Exponential relaxation to self-similarity for the superquadratic fragmentation equation

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April 12, 2016

Abstract

We consider the self-similar fragmentation equation with a superquadratic fragmentation rate and provide a quantitative estimate of the spectral gap.

Keywords: fragmentation equation, self-similarity, exponential convergence, spectral gap, long-time behavior

AMS Class. No. Primary: 35B40, 35P15, 45K05; Secondary: 82D60, 92D25

1   Introduction

The *fragmentation equation*

$$
\begin{cases}
\partial_t\, f(t,x) = \mathcal{F}f(t,x), & t \geq 0, \; x > 0, \\
f(0,x) = f_{\mathrm{in}}(x), & x > 0
\end{cases}
\tag{1}
$$

is a model that describes the time evolution of a population structured with respect to the size x of the individuals.

The key term of the model is the fragmentation operator $\mathcal{F}$, defined as

$$
\mathcal{F}f(x) := \int_x^\infty b(y,x)f(y)\,\mathrm{d}y - B(x)f(x).
\tag{2}
$$

The fragmentation operator quantifies the generation of smaller individuals from a member of the population of size x: the individuals split with a rate $B(x)$ and generate smaller individuals of size $y \in (0,x)$, whose distribution is governed by the kernel $b(x,y)$.

Models involving the fragmentation operator appear in various applications. Among them we can mention crushing of rocks, droplet breakup or combustion [2] which are pure fragmentation phenomena, but also cell division [14], protein polymerization [9] or data transmission protocols on the web [3], for which the fragmentation process occurs together with some "growth" phenomenon.

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In order to ensure the conservation of the total mass of particles which may occur during the fragmentation process, the coefficients $B(x)$ and $b(y, x)$ must be linked through the relation

$$\int_0^y x b(y, x) \, \mathrm{d}x = y B(y). \tag{3}$$

This assumption ensures, at least formally, the mass conservation law

$$\forall t > 0, \qquad \int_0^\infty x f(t, x) \, \mathrm{d}x = \int_0^\infty x f_{\text{in}}(x) \, \mathrm{d}x := \rho_{\text{in}}. \tag{4}$$

Moreover, it is well known that $x f(t, x)$ converges to a Dirac mass at $x = 0$ when $t \to +\infty$. Usually, the various contributions that are available in the literature restrict their attention to coefficients which satisfy the homogeneous assumptions (see [8] for instance)

$$B(x) = x^\gamma, \quad \gamma > 0, \qquad \text{and} \qquad b(y, x) = y^{\gamma - 1} p\left(\frac{x}{y}\right), \tag{5}$$

where $\mathrm{d}\mu(z) := p(z) \, \mathrm{d}z$ is a positive measure supported on $[0, 1]$ which satisfies

$$\int_0^1 z \, \mathrm{d}\mu(z) = 1.$$

These hypotheses guarantee that the relation (3) is verified.

From a mathematical point of view, it is convenient to perform the (mass preserving) self-similar change of variable

$$f(t, x) = (1 + t)^{2/\gamma} g\left(\frac{1}{\gamma} \log(1 + t), (1 + t)^{1/\gamma} x\right),$$

or, by writing g in terms of f,

$$g(t, x) := e^{-2t} f(e^{\gamma t} - 1, e^{-t} x).$$

It allows to deduce that $g(t, x)$ satisfies the *self-similar fragmentation equation*

$$\begin{cases} \partial_t g + \partial_x(x g) + g = \gamma \mathcal{F} g, & t \geq 0, \ x > 0, \\ g(0, x) = f_{\text{in}}(x), & x > 0. \end{cases} \tag{6}$$

Equation (6) belongs to the class of *growth-fragmentation equations* and it admits – unlike Equation (1) – positive steady-states [7, 8, 15].

Denote by G the unique positive steady-state of Equation (6) with normalized mass, i.e. the solution of

$$(x G)' + G = \gamma \mathcal{F} G, \qquad G > 0, \qquad \int_0^\infty x G(x) \, \mathrm{d}x = 1.$$

Then it has been proved (see [8, 12]) that the solution $g(t, x)$ of the self-similar fragmentation equation (6) converges to $\rho_{\text{in}} G(x)$ when $t \to +\infty$.

Coming back to the fragmentation equation (1), this result implies the convergence of $f(t, x)$ to the self-similar solution $(t, x) \mapsto \rho_{\text{in}}(1 + t)^{2/\gamma} G((1 + t)^{1/\gamma} x)$ and hence the convergence of $x f(t, x)$ to a Dirac mass δ_0.

In order to obtain more precise quantitative properties of the previous equation, one can wonder about the rate of convergence of $g(t, x)$ to the asymptotic profile $G(x)$. Many recent

works are dedicated to this question and prove, under different assumptions and with different techniques, an exponential rate of convergence for growth-fragmentation equations [1, 3, 4, 5, 6, 11, 13, 15].

Nevertheless, to our knowledge the only results about the specific case of the self-similar fragmentation equation are those provided by Cáceres, Cañizo and Mischler [4, 5]. They prove exponential convergence in the Hilbert space $L^2((x + x^k)\,\mathrm{d}x)$ for a sufficiently large exponent k in [5], and in the Banach space $L^1((x^m + x^M)\,\mathrm{d}x)$ for suitable exponents $1/2 < m < 1 < M < 2$ in [4]. For proving their results, the authors of the aforementioned articles require the measure p to be a bounded function (from above and below) and the power γ of the fragmentation rate to be less than 2.

The current paper aims to obtain a convergence result for super-quadratic rates, namely when $\gamma \geq 2$. We obtain exponential convergence to the asymptotic state by working in the weighted Hilbert space $L^2(x\,\mathrm{d}x)$, under the following assumptions:

$$\gamma \geq 2 \qquad \text{and} \qquad p(z) \equiv 2. \tag{7}$$

The fact that $p(z)$ is a constant means that the distribution of the fragments is uniform: the probability to get a fragment of size x or x' from a particle of size $y > x, x'$ is exactly the same. Then the condition $\int_0^1 z p(z)\,\mathrm{d}z = 1$ imposes this constant to be equal to 2, meaning that the fragmentation is necessarily binary. Our assumption on p is more restrictive than in [4, 5], but in return we get a stronger result in the sense that we obtain an estimate of the exponential rate. Now we state the main theorem of this paper.

Theorem 1.1. *Let $g_{\mathrm{in}} \in L^1(x\,\mathrm{d}x) \cap L^2(x\,\mathrm{d}x)$ and let $g \in C([0,\infty), L^1(x\,\mathrm{d}x))$ be the unique solution of the self-similar fragmentation equation* (6) *with initial condition g_{in} and with fragmentation coefficients satisfying* (5) *and* (7)*, that is*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \qquad \text{and} \qquad b(y,x) = 2y^{\gamma-1}.$$

Then the following estimate holds:

$$\|g(t,\cdot) - \rho_{\mathrm{in}} G\|_{L^2(x\,\mathrm{d}x)} \leq \|g_{\mathrm{in}} - \rho_{\mathrm{in}} G\|_{L^2(x\,\mathrm{d}x)}\, e^{-t}, \qquad t \geq 0.$$

2   Preliminaries

Define the suitable weighted spaces

$$\dot{L}_k^p := L^p(\mathbb{R}^+, x^k\,\mathrm{d}x) \quad \text{for } p \geq 1,\ k \in \mathbb{R}, \qquad \text{and} \qquad \dot{W}_1^{1,1} := W^{1,1}(\mathbb{R}^+, x\,\mathrm{d}x).$$

For $u \in \dot{L}_1^1$ we denote moreover by

$$M(x) := \int_0^x y u(y)\,\mathrm{d}y$$

the primitive of $xu(x)$ which vanishes at $x = 0$.

Now we recall the following existence and uniqueness result of a solution to the fragmentation equation, easily deduced from [8], Theorems 3.1-3.2 and Lemma 3.4:

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Theorem [8]. *If*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \quad \text{and} \quad b(y, x) = 2y^{\gamma-1},$$

for any $f_{\text{in}} \in \dot{L}_1^2 \cap \dot{L}_1^1$, *there exists a unique solution* $f \in C([0, \infty); \dot{L}_1^1) \cap L^1_{loc}([0, \infty); \dot{L}_{1+\gamma}^1)$ *to the fragmentation equation (1) such that the mass conservation (4) is satisfied. If, moreover,* $f_{\text{in}} \in \Xi := \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$, *the associated solution g to the self-similar fragmentation equation is such that*

$$(g(t, \cdot))_{t \geq 0} \text{ is uniformly bounded in } \Xi.$$

In the following lemma we give some useful properties of the set

$$\Xi = \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$$

and of the subset

$$\Xi_0 := \left\{ u \in \Xi, \ \int_0^\infty x u(x) \, \mathrm{d}x = 0 \right\}.$$

Lemma 2.1. *The set* $\Xi = \dot{L}_{1+\gamma}^1 \cap \dot{W}_1^{1,1}$ *satisfies*

$$G \in \Xi, \quad \Xi \subset C(0, \infty) \cap L^1 \cap \dot{L}_1^2 \cap \dot{L}_{1+\gamma}^2 \quad \text{and}$$

$$\forall u \in \Xi, \quad \lim_{x \to 0} x u(x) = \lim_{x \to +\infty} x u(x) = 0.$$

Moreover for any function $u \in \Xi_0$ the following inequality holds:

$$\forall x > 0, \quad |M(x)| \leq x^{-\gamma} \|u\|_{\dot{L}_{1+\gamma}^1}.$$

Proof. The fact that the steady-state G belongs to Ξ is a consequence of the estimates in [1]. In the case when $p(z) \equiv 2$ it can also be deduced from the explicit formula (see [7])

$$G(x) = \frac{\gamma}{\Gamma(2/\gamma)} e^{-x^\gamma}$$

where Γ is the Euler Gamma function.

For $\gamma \geq 0$ and $u \in \Xi$ such that $\int_0^\infty x u(x) \, \mathrm{d}x = 0$ we can write for $x > 0$

$$|M(x)| = \left| -\int_x^\infty y u(y) \, \mathrm{d}y \right| \leq x^{-\gamma} \int_0^\infty y^{1+\gamma} |u(y)| \, \mathrm{d}y.$$

$\square$

3 Proof of the main theorem

Define the self-similar fragmentation operator $\mathcal{L}u := -(xu)' - u + \gamma \mathcal{F}u$ and denote by

$$(u, v) := \int_0^\infty x u(x) v(x) \, \mathrm{d}x$$

the canonical scalar product in $\dot{L}_1^2$. Theorem 1.1 is a consequence of the following result.

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Theorem 3.1. *Under Assumptions* (5) *and* (7), *i.e. for*

$$B(x) = x^\gamma, \quad \gamma \geq 2 \qquad and \qquad b(y,x) = 2y^{\gamma-1},$$

we have

$$\forall u \in \Xi_0, \qquad (u, \mathcal{L}u) \leq -\|u\|_{\dot{L}_1^2}^2.$$

Proof. Using Lemma 2.1 we can deduce, for $u \in \Xi_0$,

$$(u, (xu)') = \int_0^\infty xu(x)\big(xu(x)\big)' \, \mathrm{d}x = \frac{1}{2} \int_0^\infty \big((xu(x))^2\big)' \, \mathrm{d}x = 0$$

and

$$
\begin{aligned}
(u, \mathcal{F}u) &= 2 \int_0^\infty xu(x) \int_x^\infty y^{\gamma-1} u(y) \, \mathrm{d}y \, \mathrm{d}x - \int_0^\infty x^{\gamma+1} u^2(x) \, \mathrm{d}x \\
&= 2 \int_0^\infty x^{\gamma-1} u(x) \int_0^x y u(y) \, \mathrm{d}y \, \mathrm{d}x - \int_0^\infty x^{\gamma+1} u^2(x) \, \mathrm{d}x \\
&= 2 \int_0^\infty x^{\gamma-2} M'(x) M(x) \, \mathrm{d}x - \int_0^\infty x^{\gamma+1} u^2(x) \, \mathrm{d}x \\
&= -(\gamma-2) \int_0^\infty x^{\gamma-3} M^2(x) \, \mathrm{d}x - \int_0^\infty x^{\gamma+1} u^2(x) \, \mathrm{d}x \leq 0.
\end{aligned}
$$

$\square$

Proof of Theorem 1.1. Assume first that $g_{\mathrm{in}} \in \Xi$. From Theorem 3.1 we obtain the differential inequality

$$\frac{d}{dt} \|g(t,\cdot) - \rho_{\mathrm{in}} G\|_{\dot{L}_1^2} \leq -\|g(t,\cdot) - \rho_{\mathrm{in}} G\|_{\dot{L}_1^2}$$

which gives the result. Then we may remove the additional assumption $g_{\mathrm{in}} \in \Xi$. $\square$

4   Conclusion

We have proved a spectral gap result for the self-similar fragmentation operator $\mathcal{L}$ with a superquadratic fragmentation rate $B(x)$. More precisely we have obtained that this spectral gap is larger than 1. This is a new result concerning the long-time behaviour of the fragmentation equation (1). It also allows to extend the results obtained in [10] for non-linear growth-fragmentation equations.

Acknowledgement

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