = (\mathcal{R}_{\mathcal{B}^*}(\lambda_1)\mathcal{A}^*)^N\phi_1. \tag{10.16}$$

By construction $\phi_1 \in L^2_{m^{-1}}$ and we thus conclude that $\phi_1 \in L^\infty_{m^{-1}}$. ☐

From now on, we choose the normalization convention $\|\phi_1\|_{L^\infty_{m^{-1}}} = 1$ and $\langle f_1, \phi_1\rangle = 1$.

Because of (10.9) and proceeding similarly as during the proof of condition **(H3)**, we have

$$\mathcal{A}\mathcal{R}_\mathcal{B}(\kappa) : L^1_{m_0} \to L^1_m, \quad \forall \kappa > \kappa_\mathcal{B},$$

so that

$$\mathcal{R}_{\mathcal{B}^*}(\kappa)\mathcal{A}^* : L^\infty_{m^{-1}} \to L^\infty_{m_0^{-1}}, \quad \forall \kappa > \kappa_\mathcal{B}.$$

From the first identity in (10.16), we deduce

$$\|\phi_1\|_{L^\infty_{m_0^{-1}}} \le C_{01}\|\phi_1\|_{L^\infty_{m^{-1}}},$$

with constructive constant $C_{01} \in (0,\infty)$. We may here proceed along an already used argument. Consider $0 \le f \in L^1_m$ and assume $\|f\|_{L^1_m} \le A[f]_{\phi_1}$. We then compute

$$\begin{aligned}\int f\phi_1 &= \int_{\mathcal{O}_\varrho} f\frac{\phi_1}{m}m + \int_{\mathcal{O}_\varrho^c} fm\frac{\phi_1}{m_0}\frac{m_0}{m}\\ &\le \langle f, \mathbf{1}_{\mathcal{O}_\varrho}\rangle \sup_{\mathcal{O}_\varrho} m + \|f\|_{L^1_m} C_{01} \sup_{\mathcal{O}_\varrho^c}\frac{m_0}{m}\\ &\le \langle f, \mathbf{1}_{\mathcal{O}_\varrho}\rangle \sup_{\mathcal{O}_\varrho} m + \frac12 [f]_{\phi_1}\end{aligned}$$

by choosing $\varrho = \varrho(A)$ large enough, where we denote $\mathcal{O}_\varrho := \mathbb{T}^d \times B_\varrho$. Together with Lemma 10.2, we deduce that there exists $T > 0$ and $g_A \ge 0$, $g_A \not\equiv 0$, such that

$$S_\mathcal{L}(T)f \ge g_A[f]_{\phi_1}, \tag{10.17}$$

what is nothing but the Harris condition (6.8). On the other hand, from the above regularization estimate, we have in the same time

$$1 = \|\phi_1\|_{L^\infty_{m^{-1}}} \le C_0\|\phi_1\|_{L^1_{m^{-1}}}, \quad \|\phi_1\|_{L^1_{m_0^{-1}}} \le C_1\|\phi_1\|_{L^1_{m^{-1}}},$$

for some constructive constants $C_i \in (0,\infty)$. We may thus compute

$$\begin{aligned}\int \phi_1 m^{-1} &\le \int_{B_\varrho} \phi_1 m^{-1} + \sup_{B_\varrho^c}\frac{m_0}{m}\int \phi_1 m_0^{-1}\\ &\le \int_{B_\varrho} \phi_1 m^{-1} + \sup_{B_\varrho^c}\frac{m_0}{m} C_1 \int \phi_1 m^{-1},\end{aligned}$$

so that for $\varrho > 0$ large enough, we deduce

$$C_0^{-1} \le \|\phi_1\|_{L^1_{m^{-1}}} \le 2\int_{B_\varrho} \phi_1 m^{-1}. \tag{10.18}$$

Together with the definition of g_A, we deduce that the positivity condition (6.3) holds.

Finally, as during the proof of **(H3)** above, for $0 \le f_0 \in L^1_m$ and denoting $f := S_{\mathcal L}(t) f_0$, we compute

$$\begin{aligned}\frac{d}{dt}\int f m dv dx &= \int \mathscr{K}[f] m dv dx - \int K f m dx dv \\ &\le C_0 \int f m_0 dv dx + \kappa_{\mathcal B} \int f m dx dv,\end{aligned}$$

with $C_0 := \| k m m_{0*}^{-1} \|_{L^\infty_{x v_*} L^1_v} < \infty$ and $m_0/m \to 0$ as $v \to \infty$. Observing that

$$\int f m_0 \le \int_{B_\varrho} f \phi_1 \sup_{B_\varrho} \frac{m_0}{\phi_1} + \int_{B^c_\varrho} f m \sup_{B^c_\varrho} \frac{m_0}{m},$$

for any $\kappa > \kappa_{\mathcal B}$, we may choose $\varrho > 0$ large enough in such a way that $\sup_{B^c_\varrho} \frac{m_0}{m} \le (\kappa - \kappa_{\mathcal B})/C_0$ and we deduce that

$$\frac{d}{dt}\int f m dv dx \quad \le \quad C_1 \int f \phi_1 dv dx + \kappa \int f m dx dv$$

with $C_1 = \sup_{B_\varrho} \frac{m_0}{\phi_1}$. From the Gronwall lemma, we obtain

$$\| f_t \|_{L^1_m} \le e^{\kappa t} \| f_0 \|_{L^1_m} + \frac{e^{\lambda_1 t} - e^{\kappa t}}{\lambda_1 - \kappa} C_1 \int f_0 \phi_1,$$

from what we immediately deduce that $S_{\mathcal L}$ satisfies the Lyapunov condition (6.7) for any $t > 0$. It remains to quantify the constant C_1. The dual formulation of (10.15) applied to the dual eigenfunction ϕ_1 with $t = 1$ and $\varrho_* = \varrho$ yields

$$\phi_1 = e^{-\lambda_1} S^*_{\mathcal L}(1) \phi_1 \ge e^{-\lambda_1} c\, \mathbf{1}_{\mathbb T^d \times B_\varrho} \int_{\mathbb T^d \times B_\varrho} \phi_1 \, dv_* dx_*.$$

Together with Equation (10.18), this provides the explicit bound $C_1 \le 2 C_0 e^{\lambda_1} c^{-1} \frac{\sup_{B_\varrho} m_0}{\inf_{B_\varrho} m}$.

We have established that the three conditions (6.8), (6.7) and (6.9) hold, so that we conclude the proof of Theorem 10.1 by just applying Theorem 6.3.

10.2. **The whole space case.** In this section, we assume that $\Omega := \mathbb R^d$ and we consider the kinetic equation (10.1) with an additional force field confinement $F = -\nabla_x \Phi$ associated to a potential Φ. More precisely from now-on, we assume that

$$\Phi(x) = |x|^\beta, \ \beta > 2, \quad K(v) = \langle v \rangle^\gamma, \ \gamma > 0, \tag{10.19}$$

that (10.6) holds (for $p = 2$) and that there exist $\zeta, c_\zeta > 0$ such that

$$\mathscr{K}[\mathscr{M}^\zeta] \ge c_\zeta \langle v \rangle^\gamma \mathscr{M}^\zeta, \quad \mathscr{M} := e^{-|v|^2/2}. \tag{10.20}$$

Observe that condition (10.20) is satisfied when $\mathscr K$ is the positive part of the mass conservative operator (10.3). For further references, we write $\mathcal L := \mathcal T + \mathscr C$ with

$$\mathcal T := -v \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_v, \quad \mathscr C f := \mathscr K[f] - K f$$

and we define the Hamiltonian

$$\mathcal H := \Phi(x) + \frac12 |v|^2.$$

In the sequel, we will only consider some weight functions $m = \omega(\mathcal H)$ with $\omega(y) = y^\varrho$, $\varrho \ge 0$, or $\omega(y) = e^{\kappa y}$, $\kappa \in (0,1)$, so that $\omega(\mathcal H) \sim \omega(|v|^2)\omega(\Phi)$. For $p \in [1,\infty)$, we further assume that $v \mapsto \omega^{-1}(|v|^2) \in L^{p'}$ (which imposes $\varrho > d/(2p')$ for a polynomial weight).

Theorem 10.4. *For the kinetic equation* (10.1) *in the whole space with confinement force and under conditions* (10.5)-(10.6)-(10.7)-(10.8)-(10.19)-(10.20) *for some weight function* $m = \omega(\mathcal H)$ *as discuted above, the conclusion* **(C3)** *about existence, uniqueness and positivity of the eigentriplet solution* (λ_1, f_1, ϕ_1) *holds as well as the ergodicity* **(E2)** *for the weak convergence in* $L^1_{\phi_1}$.

We are not aware of any result on the first eigentriplet problem for such linear Boltzmann like equation in the whole space. We may however compare our result to [200] where the corresponding mass conservative framework is considered. We present the proof of Theorem 10.4 in that situation by adapating the arguments presented in the previous section.

Condition (H1). Let us consider a weight function $m = \omega(\mathcal{H})$ as intruced before and let us fix $p \in [1, \infty)$. For a solution to the evolution equation (10.1), we classically compute

$$\begin{aligned} \frac{d}{dt} \int \frac{f^p}{p} m^p dv dx &= \int (\mathcal{L} f) f^{p-1} m^p dv dx \\ &= \int (f^p/p) \mathcal{T}^* m^p + \int (\mathscr{K} f) f^{p-1} m^p - \int f^p K m^p \\ &\le \frac{1}{p} \int (\mathscr{K} f)^p m^p + \int f^p \big(\frac{1}{p'} - K \big) m^p, \end{aligned}$$

by using an integration by parts and the Young inequality. For the first term, we have

$$\begin{aligned} \int (\mathscr{K} f)^p m^p dv dx &\le c_\omega \int \omega(\Phi) \Big(\int f dv \Big)^p dx \\ &= c_\omega \int \omega(\Phi) \int f^p \omega(|v|^2) dv dx \|\omega^{-1}(|v|^2)\|^p_{L^{p'}} \\ &\lesssim \int f^p m^p dv dx. \end{aligned}$$

All together and thanks to the Gronwall lemma, we have established an apriori estimate on the evolution of the norm $\|f\|_{L^p_m}$ and we deduce as in section 8 that $\mathcal{L}$ generates a positive semigroup on L^p_m. In particular, the condition **(H1)** is satisfied thanks to Lemma 2.2.

Condition (H2). We define $f_0 := e^{-\zeta \mathcal{H}}$ and we compute

$$\begin{aligned} \mathcal{L} f_0 &= \mathscr{C} f_0 = r e^{-\zeta \Phi} \mathscr{K}[e^{-\mathscr{M}^\zeta}] - K e^{-\zeta \Phi} \mathscr{M}^\zeta \\ &\ge (r c_\zeta - K_0) \langle v \rangle^\gamma e^{-\zeta \mathcal{H}} \ge 0, \end{aligned}$$

for $r > 0$ large enough. That implies that $\mathcal{I}$ is lower bounded by $\kappa_0 = 0$ by using Lemma 2.4-**(ii)**, and we have thus established that $\mathcal{L}$ satisfies **(H2)**.

Condition (H3). We introduce the collisionless operator

$$\mathcal{B} f := \mathcal{T} f - K f$$

and we define

$$\mathcal{B}^\sharp \phi := \frac{1}{2} \mathcal{T}^* \phi - K \phi.$$

Our analysis is mainly a consequence of the following moment estimate.

Lemma 10.5. *There exist some weight functions $w \lesssim \mathcal{H}$ and some real numbers $\alpha, c_\alpha, C_\alpha > 0$ such that*

$$\mathcal{B}^\sharp w \le C_\alpha w - c_\alpha \mathcal{H}^{1+\alpha}. \tag{10.21}$$

Proof of Lemma 10.5. We split the proof into two steps.

Step 1. We first assume $\gamma \le \beta - 2$ and we define

$$w := 1 + \frac{1}{2} [x]^{1+\gamma/2} \cdot v + \mathcal{H},$$

with $[x]^\delta := x |x|^{\delta - 1}$. We observe that

$$\big| [x]^{1+\gamma/2} \cdot v \big| \le \frac{1}{2} |x|^{2+\gamma} + \frac{1}{2} |v|^2 \le \frac{1}{2} \mathcal{H} + \frac{1}{2},$$

so that $w \sim \langle \mathcal{H} \rangle$. We now compute

$$\mathcal{T}^* w \le \frac{\gamma + 2}{4} |x|^{\gamma/2} |v|^2 - \frac{\beta}{2} |x|^{\beta + \gamma/2}$$

and thus

$$\mathcal{B}^\sharp w \le C_1 |x|^{\gamma/2} |v|^2 - \frac{1}{2} |x|^{\beta + \gamma/2} - \frac{1}{2} |v|^{2+\gamma}.$$

Using that

$$C_1|x|^{\gamma/2} \le C_1^2 + \frac{1}{4}|v|^\gamma,$$

if $|x| \le |v|$ and

$$C_1|v|^2 \le (4C_1)^{\beta/(\beta-2)} + \frac{1}{4}|x|^\beta,$$

if $|v| \le |x|$, we obtain

$$\mathcal{B}^\sharp w \le \big(C_1^2 + (4C_1)^{\beta/(\beta-2)}\big)\mathcal{H} - \frac{1}{4}|x|^{\beta+\gamma/2} - \frac{1}{4}|v|^{2+\gamma},$$

from what we conclude with $\alpha := \gamma/(2\beta)$.

Step 2. We now assume $\beta < \gamma + 2$ and we define

$$w := 1 + \frac{1}{2}[x]^{\beta/2} \cdot v + \mathcal{H},$$

so that again $w \sim \langle \mathcal{H} \rangle$. We easily compute

$$\mathcal{B}^\sharp w \le C_0|x|^{\beta/2-1}|v|^2 - \frac{1}{2}|x|^{\frac{3}{2}\beta-1} - \frac{1}{2}|v|^{\gamma+2}.$$

Using Young's inequality similarly as in the step 1, we get that

$$\mathcal{B}^\sharp w \le C\mathcal{H} - c|x|^{\frac{3}{2}\beta-1} - c|v|^{2+\gamma},$$

which in turn implies (10.21) with $\alpha := \min(\gamma/2, 1/2 - 1/\beta)$. □

We classically deduce the following resolvent estimate.

Lemma 10.6. *For any weight function $m_0 := \omega(\mathcal{H})$, there exists a weight function $m_1 := \omega_1(\mathcal{H})$ such that $m_1/m_0 \to \infty$ as $\mathcal{H} \to \infty$ and for any $\kappa > \kappa_\mathcal{B} > -K_0$ there holds*

$$\mathcal{R}_\mathcal{B} : L^2(m_0) \to L^2(m_1). \tag{10.22}$$

Proof of Lemma 10.6. We split the proof into two steps.

Step 1. We fix $\kappa_\mathcal{B} \in (-K_0, \kappa_0)$ and $m_0 := \omega_0(\mathcal{H})$ with ω_0 a function as defined above. We observe that

$$\mathcal{B}^\sharp m_0 = -K m_0 \le -m_0,$$

so that

$$\int (\kappa - \mathcal{B}) f (f m_0) = \int f^2 (\kappa - \mathcal{B}^\sharp) m_0 \le 0,$$

which means that $\kappa - \mathcal{B}$ is a dissipative operator in $L^2_{m_0}$. We deduce that $\mathcal{R}_\mathcal{B}(\kappa) : L^2_{m_0} \to L^2_{m_0}$ for any $\kappa > -1$.

Step 2. We take

$$m := \omega(\mathcal{H}) w, \quad \omega(\mathcal{H}) := \omega_0(\mathcal{H})/\mathcal{H},$$

where w is defined as in Step 1 of Lemma 10.5 when $\gamma \le \beta - 2$ and as in Step 2 of Lemma 10.5 when $\gamma > \beta - 2$. In any cases $m \lesssim m_0$. On the other hand, Lemma 10.5 and $\mathcal{T}^* \omega(\mathcal{H}) = 0$ imply together that

$$\mathcal{B}^\sharp m \le C_\alpha m - c_\alpha \omega(\mathcal{H}) \mathcal{H}^{1+\alpha}, \quad \alpha > 0.$$

This apriori estimate implies $\mathcal{R}_\mathcal{B}(C_\alpha) : L^2_m \to L^2_{m_1}$, with $m_1 := m_0 \mathcal{H}^\alpha$. For $g \in L^2_{m_0}$ and $\kappa > \kappa_\mathcal{B} := -K_0$, the function $f := \mathcal{R}_\mathcal{B}(\kappa) g \in L^2_{m_0} \subset L^2(m)$ also satisfies

$$(C_\alpha - \mathcal{B}) f = g + (C_\alpha - \kappa) f.$$

We deduce $\|f\|_{L^2(m_1)} \lesssim \|g\|_{L^2(m)} + \|f\|_{L^2(m)} \lesssim \|f\|_{L^2(m_0)}$, which is nothing but (10.22). □

We argue as during the proof of **(H3)** in Section 10.1. By a localization argument and the averaging lemma, we have $\mathcal{A}\mathcal{R}_\mathcal{B}(\kappa) : L^2_{m_0} \to L^2(B_R \times B_R)$ with compact injection for any $R > 0$. Togeher with Lemma 10.6, we deduce that $\mathcal{R}_\mathcal{B}(\kappa) \in \mathcal{K}(L^2_{m_0})$ for any $\kappa > \kappa_\mathcal{B}$, and we conclude exactly as in Section 10.1.

Condition (H4) and (H5′). We recall that it has been proved in [79, Lem. 4.5] that the semigroup S_t associated to the operator $\mathcal{L}$ satisfies the Harris condition: for any $T > 0$ and $\varrho > 0$, there exists $\alpha > 0$ such that

$$S_T f \geq \alpha \mathbf{1}_{B_\varrho} \int_{B_{\vartheta\varrho}} f dx dv, \quad \forall f \geq 0, \tag{10.23}$$

for some constant $\vartheta \in (0, 1)$ and where $B_r := \{(x, v) \in \mathbb{R}^{2d}; |x| < r, |v| < r\}$. Although the statement of [79, Lem. 4.5] is not written in that way, one may easily track the constants appearing in Lemmas 3.5, 3.6, 3.7 and 4.1 in [79] and one obtain (10.23) with $\vartheta := 1/2$. Now, (10.23) immediately implies **(H5′)** which in turn implies **(H4)** thanks to Lemma 4.8-(2)-(3).

Because $\mathcal{L}$ is the generator of a semigroup it also satisfies the weak maximum principle and Kato's inequalities **(H1′)**. We are then in position to apply Theorem 2.21, Theorem 4.13, Theorem 5.18 and Theorem 5.23-(3) and thus complete the proof of Theorem 10.4.

11. The kinetic Fokker-Planck equation

In this part, we consider the kinetic Fokker-Planck evolution equation associated to the operator

$$\mathcal{L} f := -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + cf, \tag{11.1}$$

on functions $f : \mathcal{O} \to \mathbb{R}$, where $\mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ is a domain, $b : \mathcal{O} \to \mathbb{R}^d$ is a given vector field and $c : \mathcal{O} \to \mathbb{R}$ is a given function. In contrast with the previous part, collisions are typically modelized by a Fokker-Planck operator $\Delta_v f + \mathrm{div}_v(vf)$ (when $b = v$ and $c = d$) which takes into account a thermal bath of (Gaussian) whitenoise instead of the integral collisional operator $\mathscr{K}[f] - Kf$ in the linear Boltzmann equation (10.1).

We will consider the case when Ω is a bounded domain and the equation is complemented with a boundary condition. More precisely, we assume the classical balance between the values of the trace γf of f on the outgoing and incoming velocities subsets of the boundary

$$(\gamma_- f)(x, v) = \mathscr{R}_x(\gamma_+ f(x, .))(v) \;\text{ on }\; \Sigma_-, \tag{11.2}$$

where in this context we define $\Sigma^x_\pm := \{v \in \mathbb{R}^3; \pm v \cdot \nu_x > 0\}$ the sets of outgoing (Σ^x_+) and incoming (Σ^x_-) velocities at point $x \in \partial\Omega$, next the sets

$$\Sigma_\pm = \{(x, v) \in \Sigma; \pm\nu_x \cdot v > 0\} = \{(x, v);\, x \in \partial\Omega,\, v \in \Sigma^x_\pm\},$$

and finally the outgoing and incoming trace functions $\gamma_\pm f := \mathbf{1}_{\Sigma_\pm} \gamma f$. Here and in the sequel, ν_x denotes the unit normal outward vector field defined on the boundary set $\partial\Omega$. We similarly define the grazing velocity set

$$\Sigma_0 = \{(x, v) \in \Sigma; \nu_x \cdot v = 0\}.$$

The reflection operator $\mathscr{R}_x$ is local in position, but can be local or nonlocal in the velocity variable, so that it writes

$$(\mathscr{R}_x g)(v) := \int_{\Sigma^x_+} r(x, v, v_*) g(v_*) v_* \cdot \nu_x \, dv_*,$$

for a reflection kernel $r : \partial\Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Some classical general assumptions on r are

$$r \geq 0, \quad \mathscr{R}^*_x 1 = 1, \quad \mathscr{R}_x \mathscr{M} = \mathscr{M}, \tag{11.3}$$

for some positive function $\mathscr{M} = \mathscr{M}(v)$, see for instance [95, 97, 98]. The second (normalisation) condition corresponds to the fact that all the particles reaching the outgoing boundary are put back on the incoming boundary (no mass is lost) while the third (reciprocity) condition means (when $\mathscr{M}$ is a Gaussian function) that the wall is in a local equilibrium state and is not influenced by the incoming particles. The normalization condition implies the local mass conservation

$$\int_{\Sigma^x_-} \mathscr{R}_x g |\nu \cdot v| dv = \int_{\Sigma^x_+} g \nu \cdot v dv, \tag{11.4}$$

while the three assumptions (11.3) on r together also imply

$$\begin{aligned}\int_{\Sigma^x_-} \big(\mathscr{R}_x g\big)^2 \mathscr{M}^{-1} |\nu\cdot v| dv &\le \int_{\Sigma^x_-} \big(\mathscr{R}_x (g^2/\mathscr{M})\big)\big(\mathscr{R}_x \mathscr{M}\big) \mathscr{M}^{-1} |\nu\cdot v| dv \\ &= \int_{\Sigma^x_+} g_*^2 \mathscr{M}_*^{-1} \, (\mathscr{R}^* 1) \nu\cdot v_* dv_*,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality (and the fact that $r \ge 0$) in the first line and the reciprocity condition in the second line. As a consequence, we have

$$\int_{\Sigma^x_-} \big(\mathscr{R}_x g\big)^2 \mathscr{M}^{-1} |\nu\cdot v| dv \le \int_{\Sigma^x_+} g^2 \mathscr{M}^{-1} \, \nu\cdot v dv, \tag{11.5}$$

where we have used the normalization condition in that last step. In the sequel, we will rather consider the possibly position dependent Maxwell boundary condition operator

$$\mathscr{R}_x g = \alpha(x) \mathcal{D}_x g + \beta(x) \Gamma_x g, \tag{11.6}$$

where the accommodation coefficients $\alpha, \beta : \partial\Omega \to [0,1]$ satisfy $\alpha(x) + \beta(x) =: \zeta(x) \le 1$, Γ_x is the specular reflection operator

$$\Gamma_x (g(x,\cdot))(v) = g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v = v - 2\nu(x)(\nu(x)\cdot v), \tag{11.7}$$

and $\mathcal{D}_x$ is the diffusive operator

$$\mathcal{D}_x (g(x,\cdot))(v) = c_{\mathscr{M}} \mathscr{M}(v) \widetilde{g}(x), \quad \widetilde{g}(x) = \int_{\Sigma^x_+} g(x,w)\, \nu(x)\cdot w \, dw. \tag{11.8}$$

Here the constant $c_{\mathscr{M}} := (2\pi)^{1/2}$ is such that $c_{\mathscr{M}} \widetilde{\mathscr{M}} = 1$ and $\mathscr{M}$ stands for the standard Maxwellian

$$\mathscr{M}(v) := (2\pi)^{-d/2} \exp(-|v|^2/2), \tag{11.9}$$

or, more generally, $\mathscr{M} = \mathscr{M}(|v|) \ge 0$ is such that

$$\mathscr{D}_x^* 1 = 1, \quad \mathscr{D}_x \mathscr{M} = \mathscr{M}, \quad \langle v\rangle^\vartheta \mathscr{M} \in L^1(\mathbb{R}^d), \tag{11.10}$$

with $\vartheta \ge 1$ (that last condition is necessary in order that the second relation above makes sense). The boundary condition (11.6) corresponds to the *pure specular reflection* boundary condition when $\beta \equiv 1$ and it corresponds to the *pure diffusive* boundary condition when $\alpha \equiv 1$. When $\zeta \equiv 1$, the Maxwell boundary condition operator (11.6) satisfies (11.3). On the contrary, when $\zeta \not\equiv 1$, the L^2 estimate (11.5) holds but not anymore the mass conservation (11.4). However, the following L^1 estimate

$$\int_{\Sigma^x_-} |\mathscr{R}_x g| |\nu\cdot v| dv \le \zeta^* \int_{\Sigma^x_+} |g| \, \nu\cdot v dv \tag{11.11}$$

holds, with $0 \le \sup \zeta \le \zeta^* \le 1$. Finally, the case $\zeta \equiv 0$ corresonds to the zero inflow problem.

Let us finally mention that similarly as in Part 8, the regularity needed on the domain Ω may be formulated in the following way: we assume that Ω is locally on one side of $\partial\Omega$ and there exists a function $\delta = \delta_\Omega \in W^{2,\infty}(\mathbb{R}^d)$ such that for all x in an interior neighborhood of $\partial\Omega$ one has $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and the vector field ν defined on $\mathbb{R}^d$ by $x \mapsto \nu(x) = \nu_x := -\nabla_x \delta(x)$ coincides with the previously defined unit outward normal vector field on $\partial\Omega$ and satisfies $\|\nu\|_{L^\infty} = 1$. We also assume that the Lebesgue measure on $\partial\Omega$ is well defined and it is denoted by $d\sigma_x$.

11.1. **The trace problem.**

We consider in this section the trace problem for a solution $g = g(x,v)$ to the stationary Vlasov-Fokker-Planck equation

$$\mathcal{M} g := v\cdot\nabla_x g - b\cdot\nabla_v g - \Delta_v g = G \ \text{ in } \mathcal{O}, \tag{11.12}$$

for a given a vector field $b = b(x,v)$, a source term $G = G(x,v)$ and for a solution $g = g(t,x,v)$ to the evolution Vlasov-Fokker-Planck equation

$$\partial_t g + v\cdot\nabla_x g - b\cdot\nabla_v g - \Delta_v g = G \ \text{ in } (0,T)\times\mathcal{O}, \tag{11.13}$$

for a given a vector field $b = b(t,x,v)$, a source term $G = G(t,x,v)$. The second problem has been considered first in [91] and next in [272, Sec. 4], where a strong (renormalized) trace function is proved to exist. In the sequel, we recall these results and slightly extending them by considering a possible L^2H^{-1} source term. We introduce some notations. We denote

$$d\xi := |\nu(x)\cdot v| dv d\sigma_x \quad \text{and} \quad d\xi^2 := (\nu(x)\cdot \hat v)^2 dv d\sigma_x$$

the measures on the boundary set Σ. We denote by $\mathcal{B}_1$ the class of renormalized functions $\beta \in W^{2,\infty}_{\rm loc}(\mathbb{R})$ such that β'' has a compact support, by $\mathcal{B}_2$ the class of functions $\beta \in W^{2,\infty}_{\rm loc}(\mathbb{R})$ such that $\beta'' \in L^\infty(\mathbb{R})$ and by $\mathcal{D}_0(\bar{\mathcal{O}})$ the space of test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$ such that $\varphi = 0$ on Σ_0. We finally define the dual operator

$$\mathcal{M}^*\varphi := -v\cdot\nabla_x\varphi + \operatorname{div}_v(b\varphi) - \Delta_v\varphi.$$

Theorem 11.1. *We consider $g, b \in L^2_{{\rm loc},x}H^1_{{\rm loc},v}$, $G \in L^2_{{\rm loc},x}H^{-1}_{{\rm loc},v}$ and we assume that g is a solution to the stationary Vlasov-Fokker-Planck equation* (11.12). *Then there exists $\gamma g \in L^2_{\rm loc}(\Sigma, d\xi^2)$ such that the following Green renormalized formula*

$$\iint_{\mathcal{O}} \big(\beta(g)\,\mathcal{M}^*\varphi - \beta''(g)\,|\nabla_v g|^2\varphi\big)\,dvdx + \langle G, \beta'(g)\varphi\rangle = \tag{11.14}$$
$$= \iint_{\Sigma} \beta(\gamma\, g)\,\varphi\;\nu(x)\cdot v\;dvd\sigma_x$$

holds for any renormalized function $\beta \in \mathcal{B}_1$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$, as well as for any renormalized function $\beta \in \mathcal{B}_2$ and any test functions $\varphi \in \mathcal{D}_0(\bar{\mathcal{O}})$. It is worth emphasizing that $\beta'(g)\varphi \in L^2_xH^1_v$ so that the duality product $\langle G, \beta'(g)\varphi\rangle$ is well defined.
If furthermore $\gamma_\mp g \in L^2_{\rm loc}(\Sigma; d\xi)$ then $\gamma_\pm g \in L^2_{\rm loc}(\Sigma; d\xi)$ and (11.14) *holds for any renormalized function $\beta \in \mathcal{B}_2$ and any test functions $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$.*

Proof of Theorem 11.1. We only allude the proof which uses very similar arguments as those presented in Section 10 and that can also be partially found in [136, 272]. In the one hand, considering the mollifier $(\rho_\varepsilon)_{\varepsilon>0}$ defined in (8.19) with $z := (x, y)$, we get that g_ε is smooth and satisfies

$$g_\varepsilon \to g \;\text{ in }\; L^2_{{\rm loc},x}H^1_{{\rm loc},v}, \quad \mathcal{M}g_\varepsilon = G_\varepsilon \to G \;\text{ in }\; L^2_{{\rm loc},x}H^{-1}_{{\rm loc},v},$$

which is nothing but a variant of [139, Lem. II.1]. The function g_ε being smooth, for any $\beta \in C^2$ such that $\beta' \in C^1_b$, we may differentiate $\beta(g_\varepsilon)$ and we get

$$\mathcal{M}\beta(g_\varepsilon) + \beta''(g_\varepsilon)|\nabla_v g_\varepsilon|^2 = \beta'(g_\varepsilon)G_\varepsilon \;\text{ in }\; \mathcal{O},$$

with. We may thus pass to the limit as $\varepsilon \to 0$ and we obtain (11.14). □

We also write without proof (since this one is similar to the proof of Proposition 8.10) a stability result that we will use several times in the sequel.

Proposition 11.2. *Let us consider three sequences (g_k), (b_k) of $L^2_{{\rm loc},x}H^1_{{\rm loc},v}$ and (G_k) of $L^2_{{\rm loc},x}H^{-1}_{{\rm loc},v}$ such that*

$$v\cdot\nabla_x g_k - b_k\cdot\nabla_v g_k - \Delta_v g_k = G_k \quad \text{in} \quad \mathcal{D}'(\mathcal{O})$$

for any $k \ge 1$ and three functions $g, b \in L^2_{{\rm loc},x}H^1_{{\rm loc},v}$ and $G \in L^2_{{\rm loc},x}H^{-1}_{{\rm loc},v}$ such that $g_k \to g$ strongly in $L^2_{{\rm loc},x}H^1_{{\rm loc},v}$, $b_k \rightharpoonup b$ weakly in $L^2_{\rm loc}(\bar{\mathcal{O}})$ and $G_k \to G$ strongly in $L^2_{{\rm loc},x}H^{-1}_{{\rm loc},v}$. Then g satisfies (11.12) *and, up to the extaction of a subsequence, $\gamma g_k \to \gamma g$ a.e. on $\Sigma\backslash\Sigma_0$.*
(2) If $g_k \rightharpoonup g$ weakly in $L^1_{\rm loc}(\bar{\mathcal{O}})$ then g satisfies (11.14) *and, up to the extraction of a subsequence, $\gamma g_k \xrightharpoonup{r} \gamma g$ on $\Sigma\backslash\Sigma_0$ (we recall that the renormalized convergence has been defined in* (8.51)*).*

11.2. **Well-posedness problem with inflow term at the boundary.** We consider the kinetic Fokker-Planck operator $\mathcal{L}$ defined in (11.1) and we start revisiting the well posedness problem

$$(\lambda - \mathcal{L})f = \mathfrak{F} \quad \text{in} \quad \mathcal{O}, \quad \gamma_- f = \mathfrak{g} \quad \text{on} \quad \Sigma_-, \tag{11.15}$$

for given data $\mathfrak{F} : \mathcal{O} \to \mathbb{R}$ and $\mathfrak{g} : \Sigma_- \to \mathbb{R}$.

For a given weight function $m : \mathbb{R}^d \to [1, \infty)$, we define the measure $d\xi_m := m^2|\nu(x)\cdot v|\,dvd\sigma_x$ on the boundary Σ. We next define $L^2H^1_m = L^2H^1_m(\mathcal{O})$ the space associated to the Hilbert norm defined by

$$\|f\|^2_{L^2H^1_m} := \|f\|^2_{L^2_m} + \|\nabla_v f\|^2_{L^2_m},$$

and we assume that m satisfies the Poincaré type inequality

$$\|f\frac{\nabla m}{m}\|_{L^2_m} \lesssim \|f\|_{L^2 H^1_m}, \quad \forall f \in L^2 H^1_m. \tag{11.16}$$

Such a Poincaré inequality is classically known to be true when $m := \mathscr{M}^{-\vartheta}$, $\mathscr{M}$ is the Maxwellian (11.9) and $\vartheta > 0$. We also define

$$L^2 H^{-1}_m := \{\mathfrak{F} = g + \mathrm{div}_v G;\ g, G_i \in L^2_m(\mathcal{O})\},$$

so that when $m = 1$ the space $L^2 H^{-1}_m$ is nothing but the space of continuous and linear mappings on $L^2 H^1$. For $\mathfrak{F} \in L^2 H^{-1}_m$ and $f \in L^2 H^1_m$, we may thus write

$$\langle \mathfrak{F}, f m^2 \rangle \le \|\mathfrak{F}\|_{L^2 H^{-1}_m} \|f\|_{L^2 H^1_m}.$$

We finally define in this context

$$W_2 := \{f \in L^2 H^1_m;\ \hat v \cdot \nabla_x f \in L^2 H^{-1}_m\},$$

and

$$W_{2,\Sigma} := \{g \in W_2;\ \gamma g \in L^2(\Sigma; d\xi_m)\},$$

with $W_{2,\Sigma} \neq W_2$ in general.

Theorem 11.3. *Let us fix a vector field $b \in H^1_{\mathrm{loc}}(\bar{\mathcal{O}})$, a function $c \in L^\infty(\mathcal{O})$, a weight function $m : \mathbb{R}^d \to [1, \infty)$ and let us assume that $b/\langle v\rangle \in L^\infty(\mathcal{O})$, that* (11.16) *holds and that*

$$\lambda^* := \text{ess sup}\, \varpi < \infty, \quad \varpi := c + \frac{\Delta m^2}{2m^2} - \frac{1}{2}\mathrm{div} b - b \cdot \frac{\nabla m}{m}. \tag{11.17}$$

For any $\mathfrak{F} \in L^2 H^{-1}_m$, $\mathfrak{g} \in L^2(\Sigma_-; d\xi_m)$ and $\lambda > \lambda^*$, there exists a unique solution $f \in W_{2,\Sigma}$ to the Dirichlet problem (11.15)*. We have furthermore $f \ge 0$ if $\mathfrak{F} \ge 0$ and $\mathfrak{g} \ge 0$.*

A similar result is proved in [129, Appendix A] in the case $\Omega = \mathbb{R}^d$. Also observe that (11.17) holds with $m := \mathscr{M}^{-1/2}$ when $\mathscr{M}$ is the standard Maxwellian (11.9) and $b(v) = \vartheta v$, with $\vartheta > 1/2$.

Proof of Theorem 11.3. We split the proof into five steps.

Step 1. A priori estimates. We argue similarly as in [90, 89]. Multiplying the first equation in (11.15) by fm^2, performing several integrations by part in the velocity variable and using the Green formula, we have

$$\int_{\mathcal{O}} (\lambda - \varpi) f^2 m^2 + \frac{1}{2}\int_\Sigma (\gamma f)^2 m^2 \nu \cdot v + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 = \langle \mathfrak{F}, f m^2\rangle.$$

Fixing $\lambda > \lambda^*$, using the Young inequality

$$\|\mathfrak{F}\|_{L^2 H^{-1}_m} \|f\|_{L^2 H^1_m} \le \Big(\frac{1}{2(\lambda - \lambda^*)} + \frac{1}{2}\Big)\|\mathfrak{F}\|^2_{L^2 H^{-1}_m} + \frac{\lambda - \lambda^*}{2}\|f\|^2_{L^2_m} + \frac{1}{2}\|\nabla_v f\|^2_{L^2_m}$$

and the boundary condition on the incoming set Σ_- in (11.15), we deduce

$$(\lambda - \lambda^*)\int_{\mathcal{O}} f^2 m^2 + \int_{\Sigma_+} (\gamma_+ f)^2 d\xi_m + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 \le \frac{1 + \lambda - \lambda^*}{\lambda - \lambda^*}\|\mathfrak{F}\|^2_{L^2 H^{-1}_m} + \int_{\Sigma_-} \mathfrak{g}^2 d\xi_m. \tag{11.18}$$

• Because of the first equation in (11.15) and the above estimate, we find

$$\hat v \cdot \nabla_x f = \frac{1}{\langle v\rangle}\big(\mathfrak{F} - \lambda f + \Delta_v f + b \cdot \nabla_v f + cf\big) \in L^2 H^{-1}_m, \tag{11.19}$$

so that $f \in W_2$.

• Multiplying the first equation in (11.15) by $f\psi$, $\psi := \nu(x) \cdot \tilde v m^2$ where here and below we use the notations $\hat v := v/\langle v\rangle$, $\tilde v := v/\langle v\rangle^2$, $\langle v\rangle^2 := 1 + |v|^2$, and using the Green formula and one integration by part in the velocity variable, we get

$$\begin{aligned}\frac{1}{2}\int_\Sigma (\gamma f)^2 (\nu \cdot \hat v)^2 m^2 \quad &= \quad \frac{1}{2}\int_{\mathcal{O}} f^2 \hat v \cdot D_x \nu_x \hat v\, m^2 - \int_{\mathcal{O}} |\nabla_v f|^2 \psi \\ &\quad + \int_{\mathcal{O}} f \nabla_v f (b\psi - \nabla_v \psi) + \int_{\mathcal{O}} f^2 \psi (c - \lambda) + \langle \mathfrak{F}, f\psi\rangle.\end{aligned}$$

Observing that

$$|\langle \mathfrak{F}, f\psi\rangle| \le \|\mathfrak{F}\|_{L^2 H^{-1}_m} \|f \nu(x) \cdot \tilde v\|_{L^2 H^1_m} \lesssim \|\mathfrak{F}\|_{L^2 H^{-1}_m} \|f\|_{L^2 H^1_m}$$

and

$$\|f\nabla_v\psi\|_{L^2} \lesssim \|f\|_{L^2H^1_m},$$

recalling that $b/\langle v\rangle \in L^\infty(\mathcal{O})$ and using the Cauchy-Schwarz inequality, we deduce

$$\|\gamma f\|^2_{L^2(\Sigma;d\xi^2_m)} \le C(1+|\lambda|)\|f\|^2_{L^2H^1_m} + C\|\mathfrak{F}\|_{L^2H^{-1}_m}\|f\|_{L^2H^1_m}, \tag{11.20}$$

for some constant $C = C(b,c,m,\nu)$, with $d\xi^2_m := (\nu\cdot\hat v)^2 m^2 dv d\sigma_x$.

• For latter reference, we establish an estimate about the behaviour of the solution near the boundary. We now introduce the following Lions-Perthame [256] type weight function

$$\psi := 2\delta(x)^{1/2}\nu(x)\cdot\tilde v, \tag{11.21}$$

and we observe that $\psi = 0$ on Σ, $\langle v\rangle\psi \in L^\infty(\mathcal{O})$, $\nabla_v\psi \in L^\infty(\mathcal{O})$ and

$$v\cdot\nabla_x\psi = \frac{1}{\delta(x)^{1/2}}(\hat v\cdot\nu(x))^2 + 2\delta(x)^{1/2}\hat v\cdot D_x\nu(x)\hat v.$$

Multiplying the first equation in (11.15) by $f\psi$, we have

$$\frac12\int_{\mathcal{O}} v\cdot\nabla_x f^2\psi - \int_{\mathcal{O}} f\,\frac{b}{\langle v\rangle}\cdot\nabla_v f\langle v\rangle\psi + \int_{\mathcal{O}}\nabla_v(f\psi)\cdot\nabla_v f + \int_{\mathcal{O}}(\lambda-c)f^2\psi = \langle F, f\psi\rangle.$$

Using Cauchy-Schwarz and Young inequalities, we deduce

$$\int_{\mathcal{O}} f^2\frac{(\hat v\cdot\nu(x))^2}{\delta(x)^{1/2}}dvdx \le C(1+|\lambda|)(\|f\|^2_{L^2H^1} + \|F\|^2_{L^2H^{-1}}), \tag{11.22}$$

for some constant $C = C(b,c,n)$.

• We finally state a somehow classical regularity estimate when $\mathcal{F} \in L^2_m(\mathcal{O})$. Taking advantage of the fact that $\mathcal{F} \in L^2_m$ and $f \in L^2H^1_m$ and localizing the problem by introducing the function $g := f\chi_\varepsilon \in L^2_xH^1_v(\mathbb{R}^d\times\mathbb{R}^d)$, $\chi_\varepsilon \in C^2_c(\mathcal{O})$, $\mathbf{1}_{\mathcal{O}_\varepsilon} \le \chi_\varepsilon \le 1$, $\mathcal{O}_\varepsilon := \{(x,v)\in\mathcal{O};\ \delta(x) > \varepsilon,\ |v| \le 1/\varepsilon\}$, we have

$$v\cdot\nabla_x g - \Delta_v g + \langle v\rangle^2 g = \mathcal{G} \text{ in } \mathcal{D}'(\mathbb{R}^d\times\mathbb{R}^d),$$

with

$$\mathcal{G} := (\mathcal{F} - \lambda f - cf - b\cdot\nabla_v f)\chi_\varepsilon - 2\nabla_v f\cdot\nabla_v\chi_\varepsilon + \langle v\rangle^2 f\chi_\varepsilon \in L^2(\mathbb{R}^d\times\mathbb{R}^d).$$

From the quantitative Hormander's hypoellipticity estimate of Hérau & Pravda-Starov [210, Prop. 2.1], we then have

$$\|D^{2/3}_x g\|_{L^2} + \|D^2_v g\|_{L^2} \lesssim \|\mathcal{G}\|_{L^2} + \|g\|_{L^2}.$$

Coming back to the function f and using the previous estimates, we deduce

$$\|D^{2/3}_x f\|_{L^2(\mathcal{O}_\varepsilon)} + \|D^2_v f\|_{L^2(\mathcal{O}_\varepsilon)} \le C(\|\mathcal{L}f\|_{L^2(\mathcal{O})} + \|f\|_{L^2(\mathcal{O})}), \tag{11.23}$$

for a constant $C = C(\lambda,\varepsilon) > 0$.

Step 2. Existence. We assume $\mathfrak{g} = 0$. A possible way for proving the existence is to use Lions' variant of the Lax-Milgram theorem [250, Chap III, §1] as in [36, 129] and as we proceed now. Defining the bilinear form $\mathcal{E} : L^2H^1_m(\mathcal{O}) \times C^1_c(\mathcal{O}\cup\Sigma_-) \to \mathbb{R}$, by

$$\begin{aligned}\mathcal{E}(f,\varphi) &= \int_{\mathcal{O}}(\lambda-\mathcal{L})f\varphi m^2\\ &:= \int_{\mathcal{O}}(\lambda f - b\cdot\nabla_v f - cf)\varphi m^2 + \nabla_v f\cdot\nabla_v(\varphi m^2) - f(v\cdot\nabla_x\varphi)m^2,\end{aligned}$$

for any $f \in L^2H^1_m(\mathcal{O})$ and $\varphi \in C^1_c(\mathcal{O}\cup\Sigma_-)$, we observe that this one is coercive, namely

$$\begin{aligned}\mathcal{E}(\varphi,\varphi) &= \int_{\mathcal{O}}(\lambda-\varpi)\varphi^2m^2 + \int_{\mathcal{O}}|\nabla_v\varphi|^2m^2 + \frac12\int_{\Sigma_-}(\gamma_-\varphi)^2d\xi_m\\ &\ge \kappa\|\varphi\|^2_{L^2H^1_m},\end{aligned}$$

for any $\varphi \in C^1_c(\mathcal{O}\cup\Sigma_-)$, with $\kappa := \min(\lambda-\lambda^*,1) > 0$. From the above mentioned Lions's theorem, for any $\mathfrak{F} \in L^2H^{-1}_m$, there exists $f \in L^2H^1_m$ such that

$$\mathcal{E}(f,\varphi) = \langle\mathfrak{F},\varphi m^2\rangle, \quad \forall\,\varphi \in C^1_c(\mathcal{O}\cup\Sigma_-). \tag{11.24}$$

In particular, f satisfies the first equation in (11.15) in the distributional sense $\mathcal{D}'(\mathcal{O})$, and thus from (11.19), we deduce that $f \in W_2$. Thanks to the trace Theorem 11.1 and the estimate (11.20), the function f admits a trace $\gamma f \in L^2(\Sigma; d\xi_m^2)$. Using the Green formula (11.14) with $\beta = \mathrm{id} \in \mathcal{B}_1$, we have

$$\iint_{\mathcal{O}} \big(f(\mathcal{L}^* - \lambda)\varphi + \mathfrak{F}\varphi\big)\, dv dx = \iint_{\Sigma} \gamma f\, \varphi\, \nu(x) \cdot v\, dv d\sigma_x, \tag{11.25}$$

for any $\varphi \in \mathcal{D}(\bar{\mathcal{O}})$. Particularizing to $\varphi \in \mathcal{D}(\mathcal{O} \cup \Sigma_-)$ and comparing with (11.24), we deduce that $\gamma_- f = 0$.

Step 3. Existence. The general case $\mathfrak{g} \in L^2(\Sigma_-; d\xi_m)$. When $\mathfrak{g} \in C^2_c(\Sigma_-)$, there exists a function $\mathfrak{h} \in C^2_c(\mathcal{O} \cup \Sigma_-)$ such that $\mathfrak{h}_{|\Sigma_-} = \mathfrak{g}$ and we consider the source term $G := \mathfrak{F} + (\mathcal{L} - \lambda)\mathfrak{h} \in L^2 H^{-1}_m$ as well as the problem

$$(\lambda - \mathcal{L}) g = G \text{ in } \mathcal{O}, \quad \gamma_- g = 0 \text{ on } \Sigma_-.$$

From Step 2, there exists a solution $g \in W_{2,\Sigma}$ to this problem and we set $f := g + \mathfrak{h}$, in such a way that $f \in W_{2,\Sigma}$ and satisfies

$$\begin{aligned} \int_{\mathcal{O}} f(\lambda - \mathcal{L}^*)\varphi &= \int_{\mathcal{O}} g(\lambda - \mathcal{L}^*)\varphi + \int_{\mathcal{O}} \mathfrak{h}(\lambda - \mathcal{L}^*)\varphi \\ &= \int_{\mathcal{O}} G\varphi + \int_{\mathcal{O}} (\lambda - \mathcal{L})\mathfrak{h}\varphi - \int_{\Sigma} \mathfrak{h}_{|\Sigma}\varphi\, \nu \cdot v, \end{aligned}$$

and thus

$$\int_{\mathcal{O}} f(\lambda - \mathcal{L}^*)\varphi = \int_{\mathcal{O}} \mathfrak{F}\varphi - \int_{\Sigma_-} \mathfrak{g}\varphi\, \nu \cdot v, \tag{11.26}$$

for any $\varphi \in C^2_c(\mathcal{O} \cup \Sigma_-)$. Together with (11.25), we get that $\gamma_- f = \mathfrak{g}$ on Σ_-. In order to deal with the general case $\mathfrak{g} \in L^2(\Sigma_-; d\xi_m)$, we introduce a sequence $(\mathfrak{g}^n)$ of $C^2_c(\Sigma_-)$ such that $\mathfrak{g}^n \to \mathfrak{g}$ in $L^2(\Sigma_-, d\xi_m)$ and we next consider the associated sequence of solutions (f^n) of $W_{2,\Sigma}$ just built above. Using the estimates exhibited in Step 1, we get that (f^n) is a Cauchy sequence in W_2, so that it converges to a limit $f \in W_{2,\Sigma}$. We may pass to the limit in (11.26) written for the sequence (f^n) and deduce that the same equation holds at the limit for f.

Step 4. Uniqueness. Consider two weak solutions $f_i \in W_2$ to the equation (11.15) in the sense that

$$\mathcal{E}(f_i, \varphi) = \langle \mathfrak{F}, \varphi m^2 \rangle, \quad \forall\, \varphi \in C^1_c(\mathcal{O} \cup \Sigma_-).$$

In particular, the difference $f := f_2 - f_1 \in W_2$ satisfies

$$\mathcal{E}(f, \varphi) = 0, \quad \forall\, \varphi \in C^1_c(\mathcal{O} \cup \Sigma_-),$$

and from the above discussion $\gamma_- f = 0 \in L^2(\Sigma_-; d\xi_m)$. Thanks to the trace Theorem 11.1, we deduce that $\gamma f \in L^2_{\mathrm{loc}}(\Sigma; d\xi_m)$ and we may choose $\beta(s) = s^2$ in the Green formula (11.14): we get

$$\int_{\mathcal{O}} f^2 \{ v \cdot \nabla_x \varphi - \mathrm{div}_v(b\varphi) + \Delta_v \varphi + 2f(c - \lambda)\varphi \} - 2|\nabla_v f|^2 \varphi = \int_{\Sigma_+} (\gamma f)^2 \nu \cdot v \varphi,$$

for any test function $\varphi \in C^2_c(\bar{\mathcal{O}})$. Choosing $\varphi = m^2 \chi_\varrho$, with $\chi_\varrho(v) := \chi(v/\varrho)$, $\chi \in C^2_c(\mathbb{R}^d)$, $\mathbf{1}_{B_1} \le \chi \le \mathbf{1}_{B_2}$, we deduce

$$\int_{\mathcal{O}} f^2 m^2 \big\{ (\lambda - \varpi)\chi_\varrho + \frac{1}{2} b \cdot \nabla \chi_\varrho - \frac{\nabla m}{m} \cdot \nabla \chi_\varrho - \Delta \chi_\varrho \big\} \le 0.$$

Because $f \in L^2 H^1_m$, we may pass to the limit $\varrho \to \infty$ thanks to the dominated convergence theorem and we obtain

$$\int_{\mathcal{O}} f^2 m^2 (\lambda - \varpi) \le 0,$$

and thus $f = 0$.

Step 5. Positivity. We assume now that $\mathfrak{F} \ge 0$ and $\mathfrak{g} \ge 0$. We proceed similarly as in the previous step by considering $\beta(s) = s_-^2$, $\varphi = m^2 \chi_M$. Letting $M \to \infty$, we deduce

$$\int_{\mathcal{O}} f_-^2 m^2 (\lambda - \varpi) \le 0,$$

and thus $f_- = 0$. □

Summing up, gathering the above estimates (11.18), (11.19), (11.20), (11.22), (11.23), we see that there exists a constant $C > 0$ such that any function $f \in D(\mathcal{L})$ satisfies

$$\begin{aligned}(11.27)\qquad &\|f\|_{L^2H^1_m} + \|\hat{v}\cdot\nabla_x f\|_{L^2H^{-1}_m} + \|f\frac{\hat{v}\cdot\nu}{\delta^{1/4}}\|_{L^2}\\ &+\|\gamma f\|_{L^2(\Sigma;d\xi^2_m)} + \|\gamma_+ f\|_{L^2(\Sigma;d\xi_m)} \le C(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2})\end{aligned}$$

and for any $\varepsilon > 0$ there exists a constant C_ε such that any function $f \in D(\mathcal{L})$ satisfies

$$\|D^{2/3}_x f\|_{L^2(\mathcal{O}_\varepsilon)} + \|D^2_v f\|_{L^2(\mathcal{O}_\varepsilon)} \le C_\varepsilon(\|f\|_{L^2} + \|\mathcal{L}f\|_{L^2}).$$

11.3. **Well-posedness problem with reflection condition at the boundary.** We consider now the well posedness problem associated to the stationary equation

$$(11.28)\qquad (\lambda-\mathcal{L})f = \mathfrak{F} \quad\text{in}\quad \mathcal{O},\quad \gamma_- f = \mathscr{R}\gamma_+ f \quad\text{on}\quad \Sigma_-,$$

for a given datum $\mathfrak{F} : \mathcal{O} \to \mathbb{R}$, where the kinetic Fokker-Planck operator $\mathcal{L}$ is still defined by (11.1) and the reflexion operator $\mathscr{R}$ is described in (11.6), (11.7), (11.8).

Theorem 11.4. *Let us fix a vector field $b \in H^1_{\rm loc}(\bar{\mathcal{O}})$ and a function $c \in L^\infty(\mathcal{O})$ which satisfy the assumptions of Theorem 11.3 with a given weight function $m : \mathbb{R}^d \to [1,\infty)$ for the pure specular reflection case $\alpha \equiv 0$ and with the weight function $m := \mathscr{M}^{-1/2}$ when $\alpha \not\equiv 0$, where $\mathscr{M}$ is the Gaussian function* (11.9) *or a more general equilibrium satsifying* (11.10)*. In that last case, we furthermore assume one of the two following conditions*

(i) $1-\zeta+\alpha^2/2 \ge \delta_* > 0$, and we observe that $L^2(\Sigma;d\xi_m) \subset L^1(\Sigma;d\xi)$,
(ii) $\langle v\rangle^2\mathscr{M} \in L^1$, and we observe that $L^2(\Sigma;d\xi^2_m) \subset L^1(\Sigma;d\xi)$,

where we recall that we have defined $d\xi_m := m^2|\nu(x)\cdot v|dvd\sigma_x$ and $d\xi^2_m := m^2(\nu(x)\cdot\hat{v})^2dvd\sigma_x$. For any $\mathfrak{F} \in L^2H^{-1}_m$ and $\lambda > \lambda^*$, there exists at least one solution $f \in W_2$ to the Dirichlet problem (11.28)*. Assuming furthermore that*

$$(11.29)\qquad \lambda^{**} := \operatorname{ess\,sup}(c - \operatorname{div} b) < \infty,$$

and $\lambda > \lambda^{**}$, the solution f is unique and $f \ge 0$ if $\mathfrak{F} \ge 0$.

It is worth emphasizing that the assumptions of Theorem 11.4 hold when $b = v$ and $m := \mathscr{M}^{-1/2}$. We also emphasize on the fact that the additional assumptions (i) or (ii) are made in order to prove the uniqueness of the solution during the proof.

Proof of Theorem 11.4. We split the proof into four steps.

Step 1. A priori estimates. We multiply the first equation in (11.28) by fm^2. As in Step 1 of the proof of Theorem 11.3, we get

$$\int_{\mathcal{O}}(\lambda-\varpi)f^2m^2 + \frac12\int_\Sigma(\gamma f)^2m^2\nu\cdot v + \int_{\mathcal{O}}|\nabla_v f|^2m^2 = \langle\mathfrak{F}, fm^2\rangle.$$

Using for instance [55, Lem. 3.1], we have

$$(11.30)\qquad \int_\Sigma(\gamma f)^2m^2\nu\cdot v \ge \int_{\Sigma_+}[(1-\zeta)(\gamma_+f)^2 + \alpha(\mathcal{D}^\perp\gamma_+f)^2]d\xi_m =: \mathcal{E}_{\zeta,\alpha}(\gamma_+f) \ge 0,$$

with $\mathcal{D}^\perp g := g - \mathcal{D}g$. Using that the contribution of the boundary is nonnegative in the first estimate, we first deduce

$$(\lambda-\lambda^*)\|f\|^2_{L^2_m} + \|\nabla f\|^2_{L^2_m} \le \|\mathfrak{F}\|_{L^2H^{-1}_m}\|f\|_{L^2H^1_m},$$

for $\lambda > \lambda^*$, so that

$$\min(\lambda-\lambda^*,1)\|f\|_{L^2H^1_m} \le \|\mathfrak{F}\|_{L^2H^{-1}_m}.$$

From the three above estimates together, for $\lambda > \lambda^*$, we obtain

$$(11.31)\qquad \int_{\mathcal{O}}(\lambda-\varpi)_+f^2m^2 + \int_{\mathcal{O}}|\nabla_v f|^2m^2 + \frac12\mathcal{E}_{\zeta,\alpha}(\gamma_+f) \le \frac{1}{\min(\lambda-\lambda^*,1)}\|\mathfrak{F}\|^2_{L^2H^{-1}_m}.$$

There is no difficulty for also getting the pieces of information (11.19), (11.20), (11.22) and (11.23), so that in particular $f \in W_2$. It is worth emphasizing here that when $\langle v\rangle^2\mathscr{M} \in L^1$, we have $L^2(d\xi^2_m) \subset L^1(\Sigma;d\xi)$ by using the Cauchy-Schwarz and (11.20), so that in particular the boundary condition is well defined.

Let us show now how the last conclusion also holds under condition (i) in the statement of the Theorem. We then assume $\vartheta = 1$ in (11.10) and we show how to establish an additionnal a priori estimate. We indeed know from (11.20) that

$$\int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 (\nu \cdot \hat{v})^2 m^2 dv d\sigma_x \le \int_{\Sigma} (\gamma f)^2 (\nu \cdot \hat{v})^2 m^2 dv d\sigma_x \le C_\lambda \|\mathfrak{F}\|^2_{L^2 H^{-1}_m},$$

and similarly as in [18] or [272, proof of Lemma 2.2] that

$$1 = \int_{\Sigma^x_-} |\nu(x) \cdot v| \mathscr{M} dv = C \int_{\Sigma^x_-} (\nu(x) \cdot \hat{v})^2 \mathscr{M} dv, \quad \forall \, x \in \partial\Omega,$$

for some constant $C \in (0, \infty)$, so that

$$\int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 d\xi_m = C \int_{\Sigma_-} (\alpha \mathcal{D}(\gamma_+ f))^2 (\nu \cdot \hat{v})^2 m^2 \le C C_\lambda \|\mathfrak{F}\|^2_{L^2 H^{-1}_m}. \tag{11.32}$$

Summing up (11.31) and (11.32), and using that

$$(\gamma_+ f)^2 \le 2(\mathcal{D}^\perp \gamma_+ f)^2 + 2(\mathcal{D}\gamma_+ f)^2,$$

we deduce that

$$\int_{\Sigma_+} [1 - \zeta + \alpha^2/2](\gamma_+ f)^2 d\xi_m \le C_\lambda \|\mathfrak{F}\|^2_{L^2 H^{-1}_m}. \tag{11.33}$$

Defining

$$f \in W_{2,\mathscr{R}} := \{g \in W_2; \ \gamma_- g = \mathscr{R}\gamma_+ g\},$$

we see that $W_{2,\mathscr{R}} = W_{2,\Sigma}$ if $1 - \zeta + \alpha^2/2 \ge \delta_* > 0$, but it is worth emphasizing that we may have $W_{2,\mathscr{R}} \ne W_{2,\Sigma}$ in the general case.

Step 2. Existence when $\mathfrak{F} \ge 0$. With the help of Theorem 11.3, we define $f_0 = 0$ and, recursively for any $n \ge 1$, we define $f_n \in W_{2,\Sigma}$ as the solution of

$$(\lambda - \mathcal{L}) f_n = \mathfrak{F} \ \text{ in } \ \mathcal{O}, \quad \gamma_- f_n = \mathscr{R}\gamma_+ f_{n-1} \ \text{ on } \ \Sigma_-. \tag{11.34}$$

It is worth emphasizing here that $\gamma_+ f_{n-1} \in L^2(\Sigma_+; d\xi_m)$ implies $\mathscr{R}(\gamma_+ f_{n-1}) \in L^2(\Sigma_-; d\xi_m)$ because of (11.5). We also notice that $f_n \ge 0$ because $\mathfrak{F} \ge 0$. By linearity

$$(\lambda - \mathcal{L})(f_{n+1} - f_n) = 0 \ \text{ in } \ \mathcal{O}, \quad \gamma_-(f_{n+1} - f_n) = \mathscr{R}\gamma_+(f_n - f_{n-1}) \ \text{ on } \ \Sigma_-,$$

and we thus show recursevily that $f_{n+1} - f_n \ge 0$. In other words, (f_n) is an increasing sequence and thus also is (γf_n). From (11.30), we have

$$\begin{aligned} \int_\Sigma (\gamma f_n)^2 d\xi_m &= \int_{\Sigma_+} (\gamma_+ f_n)^2 d\xi_m - \int_{\Sigma_-} (\mathscr{R}\gamma_- f_{n-1})^2 d\xi_m \\ &\ge \int_{\Sigma_+} (\gamma_+ f_n)^2 d\xi_m - \int_{\Sigma_-} (\mathscr{R}\gamma_- f_n)^2 d\xi_m \ge \mathcal{E}_{\zeta,\alpha}(\gamma_+ f_n), \end{aligned}$$

so that the estimate (11.31) holds true for f_n (instead of f) uniformly in $n \ge 1$. From the monotonous convergence theorem, there exists $f \in L^2 H^1_m$ satisfying (11.31), (11.33), (11.20) and such that $f_n \nearrow f$ a.e. Thanks to Proposition 11.2, we have $\gamma f_n \nearrow \gamma f$ a.e. on Σ, from what we deduce that $\mathscr{R}\gamma_+ f_n \to \mathscr{R}\gamma_+ f$ in $L^2(\Sigma_-; d\xi^2_m)$ thanks to the monotonous convergence theorem. As a consequence, we may pass to the limit in the weak formulation of (11.34), and we get that f is a solution of (11.28). We may also pass to the liminf in the estimate (11.31) written for f_n, and we thus deduce that the same estimate holds for f.

Step 3. Existence when $\mathfrak{F} \in L^2 H^{-1}_m$. When $\mathfrak{F} \in L^2_m$, we may introduce the splitting $\mathfrak{F} = \mathfrak{F}_+ - \mathfrak{F}_-$, just use the previous step for $\mathfrak{F}_\pm$ and conclude by linearity of the equation. When $\mathfrak{F} \notin L^2_m$, we proceed similarly as in [272] and in the following way. We first assume $\zeta \le \zeta^* \in [0, 1)$ and we consider the mapping $\Psi : W_{2,\Sigma} \to W_{2,\Sigma}$, $g \mapsto f = \Psi(g)$, where f is the solution to the stationary problem

$$\begin{cases} (\lambda - \mathcal{L}) f = \mathfrak{F} & \text{in} \quad \mathcal{O} \\ \gamma_- f = \mathscr{R}\gamma_+ g & \text{on} \quad \Sigma_-. \end{cases} \tag{11.35}$$

The space $W_{2,\Sigma}$ is endowed with the norm $\|\cdot\|_{W_{2,\Sigma}}$ defined by

$$\|g\|^2_{W_{2,\Sigma}} = \|g\|^2_{L^2_m} + \|\nabla_v g\|^2_{L^2_m} + \|\gamma_+ g\|^2_{L^2_m(d\xi_1)}.$$

From (11.18) and the estimate $\|\mathscr{R} g\|_{L^2(\Sigma_-;d\xi_m)} \le \zeta^* \|g\|_{L^2(\Sigma_+;d\xi_m)}$ what we obtain by repeating the proof of (11.5), we deduce

$$\begin{aligned}\frac{1}{C_\lambda}\|f\|^2_{L^2H^1_m} + \|\gamma_+ f\|^2_{L^2(\Sigma_+;d\xi_m)} &\le C_\lambda \|\mathfrak{F}\|^2_{L^2H^{-1}_m} + \|\mathscr{R}\gamma_+ g\|_{L^2(\Sigma_-;d\xi_m)} \\ &\le C_\lambda \|\mathfrak{F}\|^2_{L^2H^{-1}_m} + \zeta^* \|\gamma_+ g\|_{L^2(\Sigma_+;d\xi_m)},\end{aligned}$$

for some cosntant $C_\lambda > 0$. By linearity of (11.35), we deduce that for two functions $g_1, g_2 \in W_{2,\Sigma}$, and denoting $f_i := \Psi(g_i)$, we have

$$\frac{1}{C_\lambda}\|f_2 - f_1\|^2_{L^2H^1_m} + \|\gamma_+ f_2 - \gamma_+ f_1\|^2_{L^2(\Sigma_+;d\xi_m)} \le \zeta^* \|\gamma_+ g_2 - \gamma_+ g_1\ \|^2_{L^2(\Sigma_+;d\xi_m)},$$

so that Ψ is a contraction in $W_{2,\Sigma}$. By the Banach fixed point theorem, we deduce that there exists a solution $f \in W_{2,\Sigma}$ to the equation (11.28) in that case. Finally, in order to deal with the case $\zeta^* = 1$, we consider a sequence (ζ^*_n) of $[0,1)$ such that $\zeta^*_n \nearrow 1$ and the associated sequence (f_n) of solutions in $W_{2,\Sigma}$ associated to the equation (11.28) with the modified reflection kernel $\mathscr{R}_n g := \zeta^*_n \mathscr{R} g$. From (11.31) and (11.20), that sequence satisfies

$$\|f_n\|^2_{L^2H^1_m} + \|\gamma f_n\|^2_{L^2(\Sigma;d\xi^2_m)} + \mathcal{E}_{1,\alpha}(\gamma_+ f_n) \le C_\lambda \|\mathfrak{F}\|^2_{L^2H^{-1}_m}.$$

When $\alpha \not\equiv 0$, the above estimate or (11.33) also implies that $(\gamma_+ f_n)$ belongs to a weakly compact set of $L^1(\Sigma_+; d\xi)$. As a consequence, there exist $f \in W_2$ and $\bar\gamma_\pm$ two functions defined on $\Sigma_\pm$ such that, up to the extraction of a subsequence,

$$\begin{aligned}&f_n \rightharpoonup f \quad L^2 H^1_m, \quad \gamma_\pm f_n \rightharpoonup \bar\gamma_\pm \quad L^2(\Sigma_\pm; d\xi^2_m),\\ &\gamma_+ f_n \rightharpoonup \bar\gamma_+ \quad L^1(\Sigma_+; d\xi), \quad \mathscr{R}\gamma_+ f_n \rightharpoonup \mathscr{R}\bar\gamma_+ \quad L^1(\Sigma_-; d\xi),\end{aligned}$$

where we have used (11.11) for the last convergence. Using Proposition 11.2, we may thus pass to the limit in the equation (11.28) satisfied by f_n with modified reflection kernel $\mathscr{R}_n$ and we get that f is a solution of (11.28). In the pure specular reflection case $\alpha \equiv 0$, only the first line of convergences holds, but that it is enough in order to pass to the limit in the equations (we refer to [270, 272] for similar arguments).

Step 4. Other properties. We further assume $\lambda > \lambda^{**}$. We proceed similarly as in [261]. Consider two weak solutions $f_i \in W_2$ to the equation (11.28). In particular, the difference $f := f_2 - f_1 \in W_2$ satisfies

$$(\lambda - \mathcal{L}) f = 0 \ \text{ in } \ \mathcal{O}, \quad \gamma_- f = \mathscr{R}\gamma_+ f \ \text{ on } \ \Sigma_-.$$

Using the Green renormalized formula (11.14), we have

$$0 = \int_{\mathcal{O}} \beta'(f)(\lambda - c) f\varphi + \beta''(f)|\nabla f|^2 \varphi + \beta(f)(\mathrm{div}_v(b\varphi) - v\cdot\nabla_x\varphi - \Delta_v\varphi) + \int_\Sigma \beta(\gamma f)\nu\cdot v\varphi.$$

for any $\beta \in C^2(\mathbb{R})$, $\beta' \in C^1_b(\mathbb{R})$ and any test function $\varphi \in C^2_c(\bar{\mathcal{O}})$. We choose $\varphi = \varphi(v) \ge 0$, $\beta \ge 0$ and $\beta'' \ge 0$, so that

$$0 \ge \int_{\mathcal{O}} \beta'(f)(\lambda - c) f\varphi + \beta(f)(\mathrm{div}_v(b\varphi) - \Delta_v\varphi) + \int_\Sigma \beta(\gamma f)\nu\cdot v\varphi.$$

By an approximation argument, we may now take $\beta(s) = |s|$, and we get

$$0 \ge \int_{\mathcal{O}} |f| \big\{(\lambda - c)\varphi + (\mathrm{div}_v(b\varphi) - \Delta_v\varphi)\big\} + \int_\Sigma |\gamma f|\nu\cdot v\varphi.$$

We observe that in any cases we have $f \in L^2_m(\mathcal{O}) \subset L^1(\mathcal{O})$ and $\gamma f \in L^1(\Sigma; d\xi)$. By an approximation argument, we may now take $\varphi = 1$ and using the L^1 estimate (11.11) on $\mathscr{R}$ (with $\zeta^* = 1$), we get

$$\begin{aligned}0 &\ge \int_{\Sigma_-} |\mathscr{R}\gamma_+ f||\nu\cdot v| - \int_{\Sigma_+} |\gamma_+ f||\nu\cdot v| \\ &\ge \int_{\mathcal{O}} |f| \big\{\lambda - c + \mathrm{div}_v b\big\} \ge (\lambda - \lambda_{**}) \int_{\mathcal{O}} |f|.\end{aligned}$$

We deduce that $f = 0$. The proof of the positivity property follows the same arguments but choosing $\beta(s) = s_-$. $\square$

For latter reference, we state the counterpart of the preceding result for the kinetic Fokker-Planck evolution equation.

Theorem 11.5. *Let us make the same assumptions as in Theorem 11.4. For any $f_0 \in L^2_m$, there exists a unique solution $f \in C([0,T); L^2_m) \cap L^2(0,T; H^1_m)$ for any $T > 0$ to the kinetic Fokker-Planck evolution equation*

$$\begin{cases} \partial_t f = \mathcal{L} f & \text{in} \quad (0,\infty) \times \mathcal{O} \\ \gamma_- f = \mathscr{R} \gamma_+ f & \text{on} \quad (0,\infty) \times \Sigma_-, \end{cases} \tag{11.36}$$

with $\mathcal{L}$ defined in (11.1) *and $\mathscr{R}$ defined in* (11.6).

The proof of Theorem 11.5 is skipped since it is a mere adaptation of the proof of Theorem 8.23 and Theorem 11.4. We refer to [365, Cor. 2 7, Lem. 2.8 and Cor. 2.8] where similar well-posedness results are established (see also [272] for the existence part).

11.4. **The first eigenvalue problem in a domain with reflection at the boundary.** We consider now the first eigenvalue problem for the kinetic Fokker-Planck operator (11.1) in a domain with reflection at the boundary, namely

$$\begin{cases} \lambda f + v \cdot \nabla_x f - \Delta_v f - b \cdot \nabla_v f - cf = 0 & \text{in} \quad \mathcal{O} \\ \gamma_- f = \mathscr{R} \gamma_+ f & \text{on} \quad \Sigma_-, \end{cases} \tag{11.37}$$

and the associated dual problem. In this section, we assume that b and c satisfy the assumptions of Theorem 11.3 with the weight function $m := \mathscr{M}^{-1/2}$ when $\alpha \not\equiv 0$ and for a given weight function $m : \mathbb{R}^d \to [1,\infty)$ when $\alpha \equiv 0$ and $\mathscr{R}$ is given by (11.6). We additionnally assume that

$$\liminf_{|(x,v)| \to \infty} \varpi(x,v) = -\infty, \tag{11.38}$$

where we recall that ϖ is defined in (11.17). When $\mathscr{M}$ is the Gaussian function, we find

$$\varpi = c + \frac{|v|^2 + d}{2} - \frac{1}{2} \mathrm{div} b - b \cdot v,$$

so that (11.38) holds when b is typically a bounded perturbation of the vector field $b_0(v) = \vartheta_0 v$, $\vartheta_0 > 1/2$, and more precisely

$$\mathrm{div}_v b \in L^\infty(\mathcal{O}) \quad \text{and} \quad \inf_{x \in \Omega} \liminf_{|v| \to \infty} (b \cdot v \langle v \rangle^{-2}) \geq \vartheta_0 > 1/2.$$

The above condition is quite technical but can be seen as a compatibility condition between the thermalization due to the boundary and to the Fokker-Planck collisional operator. We are then able to work in the functional space $X := L^2_m(\mathcal{O})$.

Theorem 11.6. *Under the above conditions, the first eigentriplet problem associated to* (11.1) *has a unique solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times X'$ with $f_1 > 0$ and $\phi_1 > 0$.*

The proof of Theorem 11.6 follows from Theorem 2.21, Theorem 4.13 and Theorem 5.16 as a consequence of conditions **(H1)**–**(H5)**. We prove now that each of these conditions is satisfied. Theorem 11.6 generalizes [248, Thm. 2.12] where the same problem is tackled for the zero inflow condition ($\alpha = \beta = 0$) with $b = v - F(x)$ and $c = 1$ by using the classical Krein-Rutman theorem [238] in the space $X = C_b(\bar{\mathcal{O}})$. We also refer to [193, Thm. 6.8] for a variant and somehow generalisation of [248].

Condition (H1). From Theorem 11.4, the operator $\mathcal{L}$ satisfies **(H1)** with

$$\kappa_1 := \max(\lambda^*, \lambda^{**}),$$

with λ^* defined by (11.17) and λ^{**} defined by (11.29). For later reference, let us state more precisely the available estimates for f. On the one hand, repeating the proof of Step 1 in the proof of Theorem 11.4, we establish that for any $\lambda > \kappa_1$ and $\mathfrak{F} \in L^2_m$, the solution $f \in W_2$ to the Dirichlet problem (11.28) satisfies

$$\int_{\mathcal{O}} (\lambda - \varpi)_+ f^2 m^2 + \int_{\mathcal{O}} |\nabla_v f|^2 m^2 + \frac{1}{2} \mathcal{E}_{\zeta,\alpha}(\gamma_+ f) \leq \frac{1}{\lambda - \lambda^*} \|\mathfrak{F}\|^2_{L^2_m}. \tag{11.39}$$

On the other hand, adapting the proof of (11.22), we straightforwardly obtain

$$\int_{\mathcal{O}} f^2 \frac{(\hat{v}\cdot\nu(x))^2}{\delta(x)^{1/2}} dvdx \le C \int \mathfrak{F}^2 m^2, \tag{11.40}$$

for some constant $C = C(b, c, \nu, \lambda)$. For $\varepsilon_x, \varepsilon_v, \varrho > 0$, let us now define

$$\mathcal{U} := \{(x,v) \in \mathcal{O};\ d(x,\partial\Omega) > \varepsilon_x,\ |v| < \varrho\}, \tag{11.41}$$

and compute

$$\int_{\mathcal{U}^c} f^2 m^2 \le \int f^2 m^2 \mathbf{1}_{|v|\ge\varrho} + \int f^2 m^2 \mathbf{1}_{A_x} + \int f^2 m^2 \mathbf{1}_B,$$

with

$$A_x := \{v \in B_\varrho,\ (\hat{v}\cdot\nu(x))^2 \le \varepsilon_v^2\}, \quad B := \{(x,v);\ |v| \le \varrho,\ (\hat{v}\cdot n)^2 \ge \varepsilon_v^2,\ d(x,\partial\Omega) \le \varepsilon_x\}.$$

For the second term, we have

$$\begin{aligned} \int f^2 m^2 \mathbf{1}_{A_x} &\le \int |A_x|^{2/r'} \|f(x,\cdot)\|^2_{L^r_v} dx \\ &\lesssim (\varrho^{d-1}\varepsilon_v)^{2/r'} \|f\|^2_{L^2 H^1_m}, \end{aligned}$$

where we have used the Holder inequality with $r \in (1, 2^*/2)$ in the first line and the Sobolev inequality in the second line. For the third term, we have

$$\int f^2 m^2 \mathbf{1}_B \le m^2(\varrho) \frac{\varepsilon_x^{1/2}}{\varepsilon_v^2} \int_{\mathcal{O}} f_n^2 \frac{(\hat{v}\cdot\nu(x))^2}{\delta(x)^{1/2}}.$$

Gathering these last estimates with (11.39) and (11.40), we have established that the solution f to equation (11.28) furthermore satisfies

$$\int_{\mathcal{U}^c} f^2 m^2 \le C\Big(\frac{1}{\langle\varrho\rangle^2} + \varrho^{d-1}\varepsilon_v + m^2(\varrho)\frac{\varepsilon_x^{1/2}}{\varepsilon_v^2}\Big) \int \mathfrak{F}^2 m^2, \tag{11.42}$$

for a constant $C = C(b, c, \Omega, \lambda)$ and for any $\varepsilon_x, \varepsilon_v, \varrho > 0$.

The strong maximum principle. Let us now consider a function $0 \le f \in W_2\backslash\{0\}$ which satisfies the Dirichlet problem (11.28) associated to $\lambda > \kappa_1$ and a source term $0 \le \mathfrak{F} \in L^2_m \cap L^\infty$. In order to simplify the discussion, we assume that the normalization $\|f\|_{L^2_m} = 1$ holds. For proving the strong maximum principle, we briefly explain how we may adapt the arguments we have presented for the diffusive equation in Part 7 by taking in particular advantage of the above established estimates, the regularity results established in [181, 192] and some spreading positivity results we learnt in [353, Cor. A.20]. We proceed in three steps.

Step 1. On the one hand, from (11.42), we may choose conveniently $\varrho^{-1}, \varepsilon_v, \varepsilon_x > 0$ small enough in such a way that

$$\int_{\mathcal{U}^c} f^2 m^2 \le \frac{1}{2}\|f\|^2_{L^2_m},$$

where $\mathcal{U}$ is defined by (11.41). Because of the normalization condition, we have

$$\int_{\mathcal{U}} f^2 m^2 \ge \frac{1}{2}\|f\|^2_{L^2_m} \tag{11.43}$$

and consequently $f(x_0, v_0)^2 \ge \delta_0^2 := \|f\|^2_{L^2_m} (2\|\mathbf{1}_{\mathcal{U}}\|^2_{L^2_m})^{-1}$ for at least one point $(x_0, v_0) \in \mathcal{U}$.

Step 2. On the other hand, let us recall some integrability and regularity results established in [181] for a solution g to the kinetic Fokker-Planck evolution equation

$$\partial_t g + v\cdot\nabla_x g = \Delta_v g + B\cdot\nabla_v g + s \ \text{ in } \ \mathcal{V},$$

or a sub-solution

$$\partial_t g + v\cdot\nabla_x g \le \Delta_v g + B\cdot\nabla_v g + s \ \text{ in } \ \mathcal{V},$$

for some bounded set $\mathcal{V} \subset (0,T)\times\mathcal{O}$, $s \in L^2(\mathcal{V})$ and $B \in L^\infty(\mathcal{V})$. For that purpose, given some (t^*, x^*, v^*), we define

$$Q_r := \{(t,x,v);\ t \in (t^* - r^2, t^*],\ |x - x^* - (t-t^*)v^*| < r^3,\ |v - v^*| < r\}.$$

We claim then that there exist $2 < p < q < \infty$, $\alpha \in (0,1)$ and for any $0 < r_1 < r_0$ there exists C such that

$$\|g\|_{L^p(Q_{r_1})} \le C\,(\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^2(Q_{r_0})}) \tag{11.44}$$

for any nonnegative subsolution g on Q_{r_0} from [181, Thm. 6],

$$\|g\|_{L^\infty(Q_{r_1})} \le C\,(\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^q(Q_{r_0})}) \tag{11.45}$$

for any nonnegative subsolution g on Q_{r_0} from [181, Thm. 12] and

$$\|g\|_{C^\alpha(Q_{r_1})} \le C\,(\|g\|_{L^2(Q_{r_0})} + \|s\|_{L^\infty(Q_{r_0})}) \tag{11.46}$$

for any solution g on Q_{r_0} from [181, Thm. 3]. As a consequence of (11.44) and a classical covering argument, for any bounded set $\mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{O}$, there exist $C_0 = C_0(\mathcal{U})$ and $C_1 = C_1(\mathcal{U},\lambda)$ such that

$$\|f\|_{L^p(\mathcal{U})} \le C_0\,(\|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F} + cf - \lambda f\|_{L^2(\mathcal{O})}) \le C_1(\|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^2(\mathcal{O})}).$$

Observing that for $\varrho = p/2 > 1$, we have

$$v \cdot \nabla_x f^\varrho - \Delta_v f^\varrho - b \cdot \nabla_v f^\varrho + \varrho f^{\varrho-1}(\lambda f - cf - \mathfrak{F}) = -4\frac{(\varrho-1)}{\varrho}|\nabla(f^{\varrho/2})|^2 \le 0,$$

so that f^ϱ is a weak sub-solution to the kinetic Fokker-Planck equation, we may repeat the argument and obtain in that way that $f \in L^{p_k}(\mathcal{U})$ for any $k \ge 1$, with $p_k := \varrho^k 2$. Now, choosing k such that $p_k \ge q$ and using (11.45) (as well as again a classical covering argument), we get

$$\|f\|_{L^\infty(\mathcal{U})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F} + cf - \lambda f\|_{L^q(\mathcal{O})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^q(\mathcal{O})}.$$

Using finally (11.46), we deduce that there exists a constant $C = C(\mathcal{U},\lambda)$ such that

$$\|f\|_{C^\alpha(\mathcal{U})} \lesssim \|f\|_{L^2(\mathcal{O})} + \|\mathfrak{F}\|_{L^\infty(\mathcal{O})}.$$

Together with the conclusion of the first step, we deduce that there exists a constructive constante $r_0 > 0$ such that $f \ge \delta_0 \mathbf{1}_{B((x_0,v_0),r_0)}$.

Step 3. From [353, Cor. A.20], we deduce that for any bounded set $\mathcal{U} \subset \bar{\mathcal{U}} \subset \mathcal{O}$, there exists a constructive constant $\delta = \delta(\delta_0, r_0, \mathcal{U}) > 0$ such that

$$f(x,v) \ge \delta \quad \text{for any} \quad (x,v) \in \mathcal{U},$$

where it is worth emphasizing that the hypothesis $b, c \in C(\mathcal{O})$ made in [353, Cor. A.20] is not really necessary and can be replaced by $b, c \in L^\infty(\mathcal{U})$. Because $\mathcal{U}$ may be choosen arbitrary, we have established that $f > 0$ on $\mathcal{O}$ and the strong maximum principle.

Condition (H2). For a given function $0 \le h_0 \in C^2_c(\mathcal{O})$ normalized by $\|h_0\|_{L^2_m} = 1$, we define $f_0 \in D(\mathcal{L})$ as the solution to

$$(\kappa_1 - \mathcal{L}) f_0 = h_0 \ \text{ in } \ \mathcal{O}, \quad \gamma_- f_0 = \mathscr{R}\gamma_+ f_0 \ \text{ on } \ \Sigma_-.$$

Taking advantage of the fact that h_0 has compact support, we compute

$$1 = \int_{\mathcal{O}} h_0^2 m^2 = \int_{\mathcal{O}} (\kappa_1 - \mathcal{L}) f_0\, h_0 m^2 = \int_{\mathcal{O}} f_0 (\kappa_1 - \mathcal{L}^*)(h_0 m^2) \le C_1 \|f_0\|_{L^2_m},$$

with $C_1 := \|m^{-1}(\kappa_1 - \mathcal{L}^*)(h_0 m^2)\|_{L^2}$. On the other hand, from (11.27), we have

$$\|f_0\|_{L^2 H^1_m} + \|f_0 \frac{\hat{v}\cdot\nu}{\delta^{1/4}}\|_{L^2} \le C_2, \tag{11.47}$$

for a constant C_2 only depending on $\|h_0\|_{L^2_m}$, κ_1 and the constant C which appears in (11.27). Arguing as in (11.43), we deduce that

$$\int_{\mathcal{U}} f_0^2 m^2 \ge (2C_1)^{-1}, \quad \operatorname{supp} h_0 \subset \mathcal{U}, \tag{11.48}$$

with $\mathcal{U} = \mathcal{U}_\varrho$ defined in (11.41) and $\varrho > 0$ small enough (chosen constructively from C_2 and C_1). From the above constructive strong maximum principle, we deduce that $f_0 \ge \varepsilon \mathbf{1}_{\mathcal{U}} \ge 1/C_0 h_0$ for some $\varepsilon, C_0 > 0$. We conclude as in the second constructive argument for (H2) in Section 7.1. Coming back indeed to the equation, we have

$$\mathcal{L} f_0 = \kappa_1 f_0 - h_0 \ge \kappa_1 f_0 - \|h_0\|_{L^\infty} \mathbf{1}_{\mathcal{U}} \ge (\kappa_1 - \|h_0\|_{L^\infty} C_0) f_0,$$

so that **(H2)** holds with $\kappa_0 := \kappa_1 - \|h_0\|_{L^\infty} C_0$ from Lemma 2.4-**(ii)**.

Condition (H3). Let us fix $\kappa < \kappa_0$ arbitrary. We define $\mathcal{B}f := \mathcal{L}f - n\chi_R(v)f$ for any $f \in W_{2,\mathscr{R}}$, with $\chi_R \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{B_R} \le \chi_R \le \mathbf{1}_{B_{2R}}$ and for some given $n, R \ge 0$ to be specified below. We observe that, at least formally,

$$\int f m^2 (\mathcal{B} - \kappa) f \quad = \quad \int_{\mathcal{O}} (\varpi - \kappa - n\chi_R) f^2 m^2 - \frac{1}{2} \int_{\Sigma} (\gamma f)^2 m^2 \nu \cdot v - \int_{\mathcal{O}} |\nabla_v f|^2 m^2.$$

Thanks to (11.38), there exists a constant $R > 0$ such that

$$\sup_{v \in \mathbb{R}^d \setminus B_R} \varpi \le \kappa.$$

Choosing $n := \sup \varpi_+ - \kappa$, we deduce that $\varpi - \kappa - n\chi_R \le 0$. On the other hand, because of (11.30), the contribution of the boundary term in the above identity is non positive. We thus deduce that $(\mathcal{B} - \kappa)$ is dissipative in L^2_m. We now establish that the associated operator $\mathcal{B}$ has compact resolvent. For $\mathfrak{F} \in L^2_m$, we consider $f \in L^2_m$ the solution to

$$(11.49) \qquad -\mathcal{B}f = \mathfrak{F} \ \text{ in } \ \mathcal{O}, \quad \gamma_- f = \mathscr{R}\gamma_+ f \ \text{ on } \ \Sigma_-,$$

which existence follows from Theorem 11.4. From the above discussion (with $\kappa = -1$) and the same arguments as in Step 1 of the proof of Theorem 11.3, we have

$$(11.50) \qquad \int f^2 \langle \varpi \rangle_- m^2 + 2 \int |\nabla_v f|^2 m^2 \le \int \mathfrak{F}^2 m^2.$$

Together with the regularity estimate (11.23) and the compact imbedding $H^{2/3}(\mathcal{U}) \subset L^2(\mathcal{U})$, we conclude that $\mathcal{B}$ has compact resolvent. The operator $\mathcal{A}$ on L^2_m defined by $\mathcal{A}f := n\chi_R(v)f$ being bounbded, we may apply Lemma 2.8-(2) and we deduce that **(H3)** holds for both the primal and the dual problems.

Condition (H4) is nothing but the yet established strong maximum principle.

A variant of condition (H5). Consider (f, λ) a pair of eigenfunction and eigenvalue such that $\lambda \in \Sigma_{P+}(\mathcal{L})$. Arguing similarly as in the proof of condition **(H5)** in Section 7.1, we know that

$$\widetilde{\mathcal{L}} f = i\vartheta f, \ \vartheta \in \mathbb{R}, \quad \widetilde{\mathcal{L}}|f| = 0$$

and introducing the real and complex part decomposition $f = g + ih$, we have

$$\int_{\mathcal{O}} \frac{1}{|f|^2} |g\nabla_v h - h\nabla_v g|^2 = 0,$$

and finally $g\nabla_v h - h\nabla_v g = 0$ a.e. on $\mathcal{O}$. Because of the regularity estimate presented during the above proof of the strong maximum principle, the functions f has Hölder regularity, and thus g and h are continuous on $\mathcal{O}$. Because $|f| \not\equiv 0$, we may claim that there exists a point $(x_0, v_0) \in \mathcal{O}$ such that $h(x_0, v_0) > 0$ for instance. Denoting by ω the connected component of $\{(x, v) \in \mathcal{O};\, h(x, v) > 0\}$ containing (x_0, v_0), we have $\nabla(g/h) = 0$ on ω, and thus $g = \alpha(x)h$ on ω for some continuous function $\alpha : \Omega \to \mathbb{R}$. Coming back to the eigenvalue equation that we may write in the following system form

$$\widetilde{\mathcal{L}} g = -\vartheta h, \quad \widetilde{\mathcal{L}} h = \vartheta g,$$

we compute

$$-\vartheta h = \widetilde{\mathcal{L}}(\alpha h) = \alpha \widetilde{\mathcal{L}} h - h v \cdot \nabla_x \alpha = \alpha \vartheta g - h v \cdot \nabla_x \alpha \ \text{ on } \ \omega,$$

so that

$$-\vartheta = \alpha^2 \vartheta - v \cdot \nabla_x \alpha \ \text{ on } \ \omega.$$

We deduce that α is a constant on ω and finally $\vartheta = 0$. We have thus established that $\lambda = \lambda_1$.

At this stage, we may use Theorem 2.21, Theorem 4.13 and Theorem 5.16, in order to get the conclusions **(C1)**, **(C2)** and **(C3)** about the existence and uniqueness of the eigentriplet (λ_1, f_1, ϕ_1) which satisfies $f_1 > 0$, $\phi_1 > 0$, λ_1 is algebraically simple and on the triviality of the boundary punctual spectrum.

We briefly explain how we may deduce the stability of f_1 by adapting some arguments developped in [269] and already mentioned. On the one hand, we know from [269, Lem. 1.1] that any solution f to the rescaled evolution equation (11.36) with $\mathcal{L}$ replaced by $\widetilde{\mathcal{L}} = \mathcal{L} - \lambda_1$ satisfies

$$\partial_t (H(X) f_1 \phi_1) + \mathrm{div}_x (v H(X) f_1 \phi_1) - \mathrm{div}_v (\phi_1^2 \nabla_v (H(X) f_1 / \phi_1)) = -H''(X) f_1 \phi_1 |\nabla_v X|^2,$$

for any convex function $H:\mathbb{R}\to\mathbb{R}$ and with $X := f/f_1$. After integration, we get

$$\frac{d}{dt}\int_{\mathcal{O}} H(X) f_1\phi_1 + \int_\Sigma \nu\cdot v H(X) f_1\phi_1 = -\int_{\mathcal{O}} H''(X) f_1\phi_1 |\nabla_v X|^2, \tag{11.51}$$

When $H(s) := |s|$, the boundary term is

$$\begin{aligned}\int_\Sigma |\gamma f|\gamma\phi_1 \nu\cdot v &= \int_{\Sigma_+} |\gamma_+ f| \mathscr{R}^*\gamma_-\phi_1 \nu\cdot v - \int_{\Sigma_-} |\mathscr{R}\gamma_+ f|\gamma_-\phi_1|\nu\cdot v| \\ &\geq \int_{\Sigma_+} |\gamma_+ f|\mathscr{R}^*\gamma_-\phi_1|\nu\cdot v| - \int_{\Sigma_-} \mathscr{R}|\gamma_+ f|\gamma_-\phi_1\,|\nu\cdot v| = 0,\end{aligned}$$

from what we deduce the non expansive property

$$\int_{\mathcal{O}} |f_{t_1}|\phi_1 \leq \int_{\mathcal{O}} |f_{t_0}|\phi_1, \quad \forall\, t_1\geq t_0\geq 0. \tag{11.52}$$

On the other hand, from the Cauchy-Schwarz inequality, we have

$$(\mathscr{R}\gamma_+ f)^2 \leq (\mathscr{R}\gamma_+ f_1)\mathscr{R}(\gamma_+ f^2/\gamma_+ f_1),$$

so that

$$\int_{\Sigma_-} \frac{(\mathscr{R}\gamma_+ f)^2}{\mathscr{R}\gamma_+ f_1}\gamma_-\phi_1|\nu\cdot v| \leq \int_{\Sigma_-} \mathscr{R}(\gamma_+ f^2/\gamma_+ f_1)\gamma_-\phi_1|\nu\cdot v|$$

and finally

$$\int_\Sigma (\gamma f)^2(\gamma f_1)^{-1}\gamma\phi_1\,\nu\cdot v \leq 0.$$

When $H(s) = s^2$, the equation (11.51) and the last inequality imply

$$\frac{d}{dt}\int_{\mathcal{O}} f_1\phi_1 (f/f_1)^2 + 2\int_{\mathcal{O}} f_1\phi_1 |\nabla_v(f/f_1)|^2 \leq 0. \tag{11.53}$$

We next recall a classical compactness result.

Lemma 11.7. *Let (g_n) be a sequence of functions such that*

$$(g_n) \text{ is bounded in } L^\infty(0,T;L^2_{xv,\mathrm{loc}})\cap L^2(0,T;L^2_{x,\mathrm{loc}}H^1_{v,\mathrm{loc}})$$

and

$$\partial_t g_n + v\cdot\nabla_x g - \Delta_v g_n = G_n \text{ bounded in } L^2_{\mathrm{loc}},$$

then (g_n) belongs to a strong compact set of L^2_{loc}.

Proof of Lemma 11.7. We just sketch it. Because

$$\partial_t g_n + v\cdot\nabla_x g = \Delta_v g_n + G_n \text{ bounded in } L^2_{tx}H^{-1}_v,$$

the usual averaging lemma in [182, 137] implies that

$$(g_n * \rho) \text{ belongs to a strong compact set of } L^2_{\mathrm{loc}},$$

for any $\rho\in\mathcal{D}(\mathbb{R}^d)$. On the other hand, introducing a mollifiers sequence (ρ_ε) and writing then

$$g_n = (g_n - g_n*\rho_\varepsilon) + g_n*\rho_\varepsilon,$$

we see that the first term is small uniformly in n as $\varepsilon\to 0$ and the second term is relativelly compact thanks to the first step, from what we immediately conclude. ☐

Now, for $0\leq f_0\in L^1_{\phi_1}$, we introduce the sequence $f_{0,k} := (f_0\wedge k)\mathbf{1}_{\mathcal{U}_k}\in L^2(f_1^{-1}\phi_1)\cap L^2$, with $\mathcal{U}_k := \{(x,v)\in\mathcal{O};\,\delta(x) > 1/k,\ |v|\leq k\}$, and the associated solution $f_k\in L^\infty(0,T;L^2)\cap L^2(0,\infty;L^2_xH^1_v)$. Because of (11.53), for any increasing sequence (t_n) which converges to ∞ and for any function $\varphi_m\in\mathcal{D}(\mathcal{O})$, $\mathbf{1}_{\mathcal{U}_m}\leq\varphi_m\leq 1$, the rescaled and truncated function $g_n(t) := f_k(t+t_n)f_1^{-1}e^{-\lambda_1(t+t_n)}\varphi_m$ meet the hypothesis of Lemma 11.7, from what we classically deduce that the sequence of $\tilde f_n(t) := f_k(t+t_n)f_1^{-1}e^{-\lambda_1(t+t_n)}$ is relatively strongly compact in L^2_{loc}. Repeating the proof of Theorem 4.23 and Theorem 5.23 (see also [269, Thm. 3.2]), we deduce that $\tilde f_n(t)\to\langle f_{0,k},\phi_1\rangle f_1$ as $t\to\infty$. Together with the above non expansive property (11.52), we deduce that $f_t\to\langle f_0,\phi_1\rangle f_1$ in $L^1_{\phi_1}$ as $t\to\infty$.

We summarize our convergence result in the following theorem.

Theorem 11.8. *For any $f_0\in L^2_m$, the holds $f_t\to\langle f_0,\phi_1\rangle f_1$ in $L^1_{\phi_1}$ as $t\to\infty$.*

Theorem 11.8 generalizes [248, Thm. 2.18] for the zero inflow condition and [5, Thms. 1.6 & 1.7] for the torus case. It is worth emphasizing that in these papers the longtime convergence is established with exponantial rate (with constructive estimate in [5]). In [248] the proof is based on a representation formula for the associated semigroup S which is proved to have a kernel $p_t \in (L^1 \cap L^\infty \cap C^\infty)(\mathcal{O})$ for any $t > 0$ (see [248, Thms. 2.4 & 2.6] as well as [330, 213, 247]). One then classically deduces that $S_t \in \mathscr{K}(X)$ for any $t > 0$ and $X = L^p$, $p \in [1,\infty]$, or $X = C_0$ (see [248, Thm. 2.18]), and next one may apply Theorem 5.28. We also refer to [193, Thm. 6.8], [299] and [218, 219, 220] for related results.

We follow now a similar approach as in [248, 193]. We start with a series of technical results. Here, we make the additional assumption

$$\varpi^\sharp(x,v) := \sup_{1\le p\le\infty} w_p(x,v) \le \kappa_2 < \infty, \tag{11.54}$$

with

$$\varpi_p := \frac{(2-p)}{p}\frac{\Delta m_p}{m_p} + \frac{2}{p'}\frac{|\nabla m_p|^2}{m_p^2} + c - \frac{1}{p}\frac{\operatorname{div}(b m_p^p)}{m_p^p},$$

and $m_p := \mathscr{M}^{-1+1/p}$.

Lemma 11.9. *For any fixed $\kappa < \kappa_0$ there exists $\varrho_x > 0$, $\varrho_v > 0$ and $\kappa_2 \in \mathbb{R}$ such that defining $\mathcal{A} f := \xi_{\varrho_v}(v)\zeta_{\varrho_x}(x) f$ with $\xi_{\rho_v} \in \mathcal{D}(\mathbb{R}^d)$, $\mathbf{1}_{|v|\le\rho_v} \le \xi_{\rho_v} \le \mathbf{1}_{|v|\le 2\rho_v}$, $\zeta_{\rho_x} \in \mathcal{D}(\Omega)$, $\mathbf{1}_{\delta(x)\ge\rho_x/2} \le \zeta_{\varrho_x} \le \mathbf{1}_{\delta(x)\ge\varrho_x}$, and next $\mathcal{B} := \mathcal{L} - \mathcal{A}$, there hold*

$$\|S_\mathcal{B}(t)\|_{\mathscr{B}(L^2_m)} \lesssim e^{\kappa t}, \quad \forall\, t \ge 0, \tag{11.55}$$

$$\|S_\mathcal{B}(t)\|_{\mathscr{B}(L^p_{m_p})} \lesssim e^{\kappa_2 t}, \quad \forall\, t \ge 0, \ \forall\, p \in (2,\infty]. \tag{11.56}$$

Proof of Lemma 11.9. We first recall from Step 1 of the proof of Theorem 11.3 and (11.30) that

$$\begin{aligned}(\mathcal{L}f,f)_{L^2_m} &= -\int |\nabla f|^2 m^2 - \frac12 \int_\Sigma (\gamma f)^2 m^2 \nu\cdot v + \int f^2 \varpi m^2 \\ &\le -\int |\nabla f|^2 m^2 + \int f^2 \varpi m^2\end{aligned}$$

and, with ψ defined in (11.21),

$$\begin{aligned}(-\mathcal{L}f,f)_{L^2_\psi} &= -\frac12 \int f^2 (v\cdot\nabla_x\psi) - \int f\frac{b}{\langle v\rangle}\cdot\nabla_v f\langle v\rangle\psi + \int \nabla_v(f\psi)\cdot\nabla_v f - \int c f^2\psi \\ &\le -\int f^2 \frac{(\hat v\cdot\nu(x))^2}{\delta(x)^{1/2}} dv dx + C\int (f^2 + |\nabla f|^2).\end{aligned}$$

Defining then $\tilde m := m - \beta\psi$, with $\beta > 0$ small enough, and summing up the two previous estimates, we get

$$(\mathcal{L}f,f)_{L^2_{\tilde m}} \le -\beta \int f^2 \frac{(\hat v\cdot\nu(x))^2}{\delta(x)^{1/2}} - \frac12\int |\nabla f|^2 m^2 + \int f^2(\varpi m^2 + 1).$$

Similarly as in (11.41), we define

$$\mathcal{U} := \{(x,v) \in \mathcal{O};\ \delta(x) > \varrho_x,\ |v| < \varrho_v\},$$

and we observe that

$$\mathcal{U}^c \subset A \cup B \cup C,$$

with

$$A := \{v \in B_{\varrho_v},\ |\hat v\cdot\nu(x)| \le \varepsilon_v\}, \quad B := \{v \in B_{\varrho_v},\ |\hat v\cdot n| \ge \varepsilon_v,\ \delta(x) \le \varrho_x\},$$

for some $\varepsilon_x > 0$, and $C := B^c_{\rho_v}$. We next repeat the proof of (11.42), and we get

$$\int_{\mathcal{U}^c} f^2 m^2 \lesssim (\varrho_v^{d-1}\varepsilon_v)^{2/r'} \int |\nabla_v f|^2 + m(\varrho_v)^2 \frac{\varrho_x^{1/2}}{\varepsilon_v^2}\int f^2 \frac{(\hat v\cdot\nu(x))^2}{\delta(x)^{1/2}} + \frac{1}{\varpi_-(\rho_v)}\int f^2 \varpi_- m^2.$$

Observing that

$$\int f^2(\varpi m^2 + 1) \le \kappa \int f^2 \tilde m^2 + C_\kappa \int_\mathcal{U} f^2 m^2 + C_\kappa \int_{\mathcal{U}^c} f^2 m^2$$

with $C_\kappa := \sup(\varpi + 2 - \kappa)_+ < \infty$, and $\mathcal{A} \ge C_\kappa \mathbf{1}_\mathcal{U}$ for $n := C_\kappa$, altogether, we conclude with

$$(\mathcal{B}f, f)_{L^2_{\tilde m}} \le \kappa \|f\|_{L^2_{\tilde m}}.$$

We then classically deduce that (11.55) holds.

Similarly as for the first estimate and in the proof of [274, Lem. 3.8], for any smooth, rapidly decaying and positive function f, we have

$$\int (\mathcal{L}f) f^{p-1} m_p^p = -\int_\Sigma \frac{(m_p \gamma f)^p}{p} \nu \cdot v - (p-1) \int |\nabla (m_p f)|^2 (m_p f)^{p-2} dx, + \int f^p \varpi_p m_p^p.$$

From Darozès-Guiraud (or Jensen) inequality, we know that the first (boundary) term is nonpositive (see [124] or [272, Rem. 6.4]) and we then classically conclude to (11.56). □

Lemma 11.10. *There exists a finite family $2 = p_0 < p_1 < \cdots < p_k < \infty$ and $\alpha \in (0,1)$ such that such that both $\mathcal{C} = \mathcal{B}, \mathcal{L}$, for any $T > \tau > 0$ and $\mathcal{V} \subset\subset \mathcal{O}$,*

$$\int_\tau^T \|\mathcal{A} S_\mathcal{C}(t) f_0\|_{L^{p_j}_{m_{p_j}}} dt \le C^{p_j}_{p_{j-1}} \|f_0\|_{L^{p_{j-1}}_{m_{p_{j-1}}}}, \quad j = 1 \dots, k, \tag{11.57}$$

$$\sup_{t \in [\tau, T]} \|\mathcal{A} S_\mathcal{B}(t) f_0\|_{L^\infty} \le C^\infty_{p_k} \|f_0\|_{L^{p_k}}, \tag{11.58}$$

$$\sup_{t \in [\tau, T]} \|S_\mathcal{B}(t) f_0\|_{C^\alpha(\mathcal{V})} \le C^\alpha_\infty \|f_0\|_{L^\infty}. \tag{11.59}$$

Proof of Lemma 11.10. For $0 \le f_0 \in L^2_m$, let us denote $f := S_\mathcal{B} f_0$ which thus satisfies the PDE

$$\partial_t f - \mathcal{B} f = s := cf \quad \text{in} \quad \mathcal{D}'((0,T) \times \mathcal{O}).$$

Let us fix two open sets U_i such that $[\tau, T] \times \operatorname{supp}\xi \times \operatorname{supp}\zeta \subset U_0 \subset\subset U_1 \subset\subset (0,T) \times \mathcal{O}$. From [181, Thm. 6] and a covering lemma, there exists a constant $\bar C_0 > 0$ and $p_1 > 2$ such that

$$\|f\|_{L^{p_1}(U_0)} \le \bar C_0 \big(\|f\|_{L^2(U_1)} + \|s\|_{L^2(U_1)} \big).$$

The estimate (11.57) for $j = 1$ then follows from Theorem 11.5 (and the classical underlying energy estimate). On the other hand, [181, Thm. 12] similarly implies that there exists a constant $\bar C_k > 0$ and $p_k \in (p_1, \infty)$ such that

$$\|f\|_{L^\infty(U_0)} \le \bar C_k \big(\|f\|_{L^2(U_1)} + \|s\|_{L^{p_k}(U_1)} \big),$$

and interpolating with the previous estimate, we get

$$\|f\|_{L^{p_j}(U_0)} \le \bar C_{j-1} \big(\|f\|_{L^2(U_1)} + \|s\|_{L^{p_{j-1}}(U_1)} \big), \ \forall j, \ 2 \le j \le k-1.$$

The growth bound (11.56) and the two last estimates imply (11.58) and (11.57) for any $2 \le j \le k-1$. Finally, [181, Thm. 3] similarly implies that there exists a constant $\bar C_{k+1} > 0$ and $\alpha \in (0,1)$ such that

$$\|f\|_{C^\alpha(U_0)} \le \bar C_{k+1} \big(\|f\|_{L^2(U_1)} + \|s\|_{L^\infty(U_1)} \big),$$

from what we deduce (11.59) in the same way. □

Theorem 11.11. *Under the conditions of Theorem 11.6 and the additional assumption* (11.54)*, the conclusion* **(E3$_1$)** *holds in L^2_m with non constructive rate.*

Proof of Theorem 11.11. We introduce the splitting

$$\mathcal{A} g := M \Upsilon_\varepsilon g, \quad \Upsilon_\varepsilon g := \chi_\varepsilon g, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

with $\chi_\varepsilon \in C^2_c(\mathcal{O})$, $\mathbf{1}_{\mathcal{U}_{2\varepsilon}} \le \chi_\varepsilon \le \mathbf{1}_{\mathcal{U}_\varepsilon}$ and $\mathcal{U}_\varepsilon := \{|v| \le 1/\varepsilon, \ \delta(x) > \varepsilon\}$. We next write the iterated Duhamel formulas (with $N := k+2$)

$$S_\mathcal{L} = V + W * S_\mathcal{L},$$

with the usual notations (3.41) for V and W associated to the integer $N := k+2$ and $k \ge 1$ has been introduced in Lemma 11.10. Next for $T > 0$ large, $\tau \in (0,T)$ small and two functions (of operators) a and b, we define the modified convolution operator

$$\begin{cases} (a *_\tau b)(t) := \displaystyle\int_\tau^{t-\tau} a(t-s) b(s)\, ds & \text{if} \ \ t \in [\tau, T-\tau] \\ (a *_\tau b)(t) := 0 \ \ \text{if} \ \ t \in [\tau, T-\tau]^c, \end{cases}$$

(with these notations $*_0 = *$) and by induction $a^{*_\tau 1} := a$, $a^{*_\tau k} := a^{*_\tau (k-1)} *_\tau a$ for $k \geq 2$. With these notations, we define the new splitting

$$S_{\mathcal{L}} = V + K_1^c + K_2^c + K,$$

with

$$K := \Upsilon_\nu W_\tau *_\tau S_{\mathcal{L}}, \quad K_1^c := W * S_{\mathcal{L}} - W_\tau *_\tau S_{\mathcal{L}}, \quad K_2^c := (1-\Upsilon_\nu) W_\tau *_\tau S_{\mathcal{L}},$$

where $W_\tau := (S_{\mathcal{B}}\mathcal{A})^{*_\tau N}$ and $\nu > 0$. For later references, we also define recursively $\Xi_0 := S_{\mathcal{L}}$, $\Xi_\ell := S_{\mathcal{B}}\mathcal{A} *_\tau \Xi_{\ell-1}$ for $\ell \geq 1$, so that $K = \Upsilon_\varepsilon \Xi_N$. The sequel of the proof is split into two steps.

Step 1. On the one hand, we compute

$$\begin{aligned}
\|\Xi_N(T) f_0\|_{L^{p_1}_{m_1}} &\leq \|S_{\mathcal{B}}\|_{L^\infty(\mathscr{B}(L^{p_1}_{m_1}))} \int_\tau^{T-\tau} \Big\| \int_\tau^{t-\tau} \mathcal{A} S_{\mathcal{B}}(t-s) \mathcal{A} \Xi_{k-1}(s) ds f_0 \Big\|_{L^{p_1}_{m_1}} dt \\
&\leq C_T \int_\tau^T \int_\tau^T \|\mathcal{A} S_{\mathcal{B}}(t) \mathcal{A} \Xi_{k-1}(s) f_0\|_{L^{p_1}_m} dt ds \\
&\leq C_T C_2^{p_1} \int_\tau^T \|\mathcal{A} \Xi_{k-1}(s) f_0\|_{L^2_{m_1}} ds,
\end{aligned}$$

and thus

$$\|\Xi_N(T) f_0\|_{L^{p_1}_{m_1}} \leq C_T \|f_0\|_{L^2_m}, \tag{11.60}$$

where we have used (11.56) in the first line, the Fubini theorem in the second line, (11.57) with $j = 1$ in the third line and several times (11.55) in the last line.

For $\kappa < \kappa_0$, we may choose $\varepsilon > 0$ small enough such that (11.55) holds. From the very definition of $\mathcal{A}$ and $S_{\mathcal{B}}$, we may thus fix $\kappa_{\mathcal{B}} \in (\kappa, \kappa_0)$ arbitrary and next $T > 0$ large enough such that $\|V(T)\|_{\mathscr{B}(L^2_m)} \leq \frac{1}{3} e^{\kappa_{\mathcal{B}} T}$. We may next use (11.55) and fix $\tau > 0$ small enough such that

$$\|K_1^c(T)\|_{\mathscr{B}(L^2_m)} \leq \tau C_T \leq \tfrac{1}{3} e^{\kappa_{\mathcal{B}} T}.$$

Last, because of (11.60), we may fix $\nu > 0$ small enough, in such a way that

$$\|K_2^c(T) f_0\|_{L^2_m} \leq \eta(\nu) \|\Xi_N(T) f_0\|_{L^{p_1}_{m_1}} \leq \tfrac{1}{3} e^{\kappa_{\mathcal{B}} T} \|f_0\|_{L^2_m}.$$

The three last estimates together, we have established

$$\|(V + K_1^c + K_1^c)(T)\|_{\mathscr{B}(L^2_m)} \leq e^{\kappa_{\mathcal{B}} T}. \tag{11.61}$$

Step 2. Performing the same kind of computations as for proving (11.60) and in particular using (11.57), we get

$$\begin{aligned}
\int_0^T \|\mathcal{A} \Xi_j(s) f_0\|_{L^{p_{j+1}}_{m_{p_{j+1}}}} ds &\leq \int_0^T \int_\tau^{T-\tau} \|\mathcal{A} S_{\mathcal{B}}(t) \mathcal{A} \Xi_{j-1}(s) f_0\|_{L^{p_{j+1}}_{m_{p_{j+1}}}} dt ds \\
&\leq C_{p_j}^{p_{j+1}} \int_0^T \|\mathcal{A} \Xi_{j-1}(s) f_0\|_{L^{p_j}_{m_{p_j}}} ds,
\end{aligned}$$

for $j = 1, \dots, k$, and with $p_{k+1} := \infty$. Iterating and using (11.57) with $j = 0$, we get

$$\int_0^T \|\mathcal{A} \Xi_k(s) f_0\|_{L^\infty_m} ds \lesssim \|f_0\|_{L^2_m}.$$

Similarly, we may write

$$\begin{aligned}
\sup_{[\tau,T]} \|\mathcal{A} \Xi_{k+1} f_0\|_{L^\infty_m} &\leq \sup_{t \in [\tau,T]} \int_\tau^t \|\mathcal{A} S_{\mathcal{B}}(t-s) \mathcal{A} \Xi_k(s) f_0\|_{L^\infty_m} ds \\
&\leq \sup_{t \in [\tau,T]} \|\mathcal{A} S_{\mathcal{B}}(s)\|_{\mathscr{B}(L^\infty_m)} \int_\tau^T \|\mathcal{A} \Xi_k(s) f_0\|_{L^\infty_m} ds,
\end{aligned}$$

thanks to (11.58), and

$$\begin{aligned}
\|K f_0\|_{C^\alpha(\mathcal{O})} &\leq \int_\tau^{T-\tau} \|S_{\mathcal{B}}(T-s) \mathcal{A} \Xi_{k+1}(s) f_0\|_{C^\alpha(\mathcal{U}_\nu)} ds \\
&\leq C_\infty^\alpha T \sup_{[\tau,T]} \|\mathcal{A} \Xi_{k+1} f_0\|_{L^\infty_m},
\end{aligned}$$

thanks to (11.59). The three last estimates together and the compact support property $\mathrm{supp}\chi_\nu \subset\subset \mathcal{O}$ imply

$$\|Kf_0\|_{C^\alpha \cap L^2_{m_{p_1}}} \lesssim \|f_0\|_{L^2_m}, \quad \forall f_0 \in L^2_m,$$

from what we deduce that $K \in \mathscr{K}(L^2_m)$. We may apply Theorem 5.28 in order to conclude. □

12. A mutation-selection model

In this section, we consider the mutation-selection evolution equation associated to the mutation-selection operator

$$\mathcal{L}f := J * f - W(x)f \tag{12.1}$$

defined on functions $f : \mathbb{R}^d \to \mathbb{R}$, where J is a the mutation kernel, $*$ stands for the convolution operator and W is a confining potential.

12.1. **Almost regular mutation kernel.** We assume that the mutation kernel J is a positive finite measure of $\mathbb{R}^d$ which is lower bounded on a neighborhood of the origin, or in other words

$$0 \le J \in M^1(\mathbb{R}^d), \quad J \ge J_* \mathbf{1}_{B_r}, \tag{12.2}$$

for some constants $J_*, r > 0$. We also assume that the selection potential $W : \mathbb{R}^d \to \mathbb{R}$ is continuous and satisfies

$$W(x) > W(0) = 0, \ \forall\, x \ne 0, \qquad W(x) \to +\infty \text{ as } |x| \to \infty. \tag{12.3}$$

We finally assume the following compatibility condition between mutation and selection: there exist $\beta > 0$ and a bounded Borel set $A \subset \mathbb{R}^d$ such that

$$a := \operatorname*{ess\,inf}_{x \in A_\beta} \int_{x - A_\beta} \frac{J(dz)}{W(x-z)} > 1, \tag{12.4}$$

$$J = J_1 + J_2, \quad J_1 \in C^1_c(\mathbb{R}^d), \quad \kappa_* := \|J_2\|_1 := \int_{\mathbb{R}^d} dJ_2 < \kappa_0 := (a-1)\beta, \tag{12.5}$$

where we use the notation $A_\beta = A \cap \{W \ge \beta\}$. In the sequel, we work in the Banach lattice $X := L^1(\mathbb{R}^d)$.

Theorem 12.1. *Under the above assumptions, we have*

(1) *The first eigentriplet problem* (1.1)-(1.2) *admits a unique solution* $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$ *with the normalization* $\|\phi_1\| = \langle \phi_1, f_1 \rangle = 1$, *and this triplet additionally satisfies* $\lambda_1 \ge \kappa_0$, $0 < f_1 \in L^1_{\langle W \rangle}(\mathbb{R}^d) \cap L^\infty_{\langle W \rangle}(\mathbb{R}^d)$ *and* $0 < \phi_1 \in L^1_{\langle W \rangle}(\mathbb{R}^d) \cap L^\infty_{\langle W \rangle}(\mathbb{R}^d)$.

(2) *Moreover,* $\mathcal{L}$ *generates a semigroup* $S_{\mathcal{L}}$ *on* X *and for any* $f_0 \in X$, *there holds*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t) f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1} \le C e^{-\alpha t} \|f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1}, \tag{12.6}$$

for any $t \ge 0$ *and for some constructive constants* $C \ge 1$, $\alpha > 0$.

Let us comment on the above result.

Remark 12.2.

(1) Assumption (12.4) *is satisfies when* W *is small enough in a neighborhood of the origin. It is for instance satisfied if* $W^{-1} \notin L^1(B_1)$. *That is in particular the case in dimension* $d = 1$ *when* W *is Lipschitz, because of the condition* $W(0) = 0$.

(2) Assume $J(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x)$ *with* $\rho \in C^1_c(\mathbb{R}^d) \cap \mathcal{P}(\mathbb{R}^d)$ *and* $\rho \ge \rho_* \mathbf{1}_{B_1}$, $\rho_* > 0$, *so that* $J = J_1$ *and* $J_2 = 0$, *and* $W = W(|x|)$. *We may observe that for* $\beta > 0$ *and* $\varepsilon > 0$ *small enough*

$$\inf_{\beta \le W(x) < 2\beta} \int_{\beta \le W(y) < 2\beta} \frac{J(x-y)}{W(y)}\, dy =: a \ge \frac{\rho_*}{2\beta} \mathrm{meas}\{\mathbb{R}^d_+ \cap B_1\} > 1,$$

so that (12.4) *holds with* $A := \{W(x) < 2\beta\}$.

(3) Assumption (12.4) *is similar to* [249, Condition (2.3)], *see also* [6, Assumption 2.6] *and the comparison with* [6, Assumption 2.4], *as well as* [75, Condition (3.7)-(3.8)] *and* [77, p. 250, *Note added in proof.*]. *On the other hand, the conditions on* J *are relaxed here since* J *may have singular part in* (12.5).

(4) Optimal conditions linking J and W for the existence of a spectral gap are still unknown. In the recent paper [6], using variational methods in a L^2 framework, the authors obtain a quantified spectral gap and the associated exponential stability when the mutation kernel J is additionally assumed to be symmetric. Up to our knowledge, Theorem 12.1 is the very first result providing a quantified spectral gap for a non-symmetric mutation kernel J.

(5) Condition (12.4) *can be compared to the condition*

$$\bar a := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{J(x-y)}{W(y)} dy < 1,$$

under which no first eigenfunction may exist in X. First, we claim that $\lambda_1 \ge 0$. Indeed, considering $\varepsilon > 0$ and $f_\varepsilon = \mathbf{1}_{B_\varepsilon}$, we have

$$\mathcal{L} f_\varepsilon \ge -\big(\inf_{B_\varepsilon} W\big) f_\varepsilon,$$

so that the condition **(H2)** *holds for $\kappa_0 = -\inf_{B_\varepsilon} W$ for any $\varepsilon > 0$. Since W is continuous and $W(0) = 0$, we deduce that $\lambda_1 \ge 0$ by passing to the limit $\varepsilon \to 0$. Assume now by contradiction that there exists $f_1 \in X_+ \setminus \{0\}$ such that*

$$\lambda_1 f_1 = \mathcal{L} f_1 = J * f_1 - W f_1 \tag{12.7}$$

and define, for any $\varepsilon > 0$, the function $\varphi_\varepsilon(x) = \frac{1}{\varepsilon + W(x)} \in L^\infty(\mathbb{R}^d)$. Testing (12.7) *against φ_ε we get for any $\varepsilon \in (0,1)$*

$$\begin{aligned} 0 \le \lambda_1 \langle f_1, \varphi_1\rangle \le \lambda_1 \langle f_1, \varphi_\varepsilon\rangle &= \iint \frac{J(x-y)}{\varepsilon + W(x)} f_1(y)\,dx\,dy - \int \frac{W(x)}{\varepsilon + W(x)} f_1(x)\,dx \\ &\le \bar a \int f_1 - \int \frac{W(x)}{\varepsilon + W(x)} f_1(x)\,dx, \end{aligned}$$

and passing to the limit $\varepsilon \to 0$ we obtain the contradiction $0 \le \lambda_1\langle f_1, \varphi_1\rangle \le (\bar a - 1)\int f_1 < 0$. However, there always exists a principal eigenvector f_1 in $M^1(\mathbb{R}^d)$, which might have an atom at the origin when $\bar a < 1$, see for instance [77].

The proof of Theorem 12.1 follows from Theorem 2.21, Theorem 4.13 and Theorem 5.16 as a consequence of conditions **(H1)**–**(H5)** that we establish now. Setting $D(\mathcal{L}) := L^1_{\langle W\rangle}(\mathbb{R}^d)$, we observe that $\mathcal{L}$ is an unbounded closed operator with dense domain $D(\mathcal{L})$.

Condition (H1) and (H1′). We define the semigroup

$$S_W(t) f(x) := e^{-W(x)t} f(x), \quad \forall\, f \in L^p,\ p \in [1,\infty],$$

which is clearly a positive semigroup of contractions. We next define $S_\mathcal{L}$ as a bounded perturbation of S_W. It is also positive and it satisfies the growth estimate $\|S_\mathcal{L}(t)\|_{\mathscr{B}(L^p)} \le e^{\|J\|_1 t}$, where we recall that $\|J\|_1$ stands for the L^1 norm or the total variation norm of J. We deduce that **(H1)** holds true with $\kappa_1 := \|J\|_1$ thanks to Lemma 2.2-**(i)**. Multiplying $\mathcal{L} f$ by $\operatorname{sign} f$, for $f \in D(\mathcal{L})$, we immediately get Kato's inequality

$$(\operatorname{sign} f)\mathcal{L} f = (\operatorname{sign} f) J * f - W|f| \le J * |f| - W|f| = \mathcal{L}|f|.$$

Condition (H2). Let us define $f_0 := \frac{1}{W(x)} \mathbf{1}_{A_\beta}$, where A_β is introduced in condition (12.4). We compute

$$\begin{aligned} \mathcal{L} f_0 &= J * \Big(\mathbf{1}_{A_\beta} \frac{1}{W}\Big) - \mathbf{1}_{A_\beta} \ge \Big(J * \Big(\mathbf{1}_{A_\beta} \frac{1}{W}\Big) - 1\Big)\mathbf{1}_{A_\beta} \\ &\ge \Big(\operatorname*{ess\,inf}_{x\in A_\beta} \Big[J * (\mathbf{1}_{A_\beta} \frac{1}{W})\Big] - 1\Big)\mathbf{1}_{A_\beta} \\ &= (a-1)\mathbf{1}_{A_\beta} \ge (a-1)\frac{\beta}{W}\mathbf{1}_{A_\beta} = \kappa_0 f_0, \end{aligned}$$

where in the second equality we have used the very definition of a in assumption (12.4). We conclude that **(H2)** holds thanks to Lemma 2.4-(ii).

Condition (H3). We introduce the splitting

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} f := J_1 * f, \quad \mathcal{B} f := J_2 * f - W(x) f. \tag{12.8}$$

Arguing as in the proof of condition **(H1)**, we see that $\mathcal{B}$ is the generator a positive semigroup in $L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, with growth bound $\omega(S_\mathcal{B}) \le \kappa_*$ and thus $(\alpha - \mathcal{B})$ is invertible for any $\alpha \ge \kappa_0 > \kappa_*$, with

$$\|(\alpha - \mathcal{B})^{-1}\|_{\mathscr{B}(L^p)} \le \frac{1}{\alpha - \kappa_*}. \tag{12.9}$$

Next, observing that

$$(W + \alpha)h = (\alpha - \mathcal{B})h + J_2 * h,$$

for any $h \in \mathcal{D}(\mathcal{L})$ and $\alpha \ge \kappa_0$, we deduce that

$$(W + \alpha)(\alpha - \mathcal{B})^{-1} g = g + J_2 * ((\alpha - \mathcal{B})^{-1} g), \tag{12.10}$$

for any $g \in X$ and $\alpha \ge \kappa_0$. Together with (12.9), we deduce

$$\|(\alpha - \mathcal{B})^{-1} g\|_{L^p_W} \le \|g\|_{L^p} + \|J_2 * ((\alpha - \mathcal{B})^{-1} g)\|_{L^p} \le \frac{\alpha}{\alpha - \kappa_*}\|g\|_{L^p}, \tag{12.11}$$

for any $g \in L^p$ and $\alpha \ge \kappa_0$. Defining $\mathcal{W}(\alpha) := (\alpha - \mathcal{B})^{-1}\mathcal{A}$, we finally deduce from (12.10) the identity

$$\mathcal{W}(\alpha) f = \frac{1}{W + \alpha}\mathcal{A} f + \frac{1}{W + \alpha} J_2 * ((\alpha - \mathcal{B})^{-1}\mathcal{A} f),$$

for any $f \in X$ and $\alpha \ge \kappa_0$. We may then compute

$$\|\mathcal{W}(\alpha) f\|_{L^\infty} \le \frac{1}{\alpha}\|\mathcal{A} f\|_{L^\infty} + \frac{1}{\alpha}\|J_2\|_1 \|(\alpha - \mathcal{B})^{-1}\mathcal{A} f\|_{L^\infty},$$

and together with (12.9) for $p = \infty$ and (12.11), we deduce

$$\|\mathcal{W}(\alpha) f\|_{L^\infty} \le \|J_1\|_\infty \frac{1}{\alpha - \kappa_*}\|f\|_{L^1}, \tag{12.12}$$

for any $f \in X$ and $\alpha \ge \kappa_0$. Starting from the same identity, we prove in a similar way

$$\|\mathcal{W}(\alpha) f\|_{L^\infty_W} \le \|J_1\|_\infty \frac{\alpha}{\alpha - \kappa_*}\|f\|_{L^1}, \tag{12.13}$$

for any $f \in X$ and $\alpha \ge \kappa_0$. As a conclusion and gathering (12.9), (12.11), (12.12) and (12.13), we have established that

$$\mathcal{W}(\alpha) : L^1 \to L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}, \tag{12.14}$$

with uniform bound for any $\alpha \ge \kappa_0$. Observing that $L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle} \subset L^1$ is weakly compact and using Lemma 2.13 with $p = 1$, we deduce that **(H3)** holds. We can actually strengthen the compactness by noticing that $\mathcal{A} : L^1 \to L^1_W \cap W^{1,1}$ is bounded because of assumption (12.5). This ensures that $\mathcal{A} : L^1 \to L^1$ is compact, from what we deduce that $\mathcal{W}(\alpha) : L^1 \to L^1$ is strongly compact for all $\alpha \ge \kappa_0$. We may thus apply Lemma 2.8-(2) to infer that condition **(H3)** holds for both the primal and the dual problems.

Condition (H4). Assume that $\lambda \ge \lambda_1$ and $f \in D(\mathcal{L}) = L^1_{\langle W \rangle}$ satisfy

$$\|f\|_{L^1} = 1, \quad f \ge 0, \quad (\lambda - \mathcal{L}) f \ge 0. \tag{12.15}$$

Denoting $W_R := \inf_{B_R^c} W$, we compute

$$\int_{B_R} f \ge \int_{\mathbb{R}^d} f - \frac{1}{W_R}\int_{B_R^c} f W \ge 1 - \frac{1}{W_R}\|f\|_{L^1_{\langle W \rangle}} \ge 1/2,$$

for $R > 0$ large enough by taking advantage of the fact that $W(x)$ tend to infinity when $|x| \to \infty$. In particular, there exists $x_0^f \in B_R$ such that

$$\int_{B_{r/2}(x_0^f)} f \ge \delta := \frac{1}{2}\Big(\frac{r}{2R}\Big)^d > 0,$$

where we recall that r is defined in (12.2). We deduce that

$$(J * f)(x) \ge J_* \int_{B_{r/2}(x_0^f)} f(y)\, dy \mathbf{1}_{B_{r/2}(x_0^f)}(x) \ge J_* \delta \mathbf{1}_{B_{r/2}(x_0^f)}(x).$$

Using the equation (12.15), we obtain

$$f(x) \geq \frac{(J * f)(x)}{W(x) + \lambda} \geq \frac{J_* \delta}{W[R] + \lambda} \mathbf{1}_{B_{r/2}(x_0^f)}(x),$$

for $W[R] = \sup_{B_R} W$. With that last information and (12.2) again, we have now

$$J * f \geq \frac{J_*}{2^d} \frac{J_* \delta}{W[R] + \lambda} \mathbf{1}_{B_r(x_0^f)},$$

and, iterating the argument, we deduce

$$f \geq \frac{J_*^m}{2^{(m-1)d}(W[R] + \lambda)^{m-1}} \delta \mathbf{1}_{B_{mr/2}(x_0^f)} \geq \bar{\gamma} \mathbf{1}_{B_R},$$

with $\bar{\gamma} = \bar{\gamma}(R) > 0$ for $m = m(R)$ large enough. Choosing R an integer, we have proved that

$$f \geq h_0 := \bar{\gamma}(R) \mathbf{1}_{B_R} + \sum_{n \geq R} \bar{\gamma}(n+1) \mathbf{1}_{B_{n+1} \setminus B_n} > 0. \tag{12.16}$$

That means that the **(H4)** holds, with constructive lower bound.

Condition (H5). Let us consider $f \in L^1_{\langle W \rangle} \backslash \{0\}$ and $\lambda \in \mathbb{C}$ such that (5.16) holds, in particular

$$\mathcal{L}|f| = (\Re e \lambda)|f| \quad \text{and} \quad \mathcal{L}|f| = \Re e(\text{sign} f) \mathcal{L} f. \tag{12.17}$$

The first equality means that $\Re e\lambda$ is an eigenvalue associated to a positive eigenfunction, and Lemma 4.17 then enforces $\Re e\lambda = \lambda_1$. Lemma 4.18 subsequently ensures that $|f| \in (\text{Span}\, f_1)_+ \backslash \{0\}$, and in particular $|f| > 0$. Throwing away the term $W|f|$ in each side of the second identity in (12.17), we have

$$\Re e \frac{\bar{f}}{|f|} (J * f) = J * |f|.$$

Integrating this equation, we get

$$\int_{\mathbb{R}^{2d}} J(x-y) \Re e \Big[|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) \Big] \, dy = 0.$$

From the positivity condition (12.2) on J, we deduce

$$|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) = \Re e \Big[|f(y)| - \frac{\bar{f}(x)}{|f(x)|} f(y) \Big] = 0, \quad \forall\, x, y \in \mathbb{R}^d, \ |x-y| < r,$$

and thus $\bar{f}(x)/|f(x)| = \bar{u}$ for any $x \in \mathbb{R}^d$ for a constant $u \in \mathbb{C}$. That ends the proof of the reverse Kato's inequality **(H5)**.

Proof of theorem 12.1 part (1). We may use Theorem 2.21 in order to establish the existence of a solution $(\lambda_1, f_1, \phi_1) \in (0, +\infty) \times L^1 \times L^\infty$ to the first eigentriplet problem (1.1)-(1.2). From Theorem 4.13 and Theorem 5.16, this solution is unique, $f_1 > 0$, $\phi_1 > 0$, λ_1 is algebraically simple for both $\mathcal{L}$ and $\mathcal{L}^*$ and it is the unique eigenvalue in $\Sigma_+(\mathcal{L})$.
Due to (12.14), we actually have $f_1 \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$. Observing that $\mathcal{L}^*$ is of the same type as $\mathcal{L}$,

$$\mathcal{L}^* \phi = \check{J} * \phi - W \phi, \quad \check{J}(x) := J(-x),$$

and considering the dual problem as a primal problem in L^1, Theorem 2.21 also provides the existence of $\lambda_1^* > 0$ and $0 < \phi_1^* \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$ such that

$$\mathcal{L}^* \phi_1^* = \lambda_1^* \phi_1^*.$$

Because of Remark 4.16, we have in fact $\lambda_1^* = \lambda_1$ and the simplicity of λ_1 then yields that $\text{Span}\, \phi_1^* = \text{Span}\, \phi_1$. This ensures that $\phi_1 \in L^1_{\langle W \rangle} \cap L^\infty_{\langle W \rangle}$ and also that ϕ_1 enjoys the explicit lower bound (12.16). Besides, we can prove

$$\|\phi_1\|_{L^\infty_{\langle W \rangle}} \leq \|J_1\|_{L^1} \frac{\lambda_1}{\lambda_1 - \kappa_*} \|\phi_1\|_{L^\infty} \leq \|J_1\|_{L^1} \frac{\kappa_1}{\kappa_0 - \kappa_*} \|\phi_1\|_{L^\infty}$$

by arguing similarly as for (12.13). □

In order to prove Theorem 12.1 part (2) with constructive constants we use a Doeblin-Harris type argument

Lemma 12.3 (Lyapunov Condition)**.** *Under the above assumptions, for any $T > 0$, there are $\gamma_L \in (0,1)$ and $K > 0$ such that*

$$\|\widetilde{S}_T f\|_{L^1} \le \gamma_L \|f\|_{L^1} + K\|f\|_{\phi_1}.$$

Proof of Lemma 12.3. Writing $f_t = \widetilde{S}_t f = e^{-\lambda_1 t} S_{\mathcal{L}}(t) f$, we have, since $\lambda_1 \ge 0$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |f_t| &\le \|J\|_1 \int_{\mathbb{R}^d} |f_t| - \int_{\mathbb{R}^d} W|f_t| \\ &\le \int_{B_R^c} (\|J\|_1 - W)|f_t| + \frac{\|J\|_1}{\alpha_R}\int_{B_R} |f_t|\phi_1,\end{aligned}$$

for any $R > 0$ and some α_R the bound by below of ϕ_1 in B_R. Choosing R large enough so that $W(x) \ge \|J\|_1 + 1$ for $|x| \ge R$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} |f_t| \le -\int_{\mathbb{R}^d} |f_t| + \frac{\|J\|_1 + 1}{\alpha_R}\int_{\mathbb{R}^d} |f_t|\phi_1.$$

Since

$$\int_{\mathbb{R}^d} |f_t|\phi_1 \le \int_{\mathbb{R}^d} \widetilde{S}_t |f_0|\phi_1 = \int_{\mathbb{R}^d} |f_0|\phi_1,$$

we infer

$$\|\widetilde{S}_t f\| \le e^{-t}\|f\| + \frac{\|J\|_1 + 1}{\alpha_R}(1 - e^{-t})\|f\|_{\phi_1},$$

by Grönwall's lemma. □

Lemma 12.4 (Harris's condition)**.** *Under the assumption above, there exist $\psi_0 \in X'_{++}$, $g_0 \in X_+$ and $T > 0$ such that*

$$S_T f \ge g_0 \langle f, \psi_0\rangle, \quad \forall\, f \in X_+. \tag{12.18}$$

Proof of Lemma 12.4. Step 1. proof of (12.18). From Duhamel's formula (3.9) we have

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S_{\mathcal{L}}.$$

We note that

$$(S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) = \int_0^t S_{\mathcal{B}}(t-s)\mathcal{A}S_{\mathcal{B}}(s) f ds = \int_0^t [\mathcal{A}(f e^{W(x)s})] e^{-W(x)(t-s)} ds.$$

For any $R > r$, $x \in B_R$, it is satisfied that

$$\mathcal{A}(f e^{Ws})(x) = \int_{\mathbb{R}^d} J(x-y) f(y) e^{-W(y)s} dy \ge J_* e^{-W[2R]s}\int_{B_r(x)} f(y)dy$$

with $W[R]$ defined as in the proof of (12.16). Then we get

$$(S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) \ge \mathbf{1}_{B_R}(x) J_* t e^{-W[2R]t}\int_{B_r(x)} f(y)dy.$$

Subsequently, we obtain that

$$S_{\mathcal{B}}\mathcal{A} * (S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) \ge \mathbf{1}_{B_{R-r}}(x)\int_0^t J_* s e^{-W[2R]t}\mathcal{A}\left(\mathbf{1}_{B_R}(x)\int_{B_r(x)} f(y)dy\right) ds,$$

with

$$\mathcal{A}\left(\mathbf{1}_{B_R}(x)\int_{B_r(x)} f(y)dy\right) = \int_{\mathbb{R}^d} J(x-y)\mathbf{1}_{B_R}(y)\int_{B_r(y)} f(z)dzdy \ge J_*\int_{B_r(x)}\int_{B_r(y)} f(z)dzdy.$$

We claim that for all $a \ge r$,

$$\int_{B_r(x)}\int_{B_a(y)} f(z)dzdy \ge |B_{r/4}|\int_{B_{a+r/2}(x)} f(z)dz.$$

Indeed, we deduce

$$\int_{B_r(x)}\int_{B_a(y)} f(z)dzdy = \int_{B_r(x)}\int_{\mathbb{R}^d} \mathbf{1}_{B_a(y)}(z) f(z)dzdy = \int_{\mathbb{R}^d} f(z)\int_{B_r(x)} \mathbf{1}_{B_a(z)}(y)dy\, dz$$

and, since for all $z \in B_{a+r/2}(x)$,

$$B_{\frac{r}{4}}\left(\frac{z-x}{|z-x|}\frac{3r}{4}+x\right) \subset B_r(x) \cap B_a(z),$$

we have

$$\int_{B_r(x)} \mathbf{1}_{B_a(z)}(y)dy \geq |B_{r/4}| \mathbf{1}_{B_{a+r/2}(x)}(z),$$

and consequently,

$$\int_{B_r(x)} \int_{B_a(y)} f(z)dzdy \geq |B_{r/4}| \int_{B_{a+r/2}(x)} f(z)dz.$$

We have obtained that

$$S_{\mathcal{B}}\mathcal{A} * (S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})f(x) \geq \mathbf{1}_{B_{R-r}(x)} J_*^2 t^2/2 e^{-W[2R]t} \int_{B_{r+r/2}(x)} f(y)dy.$$

Iterating the same argument we arrive to

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)} * S_{\mathcal{B}} f(x) \geq \mathbf{1}_{B_{R-nr}}(x) J_*^n \frac{t^n}{n!} e^{-W[2R]t} \int_{B_{r+(n-1)r/2}(x)} f(y)dy.$$

In consequence, for $R = (n+1)r$, we get

$$(S_{\mathcal{B}}\mathcal{A})^{(*n)} * S_{\mathcal{B}} f(x) \geq \mathbf{1}_{B_r}(x) J_*^n \frac{t^n}{n!} e^{-W[2(n+1)r]t} \int_{B_{(n-1)r/2}(0)} f(y)dy.$$

Coming back to the Duhamel formula (3.9), we deduce

$$S_{\mathcal{L}} f(x) \geq \mathbf{1}_{B_r}(x) \sum_{n=2}^{\infty} \frac{(J_* t)^n}{n!} e^{-W[2(n+1)r]t} \int_{B_{(n-1)r/2}} f(y)dy,$$

from where (12.18) follows with

$$\psi_0 := \sum_{n=2}^{\infty} \frac{(J_* T)^n}{n!} e^{-W[2(n+1)r]T} \mathbf{1}_{B_{(n-1)r/2}}$$

and $g_0 := \mathbf{1}_{B_r}$. □

Proof of Theorem 12.1 part (2). Let us consider $A > 0$ and $f \in X_+$ such that $\|f\| \leq A[f]_{\phi_1}$. For any integer $n \geq 1$, we have

$$\begin{aligned} [f]_{\phi_1} &= \int_{B_n} f\phi_1 + \int_{B_n^c} f\phi_1 \leq \alpha_n \langle f, \psi_0 \rangle + \beta_n \|f\| \\ &\leq \alpha_n \langle f, \psi_0 \rangle + \beta_n A [f]_{\phi_1}, \end{aligned}$$

with $\alpha_n = \|\phi_1\|_{L^\infty} / \inf_{B_n} \psi_0$ and $\beta_n = \|\phi_1\|_{L^\infty_{\langle W \rangle}} / \inf_{B_n^c} W$. Choosing n_A such that $\beta_{n_A} A \leq 1/2$, we deduce the constructive estimate

$$[f]_{\phi_1} \leq 2\alpha_{n_A} \langle f, \psi_0 \rangle,$$

and thus that (6.8) holds with $g_A := (2\alpha_{n_A})^{-1} g_0$. Because of the constructive lower bound (12.16) on ϕ_1, we have

$$\langle \phi_1, g_R \rangle \geq (2\alpha_{n_A})^{-1} \langle h_0, g_0 \rangle =: r_A,$$

which provides (6.9) in a quantified way. The two above estimates and the Lyapunov condition established in Lemma 12.3 ensure that we may apply the Harris-Doblin Theorem 6.3 and thus conclude to (12.6) with constructive rate. □

12.2. **A singular mutation kernel.** Here we consider a mutation kernel supported by a set of zero Lebesgue measure, which thus does not satisfy (12.2). The kernel $J \in M^1_+(\mathbb{R}^d)$ is defined for any test function $\varphi \in C_0(\mathbb{R}^d)$ by

$$\langle J, \varphi \rangle = \varepsilon^{-1} \sum_{i=1}^d \int_{\mathbb{R}} \varphi(0, \cdots, 0, x_i, 0, \cdots, 0) J_i(\varepsilon^{-1} x_i) dx_i,$$

where $(J_i)_{1 \le i \le d}$ is a family of L^1 positive kernels on $\mathbb{R}$ and $\varepsilon > 0$ is a variance parameter. The operator $\mathcal{L}$ then reads

$$\mathcal{L} f(x) = \varepsilon^{-1} \sum_{i=1}^d \int_{\mathbb{R}} f(x - z\mathbf{e}_i) J_i(\varepsilon^{-1} z) dz - W(x) f(x),$$

where $\mathbf{e}_i$ is the i-th unit vector of the canonical basis of $\mathbb{R}^d$. This model was recently considered and studied by [350] through a probabilistic approach. It shares similarities with a model of telomere shortening which is under study in [147]. We show that the method developed in the first sections of the present paper allows us to handle this model, under similar yet slightly different assumptions on the J_i and W than in [350]. In particular we consider more general fitness functions W than quadratic ones. More precisely, we assume that W is a continuous function that satisfies (12.3) and

$$\log W(x) = O(|x|^2) \qquad \text{as } |x|^2 := \sum_{i=1}^d x_i^2 \to \infty. \tag{12.19}$$

The kernels J_i are supposed to be centered Gaussian distributions

$$J_i(z) = M_i G_{\sigma_i}(z) := \frac{M_i}{\sigma_i \sqrt{2\pi}} e^{-\frac{z^2}{2\sigma_i^2}},$$

for given masses $(M_i)_{1 \le i \le d} \in (0, +\infty)^d$ and variances $(\sigma_i)_{1 \le i \le d} \in (0, +\infty)^d$. Similarly as in Section 12.1, we work in the Banach lattice $X = L^1(\mathbb{R}^d)$ and we may prove the following result.

Theorem 12.5. *Under the above assumptions, there exists a constructive $\varepsilon_0 > 0$ small enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ the following conclusions hold*

1. *The first eigentriplet problem* (1.1)-(1.2) *admits a unique solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X_+ \times X'_+$ with the normalization $\|\phi_1\| = \langle \phi_1, f_1 \rangle = 1$, and this triplet additionally satisfies $\lambda_1 > 0$, $f_1 > 0$ and $\phi_1 > 0$.*
2. *Moreover, $\mathcal{L}$ generates a semigroup $S_{\mathcal{L}}$ on X and for any $f_0 \in X$, there holds*

$$\|e^{-\lambda_1 t} S_{\mathcal{L}}(t) f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1} \le C e^{-\alpha t} \|f_0 - \langle \phi_1, f_0 \rangle f_1\|_{L^1}, \tag{12.20}$$

for any $t \ge 0$ and for some constructive constants $C, \alpha > 0$.

Remark 12.6. *The assumption of small variance ε in Theorem 12.5 replaces* (12.4)-(12.5) *as a condition which guarantees the strict positivity of κ_0 in the condition* **(H2)**, *and so the strict positivity of λ_1. This property is fundamental for ensuring the existence of f_1 in L^1 and for the existence of a spectral gap. On the contrary, for large values of ε, there cannot exist $f_1 \in L^1$, as it is proved in Remark 12.2-(5). The reason is a concentration phenomenon which creates an atom at the origin for the principal eigenvector when the dispersion due to the mutations is too big. This is already noticed in* [350, Rk. 5.3.1], *and we refer to* [64, 77, 110] *for more details about the singularity of f_1 and the concentration phenomenon.*

For proving Theorem 12.5, we first show that the conditions **(H1)**, **(H2)** and **(H3)** are verified for the dual problem in $L^\infty = X' = (L^1)'$. Then we check that the Harris conditions are satisfied, thus ensuring the existence, uniqueness and exponential stability for the primal problem.
It is worth noticing that since the J_i are symmetric, we have $\mathcal{L}^* = \mathcal{L}$ and the only difference between the primal and dual problems is the Banach lattice in which it is posed.

Condition (H1) and (H1′). With the same proof as in Section 12.1, $\mathcal{L}$ generates a positive semigroup S in L^1 with $\omega(S) \le \|J\|_1$ and satisfies Kato's inequality. We deduce that **(H1)** and

(H1′) are verified for both $\mathcal{L}$ in X and $\mathcal{L}^*$ in X' with

$$\kappa_1 = \|J\|_1 = \sum_{i=1}^d M_i.$$

Condition (H2). In view of condition **(H3)**, we aim at verifying **(H2)** with κ_0 close enough to κ_1. More precisely, we define $\rho \in (0,1]$ the ratio between the geometric and arithmetic means of the masses M_i, namely

$$\rho := \frac{\left(\prod_{i=1}^d M_i\right)^{1/d}}{\frac{1}{d}\sum_{i=1}^d M_i},$$

we set

$$\zeta := \frac{d\prod_{i=1}^d M_i}{2\,\kappa_1^d} = \frac{d^{1-d}\rho^d}{2} \in (0,1/2],$$

and we prove that there exists ε_0 such that if $\varepsilon \in (0,\varepsilon_0)$, then **(H2)** is verified with

$$\kappa_0 = \theta\kappa_1 \qquad \text{with} \quad \theta := (1-\zeta^2)^{1/d} \in (0,1).$$

Let us fix $\eta > 0$ small enough so that

$$1 + (\eta\sigma_i)^2 \le \left(\frac{2}{1+\theta}\right)^2$$

for all $i \in \{1, \cdots, d\}$. We then define

$$f_0(x) = \prod_{j=1}^d G_{\varepsilon/\eta}(x_j),$$

and we compute

$$\begin{aligned}
\frac{\mathcal{L}f_0(x)}{f_0(x)} &= \sum_{i=1}^d M_i \frac{G_{\varepsilon/\eta} * G_{\varepsilon\sigma_i}(x_i)}{G_{\varepsilon/\eta}(x_i)} - W(x) = \sum_{i=1}^d M_i \frac{G_{\varepsilon\sqrt{\eta^{-2}+\sigma_i^2}}(x_i)}{G_{\varepsilon/\eta}(x_i)} - W(x) \\
&= \sum_{i=1}^d \frac{M_i}{\sqrt{1+(\eta\sigma_i)^2}} \exp\left(\frac{\eta^2(\eta\sigma_i)^2}{1+(\eta\sigma_i)^2}\frac{x_i^2}{2\varepsilon^2}\right) - W(x) \\
&\ge \frac{1+\theta}{2}\sum_{i=1}^d M_i \exp\left(\frac{\eta^2(\eta\sigma_i)^2}{1+(\eta\sigma_i)^2}\frac{x_i^2}{2\varepsilon^2}\right) - W(x).
\end{aligned}$$

Due to Assumptions (12.3) and (12.19) on W and using Jensen's inequality, we have

$$W(x) \le \frac{1-\theta}{2}\Big(\min_{1\le i\le d} M_i\Big)\, d\, e^{C|x|^2/d} \le \frac{1-\theta}{2}\sum_{i=1}^d M_i e^{Cx_i^2}$$

for some $C > 0$ large enough. Choosing $\varepsilon_0 > 0$ small enough so that

$$2\varepsilon_0^2 \le \frac{\eta^2(\eta\sigma_i)^2}{(1+(\eta\sigma_i)^2)C}$$

for all $i \in \{1, \cdots, d\}$, we obtain that

$$\frac{\mathcal{L}f_0(x)}{f_0(x)} \ge \theta\sum_{i=1}^d M_i e^{Cx_i^2} \ge \theta\kappa_1 = \kappa_0$$

for any $\varepsilon \in (0,\varepsilon_0]$. By virtue of Lemma 2.4-(ii), this proves the announced result.

Condition (H3) in $X' = L^\infty$. We use the splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{B}\phi = -W\phi$, and we aim at proving that (2.29) holds with $N = d$ in order to apply Lemma 2.19. More precisely, we want to find $\varphi \in L^1$ and $\gamma \in (0,1)$ such that for any $\alpha \ge \kappa_0$, there holds

$$\left\|(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^d\phi\right\|_{L^\infty} \le \gamma\|\phi\|_{L^\infty} + \int_{\mathbb{R}^d}\phi(x)\varphi(x)\,dx \tag{12.21}$$

for all $\phi \in L^\infty_+$. We have

$$\mathcal{R}_{\mathcal{B}}(\alpha)\phi = \frac{\phi}{\alpha + W} \le \frac{\phi}{\kappa_0}$$

and, defining

$$\mathcal{A}_r\phi(x) := d\,\varepsilon^{-d}\int_{\mathbb{R}^d}\phi(x-y)J^{\otimes}(y/\varepsilon)\,dy \qquad \text{with} \quad J^{\otimes}(y) := \prod_{i=1}^d J_i(y_i),$$

we have

$$\mathcal{A}^d = \mathcal{A}_r + \mathcal{A}_s$$

with both $\mathcal{A}_r$ and $\mathcal{A}_s$ positive operators. Positivity ensures that

$$\|\mathcal{A}_r\|_{\mathscr{B}(X')} = \mathcal{A}_r\mathbf{1} = d\prod_{i=1}^d M_i$$

and

$$\|\mathcal{A}_s\|_{\mathscr{B}(X')} = \mathcal{A}_s\mathbf{1} = \|\mathcal{A}^d\| - \|\mathcal{A}_r\| \le \|J\|_1^d - d\prod_{i=1}^d M_i.$$

We deduce that for any $\alpha \ge \kappa_0$,

$$\begin{aligned}\big\|(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^d\phi\big\|_{L^\infty} &\le \kappa_0^{-d}\big\|\mathcal{A}_s\phi\|_{L^\infty} + \kappa_0^{1-d}\big\|\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A}_r\phi\|_{L^\infty}\\ &\le \frac{\kappa_1^d - d\prod_{i=1}^d M_i}{\kappa_0^d}\|\phi\|_{L^\infty} + \kappa_0^{1-d}\Big\|\frac{\mathcal{A}_r\phi}{\kappa_0+W}\Big\|_{L^\infty}.\end{aligned}$$

For any $R>0$ we have

$$\begin{aligned}\frac{\mathcal{A}_r\phi(x)}{\kappa_0+W(x)} &= \frac{d\,\varepsilon^{-d}\mathbf{1}_{B_R}(x)}{\kappa_0+W(x)}\int_{B_R}\phi(x-y)J^{\otimes}(y/\varepsilon)\,dy\\ &\quad+\frac{d\,\varepsilon^{-d}\mathbf{1}_{B_R}(x)}{\kappa_0+W(x)}\int_{B_R^c}\phi(x-y)J^{\otimes}(y/\varepsilon)\,dy + \frac{d\,\varepsilon^{-d}\mathbf{1}_{B_R^c(x)}}{\kappa_0+W(x)}\int_{\mathbb{R}^d}\phi(x-y)J^{\otimes}(y/\varepsilon)\,dy\\ &\le \frac{d\,\varepsilon^{-d}}{\kappa_0}\prod_{i=1}^d\frac{M_i}{\sigma_i\sqrt{2\pi}}\int_{B_{2R}}\phi(y)\,dy + \frac{d}{\kappa_0}\int_{B^c_{R/\varepsilon_0}}J^{\otimes}(y)dy\,\|\phi\|_{L^\infty} + \frac{d\prod_{i=1}^d M_i}{\kappa_0+W_R}\,\|\phi\|_{L^\infty}\\ &\le \kappa_0^{d-1}\int_{\mathbb{R}^d}\phi(x)\varphi_R(y)\,dy + \frac{\eta_R}{\kappa_0}\|\phi\|_{L^\infty},\end{aligned}$$

where

$$\varphi_R = \frac{d\prod_{i=1}^d M_i/\sigma_i}{\sqrt{2\pi}(\varepsilon\kappa_0)^d}\mathbf{1}_{B_{2R}} \quad \text{and} \quad \eta_R = d\int_{B^c_{R/\varepsilon_0}}J^{\otimes}(y)dy + \frac{d\kappa_0}{W_R}\prod_{i=1}^d M_i,$$

and with $W_R = \inf_{B_R^c} W$. We may therefore infer that

$$\big\|(\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^d\phi\big\|_{L^\infty} \le \frac{\kappa_1^d - d\prod_{i=1}^d M_i + \eta_R}{\kappa_0^d}\|\phi\|_{L^\infty} + \langle\phi,\varphi_R\rangle.$$

Since $W(x)\to+\infty$ and $J^{\otimes}(x)\to 0$ as $|x|\to\infty$, we can find R large enough so that

$$\eta_R \le \frac{d}{2}\prod_{i=1}^d M_i = \zeta\kappa_1^d.$$

Recalling that $\kappa_0^d = (1-\zeta)\kappa_1^d$, we then obtain (12.21) with

$$\gamma = \frac{\kappa_1^d - \frac{d}{2}\prod_{i=1}^d M_i}{\kappa_0^d} = \frac{1-\zeta}{1-\zeta^2} = \frac{1}{1+\zeta} < 1 \qquad \text{and} \qquad \varphi = \varphi_R \in L^1.$$

Invoking Lemma 2.19, we deduce that **(H3)** holds true for $\mathcal{L}^* = \mathcal{L}$ in $X' = L^\infty$.

From conditions **(H1)-(H2)-(H3)**, we infer the existence of a solution to the dual problem.

Lemma 12.7. *If $\varepsilon < \varepsilon_0$, where ε_0 is defined in the paragraph about Condition* **(H2)** *above, then there exist $\lambda_1 \ge \kappa_0$ and $\phi_1 \in X'_+$, $\|\phi_1\|_{L^\infty} = 1$, such that $\mathcal{L}^*\phi_1 = \lambda_1\phi_1$. Moreover, $\phi_1 \in L^\infty_W$ and $\langle\phi_1,\varphi\rangle \ge 1-\gamma$.*

Proof. The existence of (λ_1, ϕ_1) follows from applying Theorem 2.21. The equation $\mathcal{L}\phi_1 = \lambda_1 \phi$ readily gives that

$$\|\phi_1\|_{L^\infty_W} \le \|J * \phi_1\|_{L^\infty} + \lambda_1 \|\phi_1\|_{L^\infty} \le \|J\|_1 + \lambda_1,$$

and the estimate $\langle \phi_1, \varphi \rangle \ge 1 - \gamma$ comes from Lemma 2.19 □

We now aim at verifying (6.8), (6.7) and (6.9) in order to apply Theorem 6.3.

Lemma 12.8 (Lyapunov Condition). *Under the above assumptions, for any $T > 0$, there are $\gamma_L \in (0,1)$ and $K > 0$ such that*

$$\|\widetilde{S}_T f\|_{L^1} \le \gamma_L \|f\|_{L^1} + K\|f\|_{\phi_1}.$$

Proof. The proof is exactly the same as for Lemma 12.3 in Section 12.1. □

Lemma 12.9 (Harris's condition). *Under the above assumptions, there exists $\psi_0 \in X'_{++}$, $g_0 \in X_+$ and $T > 0$ such that*

$$S_T f \ge \langle f, \psi_0 \rangle g_0, \quad \forall f \in X_+. \tag{12.22}$$

Proof. We prove the dual version of (12.22), namely

$$S_T \phi \ge \langle \phi, g_0 \rangle \psi_0, \quad \forall \phi \in X'_+, \tag{12.23}$$

where we have used that $S^*_T = S_{\mathcal{L}^*}(T) = S_{\mathcal{L}}(T) = S_T$, since $\mathcal{L}^* = \mathcal{L}$ due to the symmetry of J. The iterated Duhamel formula (3.9) and the positivity of $\mathcal{A}$ and $S_{\mathcal{B}}$ ensure that

$$S_{\mathcal{L}} \ge (S_{\mathcal{B}}\mathcal{A})^{(*d)} * S_{\mathcal{B}}.$$

We start by estimating $(S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})(t)\phi$ for $\phi \ge 0$. Since

$$\begin{aligned}
\mathcal{A}S_{\mathcal{B}}(s)\phi(x) &\ge \varepsilon^{-1} \int_{\mathbb{R}} \phi(x - z\mathbf{e}_1) e^{-sW(x - z\mathbf{e}_1)} J_1(z/\varepsilon)\, dz \\
&\ge \varepsilon^{-1} \int_{x_1-1}^{x_1+1} \phi(x - z\mathbf{e}_1) e^{-sW(x - z\mathbf{e}_1)} J_1(z/\varepsilon)\, dz \\
&\ge \varepsilon^{-1} e^{-sW[|x|+1]} J_1\Big(\frac{|x_1|+1}{\varepsilon}\Big) \int_{x_1-1}^{x_1+1} \phi(x - z\mathbf{e}_1)\, dz,
\end{aligned}$$

where we recall the notation $W[R] = \sup_{B_R} W$, we get

$$(S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})(t)\phi(x) \ge \frac{t}{\varepsilon} e^{-tW[|x|+1]} J_1\Big(\frac{|x_1|+1}{\varepsilon}\Big) \int_{x_1-1}^{x_1+1} \phi(x - z\mathbf{e}_1)\, dz.$$

Using now the part J_2 of J we obtain

$$\begin{aligned}
&\mathcal{A}(S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{B}})(s)\phi(x) \\
&\quad\ge \frac{s}{\varepsilon^2} J_1\Big(\frac{|x_1|+1}{\varepsilon}\Big) \int_{\mathbb{R}} e^{-sW[|x - z_2\mathbf{e}_2|+1]} \int_{x_1-1}^{x_1+1} \phi(x - z_1\mathbf{e}_1 - z_2\mathbf{e}_1)\, dz_1\, J_2(z_2/\varepsilon)\, dz_2 \\
&\quad\ge \frac{s}{\varepsilon^2} J_1\Big(\frac{|x_1|+1}{\varepsilon}\Big) J_2\Big(\frac{|x_2|+1}{\varepsilon}\Big) e^{-sW[|x|+2]} \int_{x_2-1}^{x_2+1} \int_{x_1-1}^{x_1+1} \phi(x - z_1\mathbf{e}_1 - z_2\mathbf{e}_1)\, dz_1 dz_2
\end{aligned}$$

and then

$$\begin{aligned}
&((S_{\mathcal{B}}\mathcal{A})^{(*2)} * S_{\mathcal{B}})(t)\phi(x) \\
&\quad\ge \frac{t^2}{2\varepsilon^2} e^{-tW[|x|+2]} J_1\Big(\frac{|x_1|+1}{\varepsilon}\Big) J_2\Big(\frac{|x_2|+1}{\varepsilon}\Big) \int_{x_2-1}^{x_2+1} \int_{x_1-1}^{x_1+1} \phi(x - z_1\mathbf{e}_1 - z_2\mathbf{e}_1)\, dz_1 dz_2.
\end{aligned}$$

Iterating and using the successive J_i's parts of J we finally get

$$\begin{aligned}
S_{\mathcal{L}}(t)\phi(x) &\ge ((S_{\mathcal{B}}\mathcal{A})^{(*d)} * S_{\mathcal{B}})(t)\phi(x) \\
&\ge \frac{t^d}{d!} \varepsilon^{-d} e^{-tW[|x|+d]} J^{\otimes}\Big(\frac{|x|+1}{\varepsilon}\Big) \int_{[-1,1]^d} \phi(y)\, dy,
\end{aligned}$$

which yields (12.23), and so (12.22), with

$$\psi_0(x) = \frac{T^d}{d!} \varepsilon^{-d} e^{-TW[|x|+d]} J^{\otimes}\Big(\frac{|x|+1}{\varepsilon}\Big)$$

and $g_0 = \mathbf{1}_{[-1,1]^d}$. □

Corollary 12.10. *For $\varepsilon \in (0, \varepsilon_0)$, there exists $f_1 \in X_+$ such that $\mathcal{L} f_1 = \lambda_1 f_1$ with $\langle f_1, \phi_1 \rangle = 1$. Moreover, the exponential convergence* (12.20) *holds for some constructive constants $C \geq 1$ and $\alpha > 0$.*

Proof. Similarly as in the proof of Theorem 12.1 part (2), we can infer from Lemma 12.9 that (6.8) holds with $g_R = C_R g_0$ where $C_R > 0$ is an explicit constant. The Lyapunov condition (6.7) is established in Lemma 12.8, and the positivity condition (6.9) readily follows from the estimate $\langle \phi_1, \varphi \rangle \geq 1 - \gamma$ established in Lemma 12.7. We can thus apply Theorem 6.3 which, together with its attached Remark 6.4, gives the conclusion. □

Proof of Theorem 12.5. It only remains to prove the uniqueness and strict positivity properties. Combining (12.22) and (12.22) with $\phi = g_0$, we get that

$$S_{2T} f = S_T(S_T f) \geq \langle f, \psi_0 \rangle S_T g_0 \geq \Big(\int g_0^2 \Big) \langle f, \psi_0 \rangle \psi_0 = 2^d \langle f, \psi_0 \rangle \psi_0.$$

for all $f \in X_+$. Since $\psi_0 > 0$, this ensures that (4.12) is verified, and then **(H4)** because of point **(4)** in Lemma 4.8. This gives the result of uniqueness and strict positivity by using Theorem 4.13. □

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