Optimal growth for linear processes with affine control

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Abstract

We analyse an optimal control with the following features: the dynamical system is linear, and the dependence upon the control parameter is affine. More precisely we consider $\dot{x}_\alpha(t) = (G+\alpha(t)F)x_\alpha(t)$, where G and F are 3×3 matrices with some prescribed structure. In the case of constant control $\alpha(t) \equiv \alpha$, we show the existence of an optimal Perron eigenvalue with respect to varying α under some assumptions. Next we investigate the Floquet eigenvalue problem associated to time-periodic controls $\alpha(t)$. Finally we prove the existence of an eigenvalue (in the generalized sense) for the optimal control problem. The proof is based on the results by [Arisawa 1998, Ann. Institut Henri Poincaré] concerning the ergodic problem for Hamilton-Jacobi equations. We discuss the relations between the three eigenvalues. Surprisingly enough, the three eigenvalues appear to be numerically the same.

Keywords. Optimal control; Infinite horizon; Hamilton-Jacobi equation; Ergodic problem; Non coercive hamiltonian

1 Introduction

We aim at optimally controlling the following 3-dimensional system in the long-time horizon,

$$\begin{cases} \dot{x}_\alpha(t) = (G + \alpha(t)F)\, x_\alpha(t)\,, \\ x_\alpha(0) = x \in (\mathbb{R}_+)^3 \setminus \{0\}\,, \end{cases} \tag{1.1}$$

where $G, F \in \mathcal{M}_3(\mathbb{R})$ are matrices with nonnegative off-diagonal entries, and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonnegative control parameter. There is no running reward. Let $T > 0$ be the final time. The final reward is the linear function $\langle m, x_\alpha(T)\rangle$, where $m \in (\mathbb{R}_+)^3 \setminus \{0\}$ is in the kernel of F^T: $m^T F = 0$.

We investigate the asymptotic behaviour of the optimal reward when $T \to +\infty$ in the three different regimes: α is constant (constant control), α is periodic (periodic control), and α is any measurable function from $(0,T)$ to $[a,A]$, where a, A are given bounds (optimal control).

Because of the linear structure of system (1.1), we expect an exponential growth of the final reward. This motivates the introduction of the following renormalized reward:

$$r_\alpha(T,x) = \frac{1}{T}\log\langle m, x_\alpha(T)\rangle\,.$$

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When α is a constant (resp. periodic) control, it is well-known that $r_\alpha(T,x)$ converges to the Perron (resp. Floquet) eigenvalue $\lambda_P(\alpha)$ (resp. $\lambda_F(\alpha)$) of system (1.1) as $T \to +\infty$. Hence, the best control is obtained by optimizing the Perron (resp. Floquet) eigenvalue. In the case of optimal control, we define the best possible reward as follows,

$$V(T,x) = \sup_\alpha \{\langle m, x_\alpha(T)\rangle \ : \ \dot{x}_\alpha(t) = (G+\alpha(t)F)x_\alpha(t) \ , \ x_\alpha(0) = x\} \ ,$$

where the supremum is taken over all measurable control functions $\alpha : (0,T) \to [a,A]$. We can resolve this optimal control problem using the dynamic programming principle. This yields a Hamilton-Jacobi equation for the value function

$$v(s,x) = \sup \{\langle m, x_\alpha(T)\rangle \ : \ \dot{x}_\alpha(t) = (G+\alpha(t)F)x_\alpha(t) \ , \ x_\alpha(s) = x\} \ ,$$

defined for intermediate times $0 < s < T$. Namely it is solution to the following Hamilton-Jacobi-Bellman equation in the sense of viscosity solutions [5],

$$\frac{\partial}{\partial s} v(s,x) + \widetilde{H}(x, D_x v(s,x)) = 0 \, ,$$

where the Hamiltonian is given by $\widetilde{H}(x,p) = \max_{\alpha\in[a,A]} \{\langle (G+\alpha F)x, p\rangle\}$ (see Section 3 for details).

Under some assumptions we prove the following ergodic result: there exists a constant λ_{HJ} such that,

$$\forall x \in (\mathbb{R}_+)^3 \setminus \{0\} \qquad \lim_{T\to+\infty} \frac{1}{T} \log V(T,x) = \lambda_{HJ} \, . \tag{1.2}$$

This constant λ_{HJ} is the analog of the Perron and Floquet eigenvalues in the case of optimal control. We have obviously,

$$\sup_\alpha \lambda_P(\alpha) \le \sup_\alpha \lambda_F(\alpha) \le \lambda_{HJ} \, .$$

Interestingly, numerical simulations show that these three eigenvalues may coincide (see Section 4 for a discussion).

Motivations and running example

From a theoretical viewpoint, the convergence result (1.2) is known as the ergodic problem for Hamilton-Jacobi equations. It appears in homogenization problems [22]. In this case the constant λ is called the effective hamiltonian [7, 16]. It can be interpreted as a nonlinear Perron eigenproblem associated with an eigenvector $\overline{u}$ which solves the following stationary Hamilton-Jacobi equation in the viscosity sense,

$$-\lambda + H(y, D_y \overline{u}) = 0 \, , \quad y \in Y \tag{1.3}$$

where $H : Y \times \mathbb{R}^n \to \mathbb{R}_+$ denotes the hamiltonian. It also appears in weak KAM theory for lagrangian dynamical systems [17, 18]. A natural way to attack this issue is to consider the following stationary Hamilton-Jacobi equation with small parameter $\epsilon > 0$,

$$-\epsilon u_\epsilon + H(x, D_x u_\epsilon) = 0 \, .$$

The asymptotic behaviour of u_ϵ, as $\epsilon \to 0$, has been investigated in several works [22, 12, 17, 25, 9, 8, 21, 6, 1, 13]. In many cases the set Y is assumed to be compact, and the Hamiltonian $H(y,p)$ is assumed to be coercive: $H(\cdot,p) \to +\infty$ as $|p| \to +\infty$. Under these assumptions it can

be proven that the function ϵu_ϵ converges to a constant λ (uniquely determined) [22, 12, 17]. Moreover the function u_ϵ converges uniformly, up to extraction, to some lipschitz function $\overline{u}$, solution of (1.3). However the function $\overline{u}$ is generally not unique. The question of convergence of u_ϵ towards $\overline{u}$ (modulo a large constant) has been investigated in [25, 9, 8].

In the context of optimal control, the coercivity of the hamiltonian $H(y,p) = \max_{\alpha\in\mathcal{A}}\{\langle b(y,\alpha), p\rangle\}$ is guaranteed under the hypothesis of uniform controllability [12, 5]: there exists a constant $\mu > 0$ such that

$$\forall y \in \overline{Y} \quad B(0,\mu) \subset \overline{\text{convex hull}}\{b(y,\alpha) \mid \alpha \in \mathcal{A}\}\,.$$

This hypothesis ensures that any two points $y, y' \in Y$ can be connected with some control $\alpha(t)$ within a time $T = O(|y - y'|)$. This yields equicontinuity of the family $(u_\epsilon)_{\epsilon>0}$, and thus compactness.

The criterion of uniform controllability is not verified in our case, hence the hamiltonian is not coercive (see Remark 3.1 below). Different approaches have been developped to circumvent the lack of coercivity. Several works rely on some partial coercivity [21, 6, 1]. In [4] the authors introduce a non-resonance condition which yields ergodicity. This condition is restricted to hamiltonian with separated variables $H(y,p) = H(p) + V(y)$. In [13] the author extends this result to the case of a non-convex hamiltonian $H(p)$, in two dimensions of space. In the context of optimal control, Arisawa [2, 3] has shown the equivalence between ergodicity and the existence of a stable subset $Z \subset Y$ which attracts the trajectories. It is required in addition that the restriction of the system to Z is controllable. The present work follows the latter approach.

From a modeling viewpoint, this work is motivated by the optimization of some experimental protocol for polymer amplification, called Protein Misfolding Cyclic Amplification (PMCA) [26]. In the system (1.1) the matrix G represents the growth of polymers in size, whereas F is the fragmentation of polymers into smaller pieces. The vector m encodes the size of the polymers. We restrict to dimension 3 for the sake of simplicity, *i.e.* three possible sizes for the polymers (small, intermediate, large). The orthogonality relation $m^T F = 0$ accounts for the conservation of the total size of polymers by fragmentation.

We now give a class of matrices which will serve as an example all along the paper. It is a simplification of the discrete growth-fragmentation process introduced in [23] to model Prion proliferation. We make the following choice:

$$G = \begin{pmatrix} -\tau_1 & 0 & 0 \\ \tau_1 & -\tau_2 & 0 \\ 0 & \tau_2 & 0 \end{pmatrix} \qquad \text{and} \qquad F = \begin{pmatrix} 0 & 2\beta_2 & \beta_3 \\ 0 & -\beta_2 & \beta_3 \\ 0 & 0 & -\beta_3 \end{pmatrix}. \tag{1.4}$$

Here $\tau_1 > 0$ denotes the rate of increase from small to intermediate polymers, and $\tau_2 > 0$ from intermediate to large polymers. The $\beta_i > 0$ denote the fragmentation rates which distribute larger polymers into smaller compartments, keeping the total size constant. The size vector is $m = (1\ 2\ 3)^T$. Finally, the rate $\alpha(t)$ accounts for *sonication*, *i.e.* externally driven intensity of fragmentation.

The continuous version of the baby model (1.1)-(1.4) consists in the following size-structured PDE with variable coefficients [20, 11, 15]

$$\partial_t f(t,\xi) + \partial_\xi(\tau(\xi) f(t,\xi)) = \alpha(t)\left(2\int_\xi^\infty \beta(\zeta)\kappa(\xi,\zeta) f(t,\zeta)\,d\zeta - \beta(\xi) f(t,\xi)\right). \tag{1.5}$$

Here, $f(t,\xi)$ represents the density of protein polymers of size $\xi > 0$ at time t. The transport term accounts for the growth of polymers in size, whereas the r.h.s is the fragmentation operator. The final reward is the total mass of polymers, namely $\int_0^\infty \xi u(T,\xi)\,d\xi$ (see [19] for more details). The Perron eigenvalue problem for (1.5) has been investigated in [15, 10].

Main assumptions

(H1) We assume that the matrices G and F are both reducible. We assume that the matrix $G + \alpha F$ is irreducible for all $\alpha > 0$.

From the Perron-Frobenius Theorem there exists a simple dominant eigenvalue $\lambda_P(\alpha)$ and a positive left- (resp. right-) eigenvector e_α (resp. ϕ_α) such that:

$$\begin{cases} (G + \alpha F)e_\alpha = \lambda_P(\alpha)e_\alpha, \quad e_\alpha > 0\,, \\ \phi_\alpha^T(G + \alpha F) = \lambda_P(\alpha)\phi_\alpha^T, \quad \phi_\alpha > 0\,. \end{cases} \tag{1.6}$$

We choose the following normalizations:

$$\langle m, e_\alpha \rangle = 1\,, \quad \langle \phi_\alpha, e_\alpha \rangle = 1\,.$$

We denote by $(\lambda_1(\alpha), \lambda_2(\alpha), \lambda_3(\alpha))$ the eigenvalues of the matrix $G + \alpha F$, where $\lambda_1(\alpha) \in \mathbb{R}$ is the dominant eigenvalue $\lambda_P(\alpha)$. We shall use repeatedly the spectral gap property of $G + \alpha F$. We denote $\mu_\alpha = \min_{i=2,3}(\lambda_1 - \Re(\lambda_i))$, where $(\lambda_i)_{i=2,3}$ are the two other (possibly complex) eigenvalues of $G + \alpha F$. By continuity and compactness, μ_α is uniformly strictly positive on compact intervals of $(0, +\infty)$.

(H2) We assume that the Perron eigenvalue $\lambda_P(\alpha)$ is bounded admits a global maximum as $\alpha \in (0, +\infty)$. This maximum is attained at $\alpha = \alpha^*$. We assume that the bounds a, A are such that $a < \alpha^* < A$.

We will show in the next Section that this assumption is satisfied for the example (1.4) provided that $\tau_2 > 2\tau_1$. This condition can be justified heuristically [10]. In fact, when growth of intermediate polymers is fast, it is interesting to have a significant fraction of intermediate polymers in the population in order to optimally increase the total size of the population. On the contrary, when $\alpha \to +\infty$ then the eigenvector e_α converges towards $(1\ 0\ 0)^T$ because fragmentation is very large.

The matrix G possesses a nonnegative eigenvector $e_0 = \lim_{\alpha \to 0} e_\alpha$, associated to a dominant eigenvalue $\lambda_P(0)$. On the other hand, from Hypothesis **(H2)** and (1.6) we deduce that e_α converges to a nonnegative eigenvector e_∞ as $\alpha \to +\infty$, with $e_\infty \in \ker F$.

(H3) We assume that both the eigenvectors e_0 and e_∞ have at least one zero coordinate.

This Hypothesis is compatible with the reducibility of both G and F. However since $G + \alpha F$ is irreducible for $\alpha > 0$, we have $e_0 \neq e_\infty$. In fact $e_\alpha > 0$ is the unique nonnegative eigenvector up to a multiplicative constant. Furthermore, e_∞ is not an eigenvector for G. In particular we have $Ge_\infty \neq 0$. In the case of the running example (1.4), we have $e_0 = (0\ 0\ 1/3)^T$ and $e_\infty = (1\ 0\ 0)^T$.

(H4)-(H5) We assume two technical conditions, related to the dynamics of the trajectories of (1.1) projected on the simplex $\mathcal{S} = \{y \geq 0 \ : \ \langle m, y \rangle = 1\}$. We refer to Section 3.2 for the statement of these conditions.

These two technical conditions are satisfied for the running example (1.4). It is not clear whether these two conditions are necessary or not for the validity of our result. We refer to Section 4 for a discussion.

In Section 2, we give conditions on the running example to ensure the existence of an optimal constant control which maximizes the Perron eigenvalue. Then we consider periodic controls

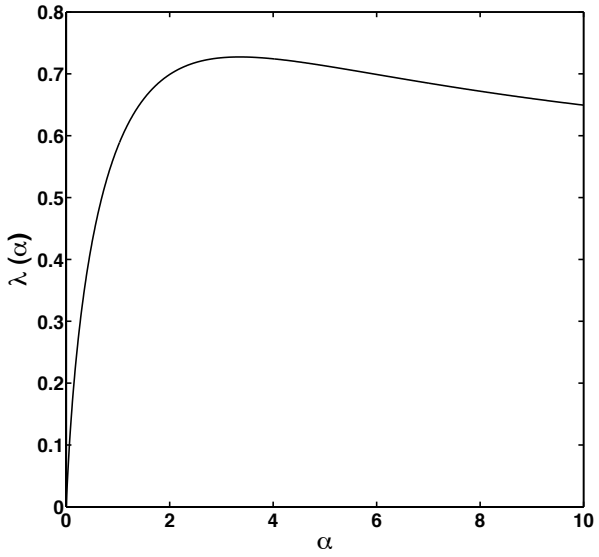


Figure 1: Dominant eigenvalue of the matrix $G + \alpha F$ as a function of the control parameter α. The function achieves a maximum for $\alpha = \alpha^* \in (0, +\infty)$.

and we investigate the variations of the Floquet eigenvalue around this best constant control. In Section 3, we turn to the full optimal control problem for which the control $\alpha(t)$ is any measurable function taking values in $[a, A]$. We prove the main result of this paper which is the convergence of the best reward when $T \to \infty$ (1.2). The technique consists in solving an ergodic problem for the Hamilton-Jacobi-Bellman equation. Finally we give numerical evidences that the ergodicity constant λ_{HJ} coincides with the best Perron eigenvalue in the case of the running example (1.4). Finally we discuss some possible ways to remove each of the assumptions **(H1-5)**.

2 Optimizing Perron and Floquet eigenvalues

We begin with a constant control $\alpha > 0$. Since the eigenvalue $\lambda_P(\alpha)$ is simple, we have the following asymptotic behaviour:

$$\lim_{t \to +\infty} x_\alpha(t) e^{-\lambda_P(\alpha) t} = \langle \phi_\alpha, x \rangle e_\alpha \,. \tag{2.1}$$

Plugging this convergence property into our optimization problem, we obtain the expansion

$$r_\alpha(T, x) = \lambda_P(\alpha) + \frac{1}{T} \log \langle \phi_\alpha, x \rangle + o\left(\frac{1}{T}\right) .$$

Thus in the class of constant controls, our optimization problem reduces to maximizing the Perron eigenvalue. The following Proposition gives an answer to this problem in the case of the running example (1.4) (see also Figure 1).

Proposition 2.1. *There exists a maximal eigenvalue $\lambda_P(\alpha^*)$ for some $\alpha^* \in (0, +\infty)$ if and only if $\tau_2 > 2\tau_1$. Furthermore we have the following alternative:*

- *either $\tau_2 \le 2\tau_1$ and $\lambda_P(\alpha)$ increases from 0 to τ_1,*
- *or $\tau_2 > 2\tau_1$ and $\lambda_P(\alpha)$ first increases from 0 to $\lambda_P(\alpha^*)$ and then decreases to τ_1.*

Proof. The characteristic polynomial of $G + \alpha F$ is

$$P(X) = X^3 + (\tau_1 + \tau_2 + \alpha(\beta_2 + \beta_3))\, X^2 + \big(\tau_1\tau_2 + \alpha\tau_1(\beta_3 - \beta_2) + \alpha^2\beta_2\beta_3\big)\, X - \alpha\tau_1\tau_2\beta_3 - \alpha^2\tau_1\beta_2\beta_3\,.$$

The Perron eigenvalue $\lambda_P(\alpha)$ is the largest zero of this polynomial, so $P(\lambda_P(\alpha)) = 0$ and, dividing by α^2 we get that that $\lim_{\alpha\to+\infty} \lambda_P(\alpha) = \tau_1$. We easily check that $P'(\lambda) > 0$ for $\lambda > \tau_1$, and we evaluate

$$P(\tau_1) = 2\tau_1^2(\tau_1 + \tau_2) + \alpha\tau_1\beta_3(2\tau_1 - \tau_2)\,.$$

We have the following alternative: either $\tau_2 < 2\tau_1$ and $P(\tau_1) > 0$ for all $\alpha > 0$, or $\tau_2 > 2\tau_1$ and $P(\tau_1) < 0$ for large α. In the former case, we have $\lambda_P(\alpha) < \tau_1$ for all $\alpha > 0$. In the latter case, we have $\lambda_P(\alpha) > \tau_1$ for large α. On the other hand we have $\lambda_P(0) = 0$. Therefore, the condition $\tau_2 > 2\tau_1$ is necessary and sufficient for λ_P to reach a global maximum.

Differentiating the relation $P(\lambda_P(\alpha)) = 0$, we get that any critical point α such that $\lambda_P'(\alpha) = 0$ satisfies

$$(\beta_2 + \beta_3)\lambda_P(\alpha)^2 + \tau_1(\beta_3 - \beta_2)\lambda_P(\alpha) + 2\alpha\beta_2\beta_3\lambda_P(\alpha) = \tau_1\tau_2\beta_2\beta_3 + 2\alpha\tau_1\beta_2\beta_3\,. \tag{2.2}$$

Differentiating twice the relation $P(\lambda_P(\alpha)) = 0$, we get that any such critical point α satisfies

$$\lambda_P''(\alpha) = \frac{2\beta_2\beta_3(\tau_1 - \lambda_P(\alpha))}{3\lambda_P(\alpha)^2 + 2(\tau_1 + \tau_2 + \alpha(\beta_2 + \beta_3))\lambda_P(\alpha) + \tau_1\tau_2 + \alpha\tau_1(\beta_3 - \beta_2) + \alpha^2\beta_2\beta_3}.$$

We claim that the denominator is always positive, despite the possible negative terme $\alpha\tau_1(\beta_3 - \beta_2)$. In fact we can factorize by $\alpha\beta_2$: $\alpha\beta_2\,(\lambda_P(\alpha) - \tau_1) \ge 0$ if $\lambda_P(\alpha) \ge \tau_1$. In the other case $\lambda_P(\alpha) \le \tau_1$, we use the relation (2.2) to get

$$\begin{aligned} 2\alpha(\beta_2 + \beta_3)\lambda_P(\alpha) + \alpha\tau_1(\beta_3 - \beta_2) + \alpha^2\beta_2\beta_3 &\ge \frac{\alpha}{\lambda_P(\alpha)}\,(\tau_1\tau_2\beta_2\beta_3 + 2\alpha\tau_1\beta_2\beta_3) - \alpha^2\beta_2\beta_3 \\ &\ge \frac{\alpha^2}{\lambda_P(\alpha)}\beta_2\beta_3\,(2\tau_1 - \lambda_P(\alpha)) \ge 0\,. \end{aligned}$$

In conclusion, $\lambda_P''(\alpha)$ has the same sign as $\tau_1 - \lambda_P(\alpha)$ so $\lambda_P(\alpha)$ can be a local minimum only if $\lambda_P(\alpha) \le \tau_1$ and a local maximum only if $\lambda_P(\alpha) \ge \tau_1$. The alternative announced in the proposition follows. □

Next we consider a periodic control $\alpha(t)$ with period $\theta > 0$. There exists a Floquet eigenvalue $\lambda_F(\alpha)$ and periodic eigenvectors $e_\alpha(t)$, $\phi_\alpha(t)$ such that

$$\begin{cases} \dfrac{d}{dt}e_\alpha(t) + \lambda_F(\alpha)e_\alpha(t) = (G + \alpha(t)F)e_\alpha(t)\,, \\ \dfrac{d}{dt}\phi_\alpha(t) + \lambda_F(\alpha)\phi_\alpha(t) = \phi_\alpha(t)(G + \alpha(t)F)\,. \end{cases}$$

These eigenfunctions are unique after normalization,

$$\frac{1}{\theta}\int_0^\theta \langle m, e_\alpha(t)\rangle\, dt = 1\,, \quad \frac{1}{\theta}\int_0^\theta \langle \phi_\alpha(t)e_\alpha(t)\rangle\, dt = 1\,.$$

Again we obtain the following expansion for the payoff,

$$r_\alpha(T,x) = \lambda_F(\alpha) + \frac{1}{T}\Big(\log\langle\phi_\alpha(0),x\rangle + \log\langle m, e_\alpha(T)\rangle\Big) + o\left(\frac{1}{T}\right).$$

A natural problem is to find periodic controls $\alpha(t)$ such that the Floquet eigenvalue is better than the optimal Perron eigenvalue $\lambda_P(\alpha^*)$. The following Proposition gives a partial answer to this question. We consider small periodic perturbations of the best constant control: $\alpha(t) = \alpha^* + \epsilon\gamma(t)$, where γ is a given θ-periodic function. For the sake of clarity we introduce the following notation for the time average over a period,

$$\langle f\rangle_\theta = \frac{1}{\theta}\int_0^\theta f(t)\,dt\,.$$

We assume that the matrix $G + \alpha^* F$ is diagonalizable (in $\mathbb{R}$). This is the case for the running example (see Appendix A). We denote by (e_1^*, e_2^*, e_3^*) and $(\phi_1^*, \phi_2^*, \phi_3^*)$ the bases of right- and left-eigenvectors associated to the eigenvalues $\lambda_1^*, \lambda_2^*, \lambda_3^*$ for the the best constant control α^*. Notice that according to previous notations we have $e_1^* = e_{\alpha^*}$ and $\lambda_1^* = \lambda_P(\alpha^*)$.

Proposition 2.2. *The directional derivative of the dominant eigenvalue vanishes at $\epsilon = 0$:*

$$\left.\frac{d\lambda_F(\alpha^* + \epsilon\gamma)}{d\epsilon}\right|_{\epsilon=0} = 0\,. \tag{2.3}$$

Hence, α^* is also a critical point in the class of periodic controls. The second directional derivative of the dominant eigenvalue writes at $\epsilon = 0$:

$$\left.\frac{d^2\lambda_F(\alpha^* + \epsilon\gamma)}{d\epsilon^2}\right|_{\epsilon=0} = 2\sum_{i=2}^{3}\langle\gamma_i^2\rangle_\theta\frac{(\phi_1^* F e_i^*)(\phi_i^* F e_1^*)}{\lambda_1^* - \lambda_i^*}\,, \tag{2.4}$$

where $\gamma_i(t)$ is the unique θ-periodic function which is solution to the relaxation ODE

$$\frac{\dot{\gamma_i}(t)}{\lambda_1^* - \lambda_i^*} + \gamma_i(t) = \gamma(t)\,.$$

The idea of computing directional derivatives has been used in a similar context in [24] for optimizing the Perron eigenvalue in a continuous model for cell division.

Taking $\gamma \equiv 1$ in Equation (2.4), we get the second derivative of the Perron eigenvalue at α^*,

$$\frac{d^2\lambda_P}{d\alpha^2}(\alpha^*) = 2\sum_{i=2}^{3}\frac{(\phi_1^* F e_i^*)(\phi_i^* F e_1^*)}{\lambda_1^* - \lambda_i^*}\,, \tag{2.5}$$

which is nonpositive since α^* is a maximum point. However we cannot conclude directly in the general case that the quantity (2.4) is nonpositive since we do not know the relative signs of the coefficients $(\phi_1^* F e_i^*)(\phi_i^* F e_1^*)$ in (2.5).

We study in [14] a variant of (1.1), where the control is not affine. We prove that there exist directions γ for which α^* is a minimum point. Alternatively speaking, periodic controls can beat the best constant control. For this, we perturb the constant control with high-frequency modes, and we compute the limit of $\frac{d^2\lambda_F}{d\epsilon^2}$ when the frequency tends to infinity. Unfortunately, this procedure gives no additional information in the case of an affine control.

Proof of Proposition 2.2. First we derive a formula for the first derivative of the Perron eigenvalue: By definition we have

$$(G+\alpha F)e_\alpha = \lambda_P(\alpha)e_\alpha\,.$$

Deriving with respect to α we get

$$\frac{d\lambda_P}{d\alpha}(\alpha)e_\alpha + \lambda_P(\alpha)\frac{de_\alpha}{d\alpha} = Fe_\alpha + (G+\alpha F)\frac{de_\alpha}{d\alpha}\,.$$

Testing against the left- eigenvector ϕ_α we obtain

$$\frac{d\lambda_P}{d\alpha}(\alpha) = \phi_\alpha F e_\alpha\,.$$

Second, we write the Floquet eigenvalue problem corresponding to the periodic control $\alpha = \alpha^* + \epsilon\gamma$:

$$\frac{\partial}{\partial t}e_\alpha(t) + \lambda_F(\alpha^*+\epsilon\gamma)e_\alpha(t) = (G+(\alpha^*+\epsilon\gamma(t))F)e_\alpha(t)\,.$$

Deriving this ODE with respect to ϵ, we get

$$\frac{\partial}{\partial t}\frac{\partial e_\alpha}{\partial\epsilon}(t) + \frac{d\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon}e_\alpha(t) + \lambda_F(\alpha^*+\epsilon\gamma)\frac{\partial e_\alpha}{\partial\epsilon}(t) = \gamma(t)Fe_\alpha(t) + (G+(\alpha^*+\epsilon\gamma(t))F)\frac{\partial e_\alpha}{\partial\epsilon}(t)\,. \tag{2.6}$$

Testing this equation against ϕ_1^* and evaluating at $\epsilon = 0$, we obtain

$$\frac{\partial}{\partial t}\left(\phi_1^*\left.\frac{\partial e_\alpha}{\partial\epsilon}\right|_{\epsilon=0}(t)\right) + \left.\frac{d\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon}\right|_{\epsilon=0} = \gamma(t)\phi_1^*Fe_1^*\,.$$

After integration over one period, we get

$$\left.\frac{d\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon}\right|_{\epsilon=0} = \left(\frac{1}{\theta}\int_0^\theta \gamma(t)\,dt\right)\phi_1^*Fe_1^* = \langle\gamma\rangle_\theta\,\frac{d\lambda_P}{d\alpha}(\alpha^*) = 0\,,$$

which is the first order condition (2.3).

Next, we test (2.6) against another left- eigenvector ϕ_i^* and we evaluate at $\epsilon = 0$. We obtain the following equation satisfied by $\gamma_i(t) = (\lambda_1^*-\lambda_i^*)\phi_i^*\dfrac{\partial e_\alpha}{\partial\epsilon}(t)(\phi_i^*Fe_1^*)^{-1}$:

$$\frac{1}{\lambda_1^*-\lambda_i^*}\frac{\partial}{\partial t}\gamma_i(t) + \gamma_i(t) = \gamma(t)\,. \tag{2.7}$$

We differentiate (2.6) with respect to ϵ. This yields

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial^2 e_\alpha}{\partial\epsilon^2}(t) + \frac{d^2\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon^2}e_\alpha(t) + 2\frac{d\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon}\frac{\partial e_\alpha}{\partial\epsilon}(t) + \lambda_F(\alpha^*+\epsilon\gamma)\frac{\partial^2 e_\alpha}{\partial\epsilon^2}(t)\\ = 2\gamma(t)F\frac{\partial e_\alpha}{\partial\epsilon}(t) + (G+(\alpha^*+\epsilon\gamma(t))F)\frac{\partial^2 e_\alpha}{\partial\epsilon^2}(t)\,.\end{aligned}$$

Testing this equation against ϕ_1^* and evaluating at $\epsilon = 0$, we find

$$\frac{\partial}{\partial t}\left(\phi_1^*\left.\frac{\partial^2 e_\alpha}{\partial\epsilon^2}\right|_{\epsilon=0}(t)\right) + \left.\frac{d^2\lambda_F(\alpha^*+\epsilon\gamma)}{d\epsilon^2}\right|_{\epsilon=0} = 2\gamma(t)\phi_1^*F\left.\frac{\partial e_\alpha}{\partial\epsilon}\right|_{\epsilon=0}(t)\,. \tag{2.8}$$

We decompose the unknown $\frac{\partial e_\alpha}{\partial \epsilon}(t)$ along the basis (e_1^*, e_2^*, e_3^*)

$$\frac{\partial e_\alpha}{\partial \epsilon}(t) = \sum_{i=1}^{3} \gamma_i(t) \frac{(\phi_i^* F e_1^*)}{\lambda_1^* - \lambda_i^*} e_i^* \,.$$

In particular, we have

$$\phi_1^* F \frac{\partial e_\alpha}{\partial \epsilon}(t) = \sum_{i=2}^{3} \gamma_i(t) \frac{(\phi_i^* F e_1^*)}{\lambda_1^* - \lambda_i^*} (\phi_1^* F e_i^*) \,,$$

since $\phi_1^* F e_1^* = 0$ by optimality. To conclude, we integrate (2.8) over one period,

$$\left. \frac{d^2 \lambda_F(\alpha^* + \epsilon\gamma)}{d\epsilon^2} \right|_{\epsilon=0} = 2 \sum_{i=2}^{3} \langle \gamma\gamma_i \rangle_\theta \frac{(\phi_i^* F e_1^*)(\phi_1^* F e_i^*)}{\lambda_1^* - \lambda_i^*} \,.$$

We conclude thanks to the following identity derived from (2.7): $\langle \gamma_i^2 \rangle_\theta = \langle \gamma\gamma_i \rangle_\theta$. □

3 Optimal control and the ergodic problem

Let a, A be some given bounds on the control parameter α, with $a < \alpha^* < A$. Let $T > 0$ be a (large) time. The optimal control problem associated to (1.1) reads as follows

$$V(T,x) = \sup_\alpha \left\{ \langle m, x_\alpha(T) \rangle \ : \ \dot{x}_\alpha(t) = (G + \alpha(t)F) x_\alpha(t) \ , \ x_\alpha(0) = x \right\} \ ,$$

where the supremum is taken over all measurable control functions $\alpha : (0,T) \to [a,A]$. The principle of dynamic programming enables to solve this optimal control problem by introducing the value function $v(s,x) = \sup \{ \langle m, x_\alpha(T) \rangle \ : \ \dot{x}_\alpha(t) = (G + \alpha(t)F) x_\alpha(t) \ , \ x_\alpha(s) = x \}$. We have $v(T,x) = \langle m, x \rangle$, $v(0,x) = V(T,x)$, and v satisfies the following Hamilton-Jacobi-Bellman equation in the viscosity sense,

$$\frac{\partial}{\partial s} v(s,x) + \widetilde{H}(x, D_x v(s,x)) = 0 \,,$$

where the Hamiltonian is given by $\widetilde{H}(x,p) = \max_{\alpha \in [a,A]} \{ \langle (G + \alpha F) x, p \rangle \}$. This equation is backward in time: the initial data is prescribed at the final time $s = T$.

Intuitively we expect an exponential growth of the reward $\langle m, x_\alpha(T) \rangle$ as $T \to +\infty$. This motivates the following reduction to a compact space/linear growth.

3.1 Reduction to a compact space/linear growth

We perform a logarithmic change of variable: $w(s,x) = \log v(s,x) - \log \langle m, x \rangle$. The reward function satisfies:

$$\frac{d}{dt} \log \langle m, x_\alpha(t) \rangle = \frac{\langle m, \dot{x}_\alpha(t) \rangle}{\langle m, x_\alpha(t) \rangle} = \langle m, (G + \alpha(t)F) y_\alpha(t) \rangle = \langle m, G y_\alpha(t) \rangle \ ,$$

where $y = \frac{x}{\langle m, x \rangle}$ denotes the projection on the simplex $\mathcal{S} = \{ y \geq 0 \ : \ \langle m, y \rangle = 1 \}$. We can write a close equation for the projected trajectory $y_\alpha(t)$ due to the linearity of the system:

$$\begin{aligned} \dot{y}_\alpha(t) &= \frac{\dot{x}_\alpha(t)}{\langle m, x_\alpha(t) \rangle} - \frac{x_\alpha(t)}{\langle m, x_\alpha(t) \rangle} \frac{\langle m, \dot{x}_\alpha(t) \rangle}{\langle m, x_\alpha(t) \rangle} \\ &= (G + \alpha(t)F) y_\alpha(t) - \langle m, (G + \alpha(t)F) y_\alpha(t) \rangle \, y_\alpha(t) \\ &= (G + \alpha(t)F) y_\alpha(t) - \langle m, G y_\alpha(t) \rangle \, y_\alpha(t) \,, \end{aligned}$$

(recall $m^T F = 0$). We introduce the following notation for the vector field on the simplex:

$$b(y,\alpha) = (G + \alpha F)y - \langle m, Gy\rangle\, y\,.$$

It is worth mentioning that the only stationary point of the vector field $b(y,\alpha)$ on the simplex $\mathcal{S}$ is the dominant eigenvector e_α: $b(e_\alpha,\alpha) = 0$. Furthermore, for any $y \in \mathcal{S}$ the trajectory starting from y with constant control $\alpha(t) \equiv \alpha$ converges to e_α as $t \to +\infty$, and leaves the simplex $\mathcal{S}$ as $t \to -\infty$. Moreover the vector field $b(y,\alpha)$ can be easily computed on each eigenvector $y = e_\beta$ due to the affine structure of the problem:

$$\begin{aligned} b(e_\beta,\alpha) &= (G+\alpha F)e_\beta - \langle m, Ge_\beta\rangle\, e_\beta \\ &= (\alpha-\beta)Fe_\beta + b(e_\beta,\beta) \\ &= (\alpha-\beta)Fe_\beta\,. \end{aligned}$$

The logarithmic value function w satisfies the following optimization problem for $x \in \mathcal{S}$:

$$w(s,x) = \sup_\alpha \left\{\log\langle m, x_\alpha(T)\rangle - \log\langle m,x\rangle\right\} = \sup_\alpha \left\{ \int_s^T \langle m, Gy_\alpha(t)\rangle\; dt \;:\; \dot y_\alpha(t) = b(y_\alpha(t),\alpha(t))\;,\; y_\alpha(s) = x \right\}$$

Finally for notational convenience, we perform the time reversal $t = T - s$, and introduce $u(t,y) = w(T-t,y)$ for $y \in \mathcal{S}$. The optimal problem is now reduced to a compact space (the simplex $\mathcal{S}$), with a running reward:

$$u(t,y) = \sup_\alpha \left\{ \int_0^t L(y_\alpha(s))\, ds \;:\; \dot y_\alpha(s) = b(y_\alpha(s),\alpha(s))\;,\; y_\alpha(0) = y \right\}\,.$$

where the reward function is linear: $L(y) = \langle m, Gy\rangle$. The function $-u$ is a viscosity solution of the following Hamilton-Jacobi equation:

$$\begin{cases} \dfrac{\partial}{\partial t}(-u(t,y)) + H(y, D_y u(t,y)) = 0\,, \\ u(0,y) = 0\,, \end{cases} \tag{3.1}$$

where the Hamiltonian is given by

$$H(y,p) = \max_{\alpha\in[a,A]} \left\{\langle b(y,\alpha), p\rangle + \langle m, Gy\rangle\right\}\,. \tag{3.2}$$

Remark 3.1. *An important observation is that this hamiltonian does not satisfy the classical coercivity assumption $H(\cdot,p) \to +\infty$ as $|p| \to +\infty$. Indeed for all $y \in \mathcal{S}$ there exists a cone $C(y)$ such that $H(y,p) \to -\infty$ if $p \in C(y)$ and $|p| \to +\infty$. In addition we have $C(y) = \emptyset$ if and only if $y = e_\beta$ for some $\beta \geq 0$. In the latter case we have*

$$H(e_\beta,p) = \max_{\alpha\in[a,A]} \left\{\langle (\alpha-\beta)Fe_\beta, p\rangle\right\} + L(e_\beta) = (A-\beta)\,\langle Fe_\beta,p\rangle_+ + (\beta-a)\,\langle Fe_\beta,p\rangle_- + \lambda_P(\beta)\,.$$

It is not coercive either.

Due to the lack of coercivity the ergodic problem is difficult to handle with. We follow the procedure described in [2, 3] to exhibit an ergodic set in the simplex. This set has to fulfill two important features: controllability, and attractivity. We construct below such an ergodic set, and we prove the two required properties.

3.2 Notations and statement of the result

Following [3] we introduce the auxiliary problem with infinite horizon:

$$u_\epsilon(y) = \sup_\alpha \left\{ \int_0^{+\infty} e^{-\epsilon t} L(y_\alpha(t))\, dt \ : \ \dot{y}_\alpha(t) = b(y_\alpha(t), \alpha(t)) \ , \ y_\alpha(0) = y \right\} . \tag{3.3}$$

The limit of the quantity $\epsilon u_\epsilon(y)$ as $\epsilon \to 0$ measures the convergence of the reward $L(y(t))$ as $t \to +\infty$ in average (convergence *à la Césaro*). The function $-u$ is a viscosity solution of the following stationary Hamilton-Jacobi equation:

$$-\epsilon u_\epsilon(y) + H(y, D_y u_\epsilon(y)) = 0 \, . \tag{3.4}$$

The next Theorem claims that the function $\epsilon u_\epsilon(y)$ converges to a constant. We interpret this constant as the (generalized) eigenvalue associated to the optimal control problem.

We introduce some notations necessary for the statement of the two technical hypotheses **(H4-5)**:

- $\mathcal{S}$: the simplex $\{y \geq 0 \ : \ \langle m, y \rangle = 1\}$,
- Θ: the (direct) orthogonal rotation with angle $\pi/2$ on the tangent space $T\mathcal{S}$,
- $\Phi_0 = \{e_\alpha \mid \alpha \in \mathbb{R}_+\} \subset \mathcal{S}$: the set of Perron eigenvectors,

(H4) We assume a first technical condition:

$$\left\langle \frac{de_\alpha}{d\alpha}, \Theta F e_\alpha \right\rangle \quad \text{has a constant sign for } \alpha \geq 0. \tag{3.5}$$

This hypothesis enables to determine the direction of the vector field $b(y, \alpha)$ accross Φ_0 at $y = e_\beta$:

$$\left\langle \Theta \left. \frac{de_\alpha}{d\alpha} \right|_{\alpha=\beta}, b(e_\beta, \alpha) \right\rangle = -(\alpha - \beta) \left\langle \left. \frac{de_\alpha}{d\alpha} \right|_{\alpha=\beta}, \Theta F e_\beta \right\rangle . \tag{3.6}$$

Due to Hypothesis **(H4)**, only the sign of $\alpha - \beta$ determines the sign of the above quantity.

We assume without loss of generality that the quantity (3.5) is negative for all $\alpha \geq 0$. This has been checked numerically in the case of the running example (1.4) (see Appendix B).

(H5) We assume a second technical condition: there exists $\delta_0 > 0$ such that for any constant control $\beta \in (A - \delta_0, A + \delta_0)$, the trajectory starting from e_0 with constant control β, which connects e_0 and e_β, does not cross the curve Φ_0 for $t > 0$. Similarly, for any constant control $\beta \in (a - \delta_0, a + \delta_0)$, the trajectory starting from e_∞ with constant control β, which connects e_∞ and e_β, does not cross the curve Φ_0.

A proper construction of some remarkable sets in the proof of the following Theorem relies on these two technical assumptions, *e.g.* $\mathcal{Z}_0, \mathcal{T}_\pm$ (Figures 2 and 5).

Theorem 3.2 (Ergodicity)**.** *Assume that hypotheses* **(H1-2-3-4-5)** *are satisfied. There exists a constant λ_{HJ} such that the following uniform convergence holds true:*

$$\lim_{\epsilon \to 0} \epsilon u_\epsilon(y) = \lambda_{HJ} \, , \quad \textit{uniformly for } y \in \mathcal{S} \, .$$

The constant λ_{HJ} is interpreted as the eigenvalue associated to the Hamiltonian $H(y,p)$. We are not able to prove the existence of an eigenvector here. Indeed we lack an equicontinuity estimate on the family $(u_\epsilon)_\epsilon$. We only obtain equicontinuity of the family $(\epsilon u_\epsilon)_\epsilon$. However we postulate such an eigenvector does exist, based on numerical evidence (see Section 4).

Corollary 3.3. *Assume that hypotheses* **(H1-2-3-4-5)** *are satisfied. Then the solution of the Hamilton-Jacobi-Bellman equation satisfies the following ergodic property:*

$$\lim_{T\to+\infty} \frac{u(T,y)}{T} = \lambda_{HJ}\,, \quad \textit{uniformly for } y \in \mathcal{S}\,.$$

Back to the original problem, Corollary 3.3 translates into (1.2), where convergence is uniform on compact subsets of $(\mathbb{R}_+)^3 \setminus \{0\}$.

3.3 Proof of Theorem 3.2

We mainly follow [3] with some refinements specific to our context. We proceed in several steps, with illustrative figures to facilitate the argumentation.

The program is the following: we identify the so-called ergodic set $\mathcal{Z}_0 \subset \mathcal{S}$ [2, 3], and we show some of its interesting properties. We begin with the controllability problem. Controllability enables to prove convergence to the constant λ_{HJ} when y belongs to the ergodic set $\mathcal{Z}_0$. Next we demonstrate the attractiveness of the ergodic set $\mathcal{Z}_0$. This enables to extend the convergence to every $y \in \mathcal{S}$.

There is a technical subtlety concerning the time required for controllability/attractiveness. This time could degenerate as we get close to the boundary of the ergodic set. We circumvent this issue by proving that close-to-optimal trajectories do not stay in the vicinity of the boundary for long time.

Step 1- Identification of some remarkable sets: the set of eigenvectors, and the ergodic set. The first remarkable subset of the simplex $\mathcal{S}$ is the set of eigenvectors $\Phi_0 = \{e_\alpha \mid \alpha \in \mathbb{R}_+\}$. It is alternatively defined as the zero level set of the cubic function

$$\varphi(y) = \langle b(y,\alpha), \Theta F y\rangle = \langle Gy - \langle m, Gy\rangle\, y, \Theta F y\rangle\ .$$

Hence we have,

$$\Phi_0 = \{e_\alpha \mid \alpha \in \mathbb{R}_+\} = \{y \in \mathcal{S}\ :\ \varphi(y) = 0\}\,.$$

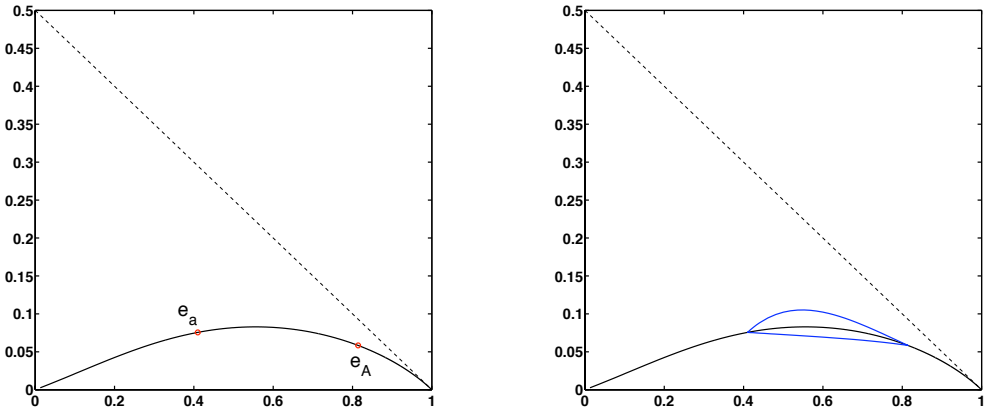
Indeed, $y \in \mathcal{S}$ is an eigenvector if and only if there exists α such that $(G+\alpha F)y - \langle m, Gy\rangle\, y = 0$, *i.e.* $Gy - \langle m, Gy\rangle\, y \in \mathrm{span}\,(Fy)$, or equivalently $Gy - \langle m, Gy\rangle\, y \perp \Theta F y$. From Hypothesis **(H3)** we deduce that Φ_0 is a curve connecting two boundary points of the simplex $\mathcal{S}$. In Figure 2 we have plotted the set of eigenvectors in the case of the running example.

The set of eigenvector splits the simplex $\mathcal{S}$ into two subsets, denoted by Φ_+ and Φ_-, respectively:

$$\Phi_+ = \{y \in \mathcal{S}\ :\ \varphi(y) \geq 0\} \quad , \quad \Phi_- = \{y \in \mathcal{S}\ :\ \varphi(y) \leq 0\}\,.$$

Notice that the sign of φ depends on the orientation of the rotation Θ. Obviously the following discussion does not depend on this convention.

We seek the ergodic set which is stable, attractant and controllable. The natural candidate is defined through its boundary as follows. The boundary consists of two curves which are joining on the set Φ_0.



(a) (b)

Figure 2: The set of eigenvectors Φ_0 (a) and the boundary of the ergodic set $\mathcal{Z}_0$ (b). The dashed line represent the boundary of the simplex.

Definition 3.4. *The ergodic set $\mathcal{Z}_0$ is the compact set enclosed by the two following curves:*

$$\gamma_a^A \begin{cases} \dot\gamma(s) = b(\gamma(s), a) \\ \gamma(0) = e_A \end{cases}, \quad \gamma_A^a \begin{cases} \dot\gamma(s) = b(\gamma(s), A) \\ \gamma(0) = e_a \end{cases}. \tag{3.7}$$

The set $\mathcal{Z}_0$ is well defined. In fact we have $\lim_{s\to+\infty} \gamma_a^A(s) = e_a$ and $\lim_{s\to+\infty} \gamma_A^a(s) = e_A$.

In Figure 2 we have plotted an example of the ergodic set $\mathcal{Z}_0$ for the running example. We list in the following Proposition some useful properties of the set $\mathcal{Z}_0$ which are derived from the very definition.

Lemma 3.5. *(i) The curves γ_a^A, γ_A^a lie on the opposite sides of Φ_0. We assume without loss of generality that the curve γ_a^A belongs to the subset Φ_+, whereas the curve γ_A^a belongs to the subset Φ_-.*

(ii) The set $\mathcal{Z}_0$ is stable: the vector fields $\{b(y,\alpha)\ ,\ \alpha \in [a, A]\}$ are all pointing inwards on the boundary of $\mathcal{Z}_0$.

Proof. (i) We denote by γ_A^0 the trajectory connecting e_0 to e_A with constant control A. Similarly we denote by γ_a^∞ the trajectory connecting e_∞ to e_a with constant control a. From Hypothesis **(H4)** these two trajectories initially start on the two opposite sides of the line of eigenvectors Φ_0. Notice that γ_a^∞ may start tangentially to Φ_0 (it is actually the case for the running example). However a continuity argument for γ_a^β with $\beta \to +\infty$ yields the statement. Hypothesis **(H5)** guarantees that they do not cross Φ_0, so that they stay on opposite sides of Φ_0 forever. We assume without loss of generality that γ_A^0 lies in Φ_- and γ_a^∞ lies in Φ_+.

Second from the bounds on the control $a \le \alpha(t) \le A$, together with Hypothesis **(H4)** (3.6), the trajectory γ_A^a starts on the same side as γ_A^0 (*i.e.* Φ_- with the above convention). Moreover it cannot cross the line of eigenvectors Φ_0 at e_β for $\beta \in (0, A)$. From the uniqueness theorem for ODE, it cannot cross the trajectory γ_A^0 either. As a conclusion, the trajectory γ_A^a is sandwiched between the portion of Φ_0 between e_0 and e_A, and γ_A^0. Therefore it belongs to Φ_-.

The same holds true for $\gamma_a^A \subset \Phi_+$.

(ii) We denote by $n(z)$ the unit vector normal and exterior to the boundary of $\mathcal{Z}_0$. We have more precisely:

$$n_a(z) = \frac{\Theta b(z,a)}{|\Theta b(z,a)|} \quad \text{on } \gamma_a^A \quad , \quad n_A(z) = \frac{\Theta b(z,A)}{|\Theta b(z,A)|} \quad \text{on } \gamma_A^a \, .$$

We have on the one side γ_a^A: $\forall \alpha \in [a,A]$ $\langle b(z,\alpha), n_a(z)\rangle \leq \langle b(z,a), n_a(z)\rangle = 0$. Indeed we have

$$\begin{aligned}
\langle b(z,\alpha) - b(z,a), n_a(z)\rangle &= (\alpha - a)\,\langle Fz, n_a(z)\rangle \\
&= (\alpha - a)\,\langle \Theta Fz, \Theta n_a(z)\rangle \\
&= (\alpha - a)\left\langle \Theta Fz, -\frac{b(z,a)}{|b(z,a)|}\right\rangle \\
&= \frac{(a-\alpha)}{|b(z,a)|}\,\langle \Theta Fz, Gz - \langle m, Gz\rangle\, z\rangle \leq 0\,.
\end{aligned}$$

We have on the other side γ_A^a:

$$\langle b(z,\alpha) - b(z,A), n_A(z)\rangle = (\alpha - A)\,\langle Fz, n_A(z)\rangle = \frac{(A-\alpha)}{|b(z,A)|}\varphi(z) \leq 0\,.$$

□

Step 2- Exact controllability inside the ergodic set. The purpose of this step is to prove the following Lemma 3.6, which asserts exact controllability in the ergodic set, except a narrow band. First we introduce some notations:

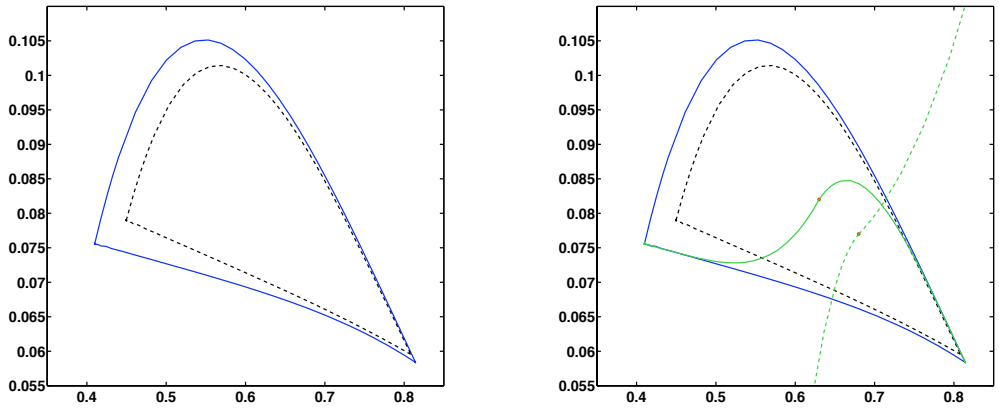
- $\mathcal{Z}_{-\delta}$ denotes the ergodic set, from which we have substracted a narrow band close to the boundary $\gamma_a^A \cup \gamma_A^a$ (Figure 3). It is defined similarly as $\mathcal{Z}_0$ by the delimitation of the two following curves,

$$\gamma_{a+\delta}^{A-\delta} \begin{cases} \dot\gamma(s) = b(\gamma(s), a+\delta) \\ \gamma(0) = e_{A-\delta} \end{cases} \quad , \quad \gamma_{A-\delta}^{a+\delta} \begin{cases} \dot\gamma(s) = b(\gamma(s), A-\delta) \\ \gamma(0) = e_{a+\delta} \end{cases} \, .$$

- $\mathcal{Z}_{+2\delta}$ denotes the ergodic set, to which we have added a narrow band close to the boundary $\gamma_a^A \cup \gamma_A^a$ (Figure 5). It is defined similarly as $\mathcal{Z}_0$ by the delimitation of the two following curves,

$$\gamma_{a-2\delta}^{A+2\delta} \begin{cases} \dot\gamma(s) = b(\gamma(s), a-2\delta) \\ \gamma(0) = e_{A+2\delta} \end{cases} \quad , \quad \gamma_{A+2\delta}^{a-2\delta} \begin{cases} \dot\gamma(s) = b(\gamma(s), A+2\delta) \\ \gamma(0) = e_{a-2\delta} \end{cases} \, .$$

Lemma 3.6. *The system $\dot y(t) = b(y(t),\alpha)$ is exactly controllable in the subset $\mathcal{Z}_{-\delta}$. For any z, z' in $\mathcal{Z}_{-\delta}$ there exists a time $T = T(z,z')$ and $\alpha(t) : [0,T] \to [a,A]$ such that $y(0) = z$ and $y(T) = z'$. Moreover it is possible to construct the control $\alpha(t)$ following a bang-bang procedure: $\alpha(t) : [0,T] \to \{a, A\}$. Last, the minimal time T needed to connect z and z' is uniformly bounded for $z, z' \in \mathcal{Z}_{-\delta}$.*



(a) (b)

Figure 3: (a) The ergodic set $\mathcal{Z}_0$ and the set $\mathcal{Z}_{-\delta}$ (dashed line). (b) Illustration for the proof of the exact controllability in the set $\mathcal{Z}_{-\delta}$: the two curves Y (plain line) and Y' (dashed line) necessarily intersect in $\mathcal{Z}_0$.

Proof. We aim at sending z onto z'. We define the trajectories y_a and y_A starting from y as follows,

$$y_a \begin{cases} \dot{y}_a(t) = b(y_a(t), a) \\ y_a(0) = z \end{cases}, \quad y_A \begin{cases} \dot{y}_A(t) = b(y_A(t), A) \\ y_A(0) = z \end{cases}.$$

When we concatenate the two trajectories $Y = y_a \cup y_A$, we obtain a curve which connects a and A. Reversing time, we define two other trajectories starting from z',

$$y'_a \begin{cases} \dot{y}'_a(t) = -b(y'_a(t), a) \\ y'_a(0) = z' \end{cases}, \quad y'_A \begin{cases} \dot{y}'_A(t) = -b(y'_A(t), A) \\ y'_A(0) = z' \end{cases}.$$

Concatening the two trajectories we obtain a curve $Y' = y'_a \cup y'_A$ which leaves the simplex $\mathcal{S}$. Moreover it leaves the ergodic set $\mathcal{Z}_0$ by the two opposite sides γ_A^a and γ_a^A. Indeed y'_a cannot intersect γ_a^A, and y'_A cannot intersect γ_A^a by the uniqueness theorem for ODE. It necessarily intersects the curve Y since the latter connects a and A.

Let $z'' \in Y \cap Y'$. We assume without loss of generality that $z'' = y_a(T)$ and also $z'' = y'_A(T')$. Observe that we do not necessarily have $z'' \in \mathcal{Z}_{-\delta}$. However we do have $z'' \in \mathcal{Z}_0$ since the set $\mathcal{Z}_0$ is stable. Finally we construct the control α as follows: (i) from time $t = 0$ to $t = T$ we set $\alpha(t) \equiv a$, (ii) from $t = T$ to $t = T + T'$ we set $\alpha(t) \equiv A$. This control sends z onto z' within time $T + T'$.

The time T' is clearly uniformly bounded. To prove that the time T is also uniformly bounded for $z, z' \in \mathcal{Z}_{-\delta}$, it is sufficient to avoid small neighbourhoods of the stationary points e_a and e_A. The image of $\mathcal{Z}_{-\delta}$ by the backward trajectories y'_a and y'_A inside $\mathcal{Z}_0$ is a compact set which does not contain e_a nor e_A. Therefore the time T is uniformly bounded. □

Step 3- Proof of the convergence towards λ_{HJ} in the set $\mathcal{Z}_{-\delta}$. We are now ready to prove an important lemma, which is a weaker version of Theorem 3.2.

Lemma 3.7. *Under the same assumptions as above, there exists a constant λ_{HJ} such that the following uniform convergence holds true:*

$$\lim_{\epsilon\to 0} \epsilon u_\epsilon(z) = \lambda_{HJ}\,, \quad \text{uniformly for } z \in \mathcal{Z}_{-\delta}\,.$$

Proof. First we observe that the function $\epsilon u_\epsilon(y)$ is uniformly bounded on the simplex $\mathcal{S}$:

$$\left|\int_0^\infty \epsilon e^{-\epsilon t} L(y_\alpha(t))\,dt\right| \le \|L\|_\infty \int_0^\infty \epsilon e^{-\epsilon t}\,dt = \|L\|_\infty\,. \tag{3.8}$$

Taking the supremum over all possible controlled trajectories, we end up with $\|\epsilon u_\epsilon\|_\infty \le \|L\|_\infty$.

Second, we show that the family $(\epsilon u\epsilon)_\epsilon$ is equicontinuous. Let $z, z' \in \mathcal{Z}_{-\delta}$. From Lemma 3.6 there exists a time $T(z,z')$ and a controlled trajectory $z_\alpha(t)$ sending z onto z' within time $T(z,z')$. By the dynamic programming principle, we have

$$\begin{aligned}\epsilon u_\epsilon(z') - \epsilon u_\epsilon(z) &\le \epsilon u_\epsilon(z') - \epsilon \int_0^{T(z,z')} e^{-\epsilon t} L(z_\alpha(t))\,dt - \epsilon e^{-\epsilon T(z,z')} u_\epsilon(z') \\ &\le \epsilon u_\epsilon(z')\left(1 - e^{-\epsilon T(z,z')}\right) + \|L\|_\infty \left(1 - e^{-\epsilon T(z,z')}\right)\,.\end{aligned}$$

Exchanging the roles of z and z', we obtain the following uniform bound:

$$|\epsilon u_\epsilon(z') - \epsilon u_\epsilon(z)| \le 2\|L\|_\infty \epsilon T(z,z')\,. \tag{3.9}$$

This simple estimate proves that (up to extraction), $\epsilon u_\epsilon(z)$ converges uniformly towards some constant. We learn from [3] that it converges in fact to a unique constant, not depending on the choice of the extraction (see the conclusion of the proof in Step 7). □

Remark 3.8. *Any refinement in the estimate of the minimal control time $T(z,z')$ would bring additional information about the convergence of u_ϵ. For instance, if we were able to prove that $T = \mathcal{O}(|z-z'|)$, then we would be able to conclude that u_ϵ is uniformly Lipschitz* (3.9)*. Hence the Ascoli-Arzela theorem would yield convergence of $u_\epsilon - \epsilon^{-1}\lambda_{HJ}$ towards some eigenvector $\overline{u}$ up to extraction [12].*

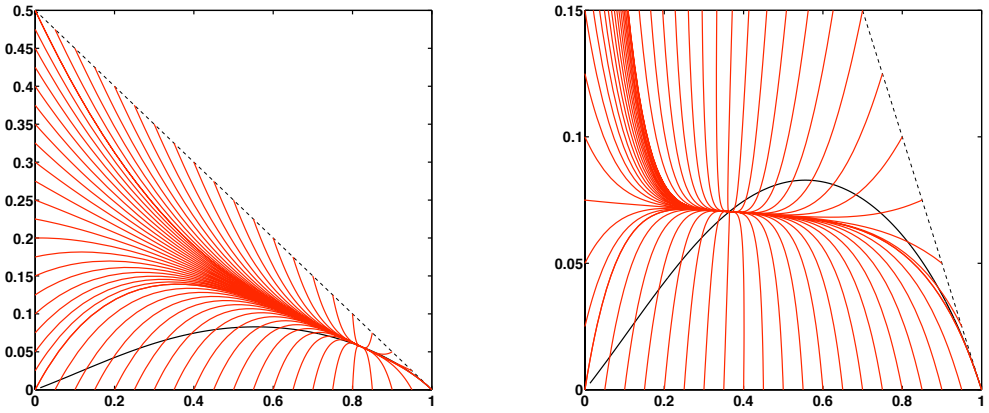
Step 4- Flow of trajectories and local charts. In order to prove attractiveness of the ergodic set, we shall use two couples of charts to cover parts of the simplex $\mathcal{S}$. The first couple consists in trajectories starting from the boundary of the simplex $\partial\mathcal{S}$, and driven by constant controls $a-\delta$ or $A+\delta$. The second couple consists in trajectories starting from the line of eigenvectors Φ_0, and driven by constant controls $a-\delta$ or $A+\delta$ too.

We shall perform all the computation for one given set of charts (starting from the boundary $\partial\mathcal{S}$ with constant control $A+\delta$). The other cases are similar. We introduce the following family of curves,

$$\Gamma_{A+\delta} \begin{cases} \frac{\partial}{\partial s}\Gamma(\theta,s) = b(\Gamma(\theta,s), A+\delta) \\ \Gamma(\theta,0) = Y_\theta \in \partial\mathcal{S} \end{cases}\,, \tag{3.10}$$

where the scalar θ is a parametrization of the boundary $\partial\mathcal{S}$. More precisely we adopt a piecewise linear parametrization with the direct orientation. This set of charts covers the whole simplex $\mathcal{S}$, see Figure 4a.

It is worth recalling that all these trajectories converge towards the eigenvector $e_{A+\delta}$ which lies on Φ_0. We denote $n_{A+\delta}(y) = \frac{\Theta b(y,A+\delta)}{|\Theta b(y,A+\delta)|}$ the unitary vector normal to the curve $\Gamma_{A+\delta}(\theta,\cdot)$ passing through $y = \Gamma_{A+\delta}(\theta,s) \in \mathcal{S}$. We denote $\Gamma_{A+\delta} = \Gamma$ and $n_{A+\delta} = n$ throughout this step.



(a) (b)

Figure 4: (a) Grid of trajectories starting from the boundary of the simplex $\mathcal{S}$, with constant control $\alpha(t) \equiv A + \delta$. (b) Similar figure with constant control $\alpha(t) \equiv a - \delta$.

The key observation is that any trajectory $\dot{y}_\alpha(t) = b(y_\alpha(t), \alpha(t))$ is in fact moving monotically along the family of curves $\{\Gamma(\theta, \cdot)\}_\theta$ on the sets Φ_- and Φ_+. We are able to quantify this phenomenon. Let $\alpha(t)$ be any measurable control which takes values in $[a, A]$. We have

$$\begin{aligned}\langle \dot{y}_\alpha(t), n(y_\alpha(t))\rangle &= \langle (\alpha(t) - (A+\delta))Fy_\alpha(t) + b(y_\alpha(t), A+\delta), n(y_\alpha(t))\rangle \\ &= \frac{((A+\delta) - \alpha(t))}{|\Theta b(y_\alpha(t), A+\delta)|} \langle \Theta F y_\alpha(t), b(y_\alpha(t), A+\delta)\rangle \\ &= \frac{((A+\delta) - \alpha(t))}{|b(y_\alpha(t), A+\delta)|} \varphi(y_\alpha(t))\,.\end{aligned}$$

We rewrite $y_\alpha(t)$ using the local chart $y_\alpha(t) = \Gamma(\theta(t), s(t))$. We have accordingly,

$$\dot{y}_\alpha(t) = \dot{\theta}(t)\frac{\partial}{\partial\theta}\Gamma(\theta(t), s(t)) + \dot{s}(t)\frac{\partial}{\partial s}\Gamma(\theta(t), s(t))\,.$$

By construction, we have $\frac{\partial}{\partial s}\Gamma(\theta, s) = b(\Gamma(\theta, s), A+\delta)$. Consequently,

$$\langle \dot{y}_\alpha(t), n(y_\alpha(t))\rangle = \dot{\theta}(t)\left\langle \frac{\partial}{\partial\theta}\Gamma(\theta(t), s(t)), n(\Gamma(\theta(t), s(t)))\right\rangle + 0\,.$$

We compute the evolution of the quantity $\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\rangle$ along the curve $\Gamma(\theta, \cdot)$. We aim at showing that this quantity has constant sign. This is intuitively clear, since trajectories cannot cross. We make it quantitative in the following calculation. For the sake of clarity, we write temporarily $b(\Gamma, A+\delta) = b(\Gamma)$.

$$\begin{aligned}\frac{\partial}{\partial s}\left\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle &= \left\langle \frac{\partial^2\Gamma}{\partial\theta\partial s}, n(\Gamma)\right\rangle + \left\langle \frac{\partial\Gamma}{\partial\theta}, \Theta Db(\Gamma)\frac{\partial\Gamma}{\partial s}\right\rangle - \left\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle\left\langle n(\Gamma), \frac{1}{|\Theta b|}\Theta Db(\Gamma)\frac{\partial\Gamma}{\partial s}\right\rangle \\ &= \left\langle Db(\Gamma)\frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle + \left\langle \frac{\partial\Gamma}{\partial\theta}, \Theta Db(\Gamma)b(\Gamma)\right\rangle - \left\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle\left\langle n(\Gamma), \Theta Db(\Gamma)\frac{b(\Gamma)}{|\Theta b(\Gamma)|}\right\rangle \\ &= \left\langle \frac{\partial\Gamma}{\partial\theta}, \left[Db(\Gamma)^T\Theta + \Theta Db(\Gamma)\right]\frac{b(\Gamma)}{|\Theta b(\Gamma)|}\right\rangle - \left\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle\langle n(\Gamma), \Theta Db(\Gamma)\Theta^{-1}n(\Gamma)\rangle\end{aligned}$$

Notice that the matrix $[Db(\Gamma)^T\Theta + \Theta Db(\Gamma)]$ is everywhere skew-symmetric, so is the matrix Θ:

$$\left[Db(\Gamma)^T\Theta + \Theta Db(\Gamma)\right]^T = -\Theta Db(\Gamma) - Db(\Gamma)^T\Theta\,.$$

Therefore it is equal to Θ up to a scalar factor:

$$\left[Db(\Gamma)^T\Theta + \Theta Db(\Gamma)\right] = \widetilde{\omega}(\theta, s)\Theta\,.$$

To conclude, the scalar product $\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\rangle$ cannot vanish. In addition, we have the semi-explicit formula:

$$\left\langle \frac{\partial\Gamma}{\partial\theta}, n(\Gamma)\right\rangle = \exp\left(\int_0^s \omega(\theta, s')\,ds'\right)\left\langle \frac{d}{d\theta}Y_\theta, n(Y_\theta)\right\rangle\,,$$

where the scalar ω is given by

$$\begin{aligned}\omega &= \widetilde{\omega} - \left\langle n(\Gamma), \Theta Db(\Gamma)\Theta^{-1}n(\Gamma)\right\rangle\\ &= \langle n(\Gamma), \widetilde{\omega}n(\Gamma)\rangle - \left\langle n(\Gamma), \Theta Db(\Gamma)\Theta^{-1}n(\Gamma)\right\rangle\\ &= \left\langle n(\Gamma), \left[Db(\Gamma)^T + \Theta Db(\Gamma)\Theta^{-1}\right]n(\Gamma)\right\rangle - \left\langle n(\Gamma), \Theta Db(\Gamma)\Theta^{-1}n(\Gamma)\right\rangle\\ &= \left\langle n(\Gamma), Db(\Gamma)^T n(\Gamma)\right\rangle\,.\end{aligned}$$

To conclude, we have proven the following Lemma.

Lemma 3.9 (A monotonicity formula)**.** *Let $(\theta_{A+\delta}, s_{A+\delta})$ be some parametrization of the simplex, given by* (3.10)*. The evolution of $\theta_{A+\delta}$ along the trajectories of $\dot{y}_\alpha(t) = b(y_\alpha(t), \alpha(t))$ is given by the following formula:*

$$\dot{\theta}_{A+\delta}(t) = ((A+\delta) - \alpha(t))\varphi(y_\alpha(t))\exp\left(-\int_0^{s(t)}\omega(\theta, s')\,ds'\right)\left\langle \frac{d}{d\theta}Y_\theta, n_{A+\delta}(Y_\theta)\right\rangle^{-1}\,,$$

where the scalar ω is defined by

$$\omega(\theta, s) = \langle D_y b(\Gamma(\theta, s), A+\delta)n(\Gamma(\theta, s)), n(\Gamma(\theta, s))\rangle\,.$$

In particular ω is uniformly bounded: $\|\omega\|_\infty \leq \|D_y b\|_\infty$.

We build the same chart $(\theta_{a-\delta}, s_{a-\delta})$ with the constant control $\alpha(t) \equiv a - \delta$,

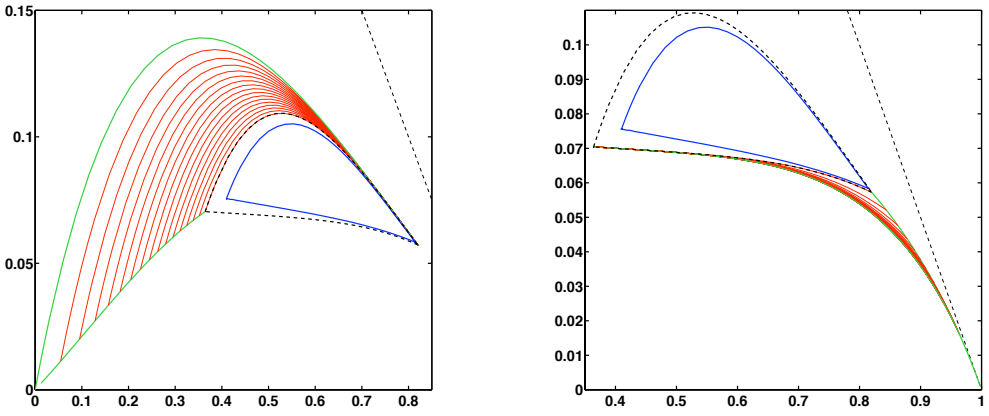
$$\Gamma_{a-\delta}\begin{cases}\frac{\partial}{\partial s}\Gamma(\theta, s) = b(\Gamma(\theta, s), a-\delta)\\ \Gamma(\theta, 0) = Y_\theta \in \partial\mathcal{S}\end{cases}\,. \tag{3.11}$$

We have similarly

$$\dot{\theta}_{a-\delta}(t) = ((a-\delta) - \alpha(t))\varphi(y_\alpha(t))\exp\left(-\int_0^{s(t)}\omega(\theta, s')\,ds'\right)\left\langle \frac{d}{d\theta}Y_\theta, n_{a-\delta}(Y_\theta)\right\rangle^{-1}\,.$$

We recall that we adopt a piecewise linear parametrization with the direct orientation for the boundary $\partial\mathcal{S}$. Since the fields $b(y, \alpha)$ are all pointing inward the simplex $\mathcal{S}$, this guarantees that we have in both cases ($A+\delta$ and $a-\delta$),

$$\forall\theta \quad \left\langle \frac{d}{d\theta}Y_\theta, n(Y_\theta)\right\rangle \leq 0\,.$$



(a) (b)

Figure 5: Tunneling effect: the trajectories are trapped in the areas delimited by the green curves. They are forced to enter the approximated ergodic set $\mathcal{Z}_{+\delta}$ due to the monotonicity formulas (3.13)-(3.14). (a) The set $\mathcal{T}_- \subset \Phi_-$; (b) The set $\mathcal{T}_+ \subset \Phi_+$.

Similarly we build the second couple of charts, defined through the following families of curves, parametrized by the starting point $e_\beta \in \Phi_0$, see Figure 5,

$$\Gamma_{A+\delta} \begin{cases} \frac{\partial}{\partial s}\Gamma(\beta,s) = b(\Gamma(\beta,s), A+\delta) \\ \Gamma(\beta,0) = e_\beta \in \Phi_0 \end{cases} \quad , \quad \Gamma_{a-\delta} \begin{cases} \frac{\partial}{\partial s}\Gamma(\beta,s) = b(\Gamma(\beta,s), a-\delta) \\ \Gamma(\beta,0) = e_\beta \in \Phi_0 \end{cases} \quad . \tag{3.12}$$

We obtain as in Lemma 3.9,

$$\dot\beta_{A+\delta}(t) = ((A+\delta) - \alpha(t))\varphi(y_\alpha(t)) \exp\left(-\int_0^{s(t)} \omega(\beta,s')\,ds'\right) \left\langle \frac{d}{d\beta} e_\beta, n_{A+\delta}(e_\beta) \right\rangle^{-1}, \tag{3.13}$$

$$\dot\beta_{a-\delta}(t) = ((a-\delta) - \alpha(t))\varphi(y_\alpha(t)) \exp\left(-\int_0^{s(t)} \omega(\beta,s')\,ds'\right) \left\langle \frac{d}{d\beta} e_\beta, n_{a-\delta}(e_\beta) \right\rangle^{-1}. \tag{3.14}$$

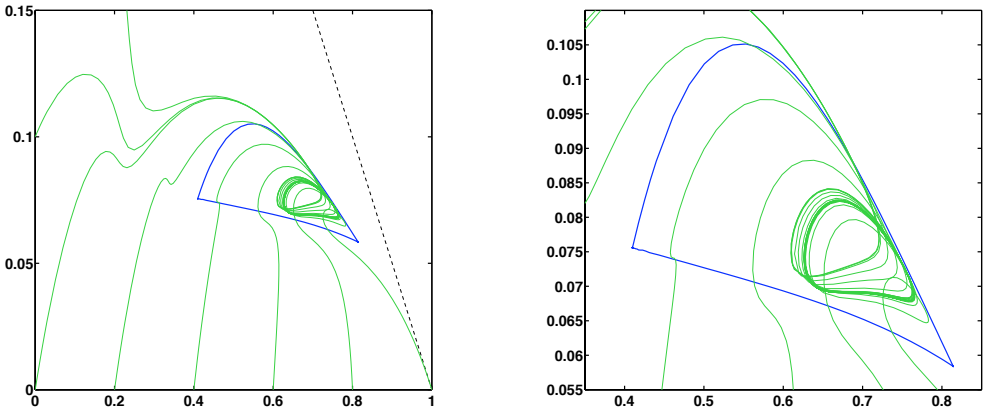
In this case, we have $b(e_\beta, A+\delta) = ((A+\delta)-\beta)Fe_\beta$. Therefore,

$$\forall \beta \quad \left\langle \frac{d}{d\beta} e_\beta, n_{A+\delta}(e_\beta) \right\rangle = \text{sign}\left((A+\delta)-\beta\right) \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle . \tag{3.15}$$

Assumption **(H4)** (3.5) allows to guarantee that this quantity has a constant sign for $\beta < A+\delta$. Notice that when the charts are defined with the control $a-\delta$, the formula (3.15) is replaced with the following:

$$\forall \beta \quad \left\langle \frac{d}{d\beta} e_\beta, n_{a-\delta}(e_\beta) \right\rangle = \text{sign}\left((a-\delta)-\beta\right) \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle .$$

Again this has constant sign for $\beta > a-\delta$. Unfortunately the second set of charts does not cover the whole simplex $\mathcal{S}$ (Figure 5). It is the reason why we introduce the transient set of charts parametrized by $Y_\theta \in \partial\mathcal{S}$. We are particularly interested in the two tunnels $\mathcal{T}_-, \mathcal{T}_+$ defined by the following curves:



(a) (b)

Figure 6: Attractiveness and stability of the ergodic set $\mathcal{Z}_0$. Several trajectories with arbitrary control $\alpha(t)$ (here, periodic) have been plotted. They all enter the ergodic set and stay there. The right figure is a zoom of the left one.

- the subset $\mathcal{T}_-$ is defined by its boundary made of 3 pieces (Figure 5a): the trajectory starting from the corner $e_0 = (0\ 0\ 1/3)$ with constant control $A+\delta$; the portion of Φ_0 from e_0 to $e_{a-\delta}$; and the trajectory starting from $e_{a-\delta}$ with constant control $A+\delta$. We have $\mathcal{T}_- \subset \Phi_-$,
- the subset $\mathcal{T}_+$ is defined similarly (Figure 5b): it is enclosed by the trajectory starting from the corner $e_\infty = (1\ 0\ 0)$ with constant control $a-\delta$; the portion of Φ_0 from e_∞ to $e_{A+\delta}$; and the trajectory starting from $e_{A+\delta}$ with constant control $a-\delta$. We have $\mathcal{T}_+ \subset \Phi_+$.

By comparison arguments we get easily that the second set of charts covers the tunnel $\mathcal{T}_-$ (resp. $\mathcal{T}_+$) with the constant control $A+\delta$ (resp. $a-\delta$). The main feature of these two tunnels is the following: due to Hypothesis **(H4)** (3.5) and Lemma 3.9, any trajectory that enters one of these tunnels can exit only by entering $\mathcal{Z}_{+\delta}$. Moreover it follows monotonically the parametrization of the second chart (3.13)-(3.14). This enables to prove that the trajectory eventually reaches the end of the tunnel: the ergodic set.

Step 5- Attractiveness of the ergodic set. We state the main result of this step: attractiveness of the approximated ergodic set $\mathcal{Z}_{+2\delta}$.

Lemma 3.10. *The set $\mathcal{Z}_{+2\delta}$ is attractive: any trajectory $y_\alpha(t)$ enters the set $\mathcal{Z}_{+2\delta}$ after finite time, at most $T_0(\delta)$. The maximal time $T_0(\delta)$ does not depend on ϵ.*

As this property holds true for any $\delta > 0$ we can conclude that the set $\mathcal{Z}_0$ is approximately attractive, but we get no estimate about the maximal time $T_0(\delta)$ as $\delta \to 0$. In fact we will prove in the next step that close-to-optimal trajectories enter the ergodic set $\mathcal{Z}_0$ (and even the smaller set $\mathcal{Z}_{-\delta}$) in finite time: this is a stronger property. The attractiveness property is illustrated in Figure 6: trajectories necessarily enter the set $\mathcal{Z}_{+2\delta}$.

Proof. Let $y_\alpha(t)$ be any trajectory of the system $\dot{y}_\alpha(t) = b(y_\alpha(t), \alpha(t))$, where the control $\alpha(t)$ takes values in $[a, A]$. We assume without loss of generality that $y_\alpha(0) \in \Phi_+ \setminus \mathcal{Z}_{+2\delta}$. We define

t_0 as the first exit time:

$$t_0 = \inf_t \{t \mid y_\alpha(t) \notin \Phi_+ \setminus \mathcal{Z}_{+2\delta}\} \ .$$

We face the following alternative: either $y_\alpha(0) \notin \mathcal{T}_+$ then we shall use the first set of charts parametrized by $Y_\theta \in \partial\mathcal{S}$ (3.11), or $y_\alpha(0) \in \mathcal{T}_+$ then we shall use the second set of charts parametrized by $e_\beta \in \Phi_0$ (3.12). We begin with the latter case.

The only way to exit the tunnel $\mathcal{T}_+$ is to enter the set $\mathcal{Z}_{+\delta} \subset \mathcal{Z}_{+2\delta}$. The quantity $\beta_{a-\delta}(t)$ is monotonic. More precisely we have

$$\forall t \in [0, t_0) \quad \dot\beta_{a-\delta}(t) \leq -\delta\varphi(y_\alpha(t)) \exp\left(-s(t)\|D_y b\|_\infty\right) \left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|^{-1} .$$

Recall that the time $s(t)$ is given by the parametrization of the trajectory $y_\alpha(t) = \Gamma_{a-\delta}(\beta_{a-\delta}(t), s(t))$. Hence $s(t)$ is uniformly bounded on $\Phi_+ \setminus \mathcal{Z}_{+2\delta}$. This is the reason why we use the set $\mathcal{Z}_{+2\delta}$ rather than the set $\mathcal{Z}_{+\delta}$ for which we could have $s(t) \to +\infty$ because $e_{a-\delta} \in \partial\mathcal{Z}_{+\delta}$. Therefore the r.h.s of (3.18) can vanish only when y_α gets close to the set Φ_0. Indeed the factor $\left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|^{-1}$ is clearly bounded from below since $\left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|$ is bounded from above.

We have a precise description of the dynamics close to Φ_0:

$$\begin{aligned} \frac{d}{dt}\varphi(y_\alpha(t)) &= \langle \dot y_\alpha(t), \nabla\varphi(y_\alpha(t))\rangle \\ &= \langle b(y_\alpha(t), \alpha(t)), \nabla\varphi(y_\alpha(t))\rangle \\ &= \langle b(e_\beta, \alpha(t)), \nabla\varphi(e_\beta)\rangle + o(|y_\alpha(t) - e_\beta|) \\ &= (\alpha(t) - \beta)\, \langle F e_\beta, \nabla\varphi(e_\beta)\rangle + o(|y_\alpha(t) - e_\beta|)\,. \end{aligned} \tag{3.16}$$

Hypothesis **(H4)** guarantees that the quantity $\langle F e_\beta, \nabla\varphi(e_\beta)\rangle$ is negative for all finite β:

$$\forall \beta < +\infty \quad \langle F e_\beta, \nabla\varphi(e_\beta)\rangle = \left\langle F e_\beta, -\chi_\beta \Theta \frac{de_\beta}{d\beta} \right\rangle = \chi_\beta \left\langle \Theta F e_\beta, \frac{de_\beta}{d\beta} \right\rangle < 0\,,$$

where χ_β is a positive scalar (coming from the discrepancy between the two vectors pointing in the normal direction to Φ_0: $\nabla\varphi(e_\beta)$ and $-\Theta\frac{de_\beta}{d\beta}$). Unfortunately $\langle b(e_\infty, \alpha(t)), \nabla\varphi(e_\infty)\rangle$ may vanish (it is actually the case for the running example: the field $b(e_\infty, \alpha)$ is independent of α and it is tangent to Φ_0). Therefore we have to isolate the large parameters $\beta \gg 1$ (trajectory starting close to e_∞). To circumvent this issue we investigate the behaviour of $y_\alpha(t)$ in the neighbourhood of e_∞, say a ball $\mathcal{B}(e_\infty, \eta')$. We have the following linearization around e_∞,

$$b(y, \alpha) = b(e_\infty, \alpha) + o(|y - e_\infty|) = G e_\infty + o(\eta')\,, \tag{3.17}$$

since $F e_\infty = 0$ (see Hypothesis **(H2)** and subsequent discussion). Moreover we have $G e_\infty \neq 0$ (see Hypothesis **(H3)** and subsequent discussion). The linearization is uniform with respect to $\alpha \in [a, A]$. Therefore the trajectory is pushed away from e_∞ in finite time, uniformly with respect to the initial data $y_\alpha(0) \in \mathcal{B}(e_\infty, \eta')$.

We choose $\eta' > 0$ such that the higher-order terms in the linearization (3.17) are negligible in front of $G e_\infty$. We choose the largest possible B such that the portion $\{e_\beta : \beta \geq B\} \subset \mathcal{B}(e_\infty, \eta')$. There exists $\nu > 0$ depending on B such that the following inequality holds,

$$\forall \beta \leq B \quad -\langle F e_\beta, \nabla\varphi(e_\beta)\rangle > \nu > 0\,.$$

Accordingly we choose a small $\delta' > 0$ such that the error term in the linearization (3.16) is uniformly smaller than $\delta\nu$ for $|y_\alpha(t) - e_\beta| < \delta'$ and $\beta \leq B$. Thus we have

$$\forall t \in [0, t_0) \quad \frac{d}{dt}\varphi(y_\alpha(t)) \geq 2\delta\nu - \delta\nu = \delta\nu\,.$$

This defines a narrow band of size δ' around the portion $\{e_\beta : A + 2\delta \leq \beta \leq B\} \subset \Phi_0$ in which the quantity $\varphi(y_\alpha(t))$ is increasing, uniformly in time. Therefore the trajectory stays away from Φ_0. Consequently the quantity $\beta_{a-\delta}(t)$ is decreasing, uniformly in time, and $\beta_{a-\delta}(t)$ gets smaller than $A + 2\delta$ in finite time. This means that $y_\alpha(t)$ enters $\mathcal{Z}_{+2\delta}$ in finite time when $y_\alpha(0) \in \mathcal{T}_+$.

We now consider the case $y_\alpha(0) \notin \mathcal{T}_+$. We choose the parametrization (3.11) given by the charts associated to the constant control $a - \delta$, and the initial point Y_θ lying on $\partial\mathcal{S}$. From (3.13) we deduce that $\theta_{a-\delta}(t)$ is nonincreasing up to t_0:

$$\forall t \in [0, t_0) \quad \dot\theta_{a-\delta}(t) \geq \delta\varphi(y_\alpha(t)) \exp\left(-s(t)\|D_y b\|_\infty\right) \left|\left\langle \frac{d}{d\theta} Y_\theta, n_{a-\delta}(Y_\theta)\right\rangle\right|^{-1}\,. \tag{3.18}$$

The function $\theta_{a-\delta}(t)$ is naturally bounded for $t \in [0, t_0)$. In fact, the point Y_θ cannot make a complete turn around the boundary $\partial\mathcal{S}$. For instance, when $Y_\theta = e_0$, we have necessarily $y_\alpha(t) \in \Phi_-$, since the trajectory $\gamma^0_{a-\delta}$ lies in Φ_-. We denote by $\overline\theta$ a bound from above for $\theta_{a-\delta}(t)$.

Recall that the time $s(t)$ is uniformly bounded on $\Phi_+ \setminus \mathcal{Z}_{+2\delta}$. As previously the factor $\left|\left\langle \frac{d}{d\theta} Y_\theta, n_{a-\delta}(Y_\theta)\right\rangle\right|^{-1}$ is clearly bounded from below since $\left|\left\langle \frac{d}{d\theta} Y_\theta, n_{a-\delta}(Y_\theta)\right\rangle\right|$ is bounded from above.

The descriptions of the dynamics close to Φ_0 (3.16) and e_∞ (3.17) are still valid. From (3.16) we deduce that the following estimate holds true for $|y_\alpha(t) - e_\beta| < \delta'$ and $\beta \leq a - 2\delta$,

$$\frac{d}{dt}\varphi(y_\alpha(t)) \leq -2\delta\nu + \delta\nu = -\delta\nu\,.$$

This defines a narrow band of size δ' around the portion $\{e_\beta : 0 \leq \beta \leq a - 2\delta\} \subset \Phi_0$ in which the quantity $\varphi(y_\alpha(t))$ is uniformly decreasing. Once it enters this narrow band, the trajectory necessarily crosses Φ_0 in finite time. It leaves Φ_+ and it enters the tunnel $\mathcal{T}_-$.

On the other hand, we build an analogous narrow band around the portion $\{e_\beta : A + 2\delta \leq \beta \leq +\infty\} \subset \Phi_0$. It is the union of the tunnel $\mathcal{T}_+$ and the ball $B(e_\infty, \eta')$.

If $y_\alpha(0)$ starts within $\mathcal{B}(e_\infty, \eta')$, the trajectory will leave the ball after a finite time of order $\tau' = 2|Ge_\infty|\eta'$. We cannot rule out the fact that it comes back inside the ball several times. Alternatively, we can evaluate the time the trajectory y_α spends inside the balls $\mathcal{B}(e_\infty, \eta')$ and $\mathcal{B}(e_\infty, 2\eta')$ each time it travels through $\mathcal{B}(e_\infty, \eta')$. Clearly the trajectory spends a time at most $\tau' = 2|Ge_\infty|\eta'$ inside the first ball, and a time at least τ' in the annulus $\mathcal{B}(e_\infty, 2\eta') \setminus \mathcal{B}(e_\infty, \eta')$. Outside the narrow bands, the quantity $\theta_{a-\delta}$ is monotonic with the following estimate:

$$\dot\theta_{a-\delta}(t) \geq c_{\eta'}\,,$$

for some constant $c_{\eta'} > 0$. As a conclusion, if we denote by N the number of times the trajectory y_α travels through $\mathcal{B}(e_\infty, \eta')$, we have

$$\theta_{a-\delta}(t_0) - \theta_{a-\delta}(0) \geq \underbrace{N\,(\tau'.0 + \tau' c_{\eta'})}_{\mathcal{B}(e_\infty, 2\eta')} + \underbrace{(t_0 - 2N\tau')c_{\eta'}}_{\text{outside the narrow bands}}\,.$$

Therefore, the number N is bounded. Moreover we have $N\tau' \leq \frac{\overline\theta}{c_{\eta'}}$. Consequently, the time t_0 is also bounded, and we have

$$t_0 \leq 2N\tau' + \frac{\overline\theta}{c_{\eta'}} = 3\frac{\overline\theta}{c_{\eta'}}\,.$$

We have finally proven that the trajectory $y_\alpha(t)$ leaves the set $\Phi_+ \setminus \mathcal{Z}_{+2\delta}$ after finite time. A precise follow-up of the time spent in this zone shows that it is uniformly bounded with respect to the initial data $y_\alpha(0)$. We now face the following alternative: either the trajectory enters the set $\mathcal{Z}_{+2\delta}$ at time t_0, or it enters the set $\Phi_- \setminus \mathcal{Z}_{+2\delta}$. In the former case there is nothing more to say. In the latter case, the trajectory crosses the line Φ_0. It does so on the portion $\{e_\beta \mid 0 \leq \beta \leq a - 2\delta\}$ (otherwise the vector field $b(e_\beta, \alpha(t_0))$ points in the wrong direction, from Φ_- to Φ_+, and the trajectory lies in the tunnel $\mathcal{T}_+$). We conclude that the trajectory enters the tunnel $\mathcal{T}_-$ at $t = t_0$.

From $t = t_0$ and $y_\alpha(t_0) \in \mathcal{T}_-$ we repeat the same procedure as for $y_\alpha(0) \in \mathcal{T}_+$, except that we change the control determining the charts: we switch from $\Gamma_{a-\delta}$ to $\Gamma_{A+\delta}$. Again, the quantity $\beta_{A+\delta}(t)$ is monotonic, and we have

$$\forall t \geq t_0 \quad \dot{\beta}_{A+\delta}(t) \geq \delta \left|\varphi(y_\alpha(t))\right| \exp\left(-s(t)\|D_y b\|_\infty\right) \left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|^{-1} .$$

This quantity is positive, and we can prove as above that the quantity $\varphi(y_\alpha(t))$ is uniformly decreasing around the portion $\{e_\beta : 0 \leq \beta \leq a - 2\delta\}$. We deduce as previously that the trajectory necessarily enters the set $\mathcal{Z}_{+2\delta}$ in finite time $T_0(y)$. The time is uniformly bounded with respect to the trajectory y_α, independently of ϵ. □

Step 6- Close-to-optimal trajectories do not stay close to the boundary of the ergodic set. Let $T_0(\delta)$ be the maximal time of entry in the set $\mathcal{Z}_{+2\delta}$ for the trajectory $\dot{y}_\alpha(t) = b(y_\alpha(t), \alpha(t))$. Let us mention that the set $\mathcal{Z}_{+2\delta}$ is stable (The proof is similar to the stability of $\mathcal{Z}_0$, see Lemma 3.5).

The final step in the proof of Theorem 3.2 consists in carefully following trajectories running in the layer $\Delta_\delta = \mathcal{Z}_{+2\delta} \setminus \mathcal{Z}_{-\delta}$. It is a narrow band surrounding the boundary of the ergodic set $\partial\mathcal{Z}_0$. Heuristically, if the trajectory stays in this band forever, then it follows closely the two trajectories defining $\partial\mathcal{Z}_0$ (3.7). Thus it spends most of the time in the neighbourhood of e_a and e_A, where the reward values are respectively $\lambda(a)$ and $\lambda(A)$. From Hypothesis **(H2)**, and particularly $\lambda(\alpha^*) > \max(\lambda(a), \lambda(A))$, we deduce that such a trajectory is far from optimality.

In the next lines we develop this argumentation with quantitative bounds. We will pay much attention to proving that the maximal exit time $T_1(\epsilon, \delta)$ outside Δ_δ is such that $\epsilon T_1(\epsilon, \delta) \ll 1$, uniformly.

The width of the narrow band $\Delta_\delta = \mathcal{Z}_{+2\delta} \setminus \mathcal{Z}_{-\delta}$ is of order δ. We introduce an intermediate scale η, with $\delta \ll \eta \ll 1$. We denote $B_a = B(e_a; 2\eta)$ and $B_A = B(e_A; 2\eta)$ the balls of radius η with respective centers a, A. The scales δ and η are to be determined later. Roughly speaking, the scale η is determined such that $\eta \ll \lambda(\alpha^*) - \max(\lambda(a), \lambda(A))$, and δ is adjusted accordingly.

Our claim is the following: if δ is chosen small enough, then close-to-optimal trajectories cannot stay in Δ_δ forever. Let us mention that the set $\mathcal{Z}_{-\delta}$ is not stable. However we require only that the trajectory enters once in $\mathcal{Z}_{-\delta}$ where we can use the uniform controllability Lemma 3.6.

Lemma 3.11. *Let $y_\alpha(t)$ be a close-to-optimal trajectory in the following sense:*

$$\int_0^{+\infty} e^{-\epsilon t} L(y_\alpha(t))\, dt \geq u_\epsilon(y) - O(1) .$$

This trajectory must leave the narrow band Δ_δ before some maximal time $T_1(\epsilon, \delta)$. In addition we have $\epsilon T_1(\epsilon, \delta) = O(\eta)$ as $\epsilon \to 0$, uniformly for $y \in \mathcal{S}$.

Proof. We begin with some considerations about optimality of trajectories. A first important remark is that for all $y \in \mathcal{S}$ we have $\liminf_{\epsilon \to 0} \epsilon u_\epsilon(y) \geq \lambda(\alpha^*)$, uniformly in $\mathcal{S}$. Indeed we may

choose the constant value control $\alpha(t) \equiv \alpha^*$ in the optimization problem (3.3). We know that exponential convergence occurs towards the eigenvector e_{α^*}:

$$|y_{\alpha^*}(t) - e_{\alpha^*}| \leq C|y - e_{\alpha^*}|e^{-\mu_{\alpha^*} t}\,,$$

where μ_{α^*} is the spectral gap and C denotes some absolute constant. This yields the estimate

$$\begin{aligned}
\epsilon \int_0^{+\infty} e^{-\epsilon t} L(y_{\alpha^*}(t))\, dt &= \epsilon \int_0^{+\infty} e^{-\epsilon t} L(e_{\alpha^*})\, dt + \epsilon \int_0^{+\infty} e^{-\epsilon t} \left(L(y_{\alpha^*}(t)) - L(e_{\alpha^*})\right) dt \\
&\geq \lambda(\alpha^*) - \|DL\|_\infty \epsilon \int_0^{+\infty} e^{-\epsilon t} |y_{\alpha^*}(t) - e_{\alpha^*}|\, dt \\
&\geq \lambda(\alpha^*) - C\|DL\|_\infty |y - e_{\alpha^*}| \epsilon \int_0^{+\infty} e^{-\epsilon t} e^{-\mu_{\alpha^*} t}\, dt \\
&\geq \lambda(\alpha^*) - C\|DL\|_\infty \frac{\epsilon}{\epsilon + \mu_{\alpha^*}}\,.
\end{aligned}$$

By letting $\epsilon \to 0$ we deduce the estimate

$$\liminf_{\epsilon \to 0} \epsilon u_\epsilon(y) \geq \lambda(\alpha^*)\,, \quad \text{uniformly for } y \in \mathcal{S}\,. \tag{3.19}$$

Now we examine the reward of trajectories during their passage in Δ_δ. Let y_α be such that

$$\forall t \in (T_0(y), T_1(y)) \quad y_\alpha(t) \in \Delta_\delta\,,$$

(note that we may *a priori* have $T_1(y) = +\infty$). We distinguish between the different parts of Δ_δ. Either $y_\alpha(t)$ belongs to $\Delta_\delta \cap (B_a \cup B_A)$ or it belongs to the complementary set. We begin with the latter case. We assume that

$$\forall t \in (t_1, t_2) \quad y_\alpha(t) \in \Delta_\delta \setminus (B_a \cup B_A)\,.$$

We first prove that, outside the balls B_a and B_A, $y_\alpha(t)$ is close to either the curve γ_a or the curve γ_A. We deal with the upper part $\Delta_\delta \cap \Phi_-$, where the curve γ_A lies, without loss of generality. We use Lemma 3.9 for this purpose. We recall (3.13) adapted to the constant value control A (instead of $A + \delta$):

$$\dot\beta_A(t) \geq (A - \alpha(t))\, |\varphi(y_\alpha(t))| \exp\left(-s(t)\|D_y b\|_\infty\right) \left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|^{-1}\,.$$

Outside the balls B_a and B_A, the time $s(t)$ is uniformly bounded, and $|\varphi|$ is bounded from below. We have

$$|\varphi(y_\alpha(t))| \exp\left(-s(t)\|D_y b\|_\infty\right) \left| \left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle \right|^{-1} \geq c_\eta\,,$$

for some positive constant c_η. Thus,

$$c_\eta \int_{t_1}^{t_2} (A - \alpha(t))\, dt \leq \beta(t_2) - \beta(t_1) \leq 3\delta\,.$$

We deduce that

$$\begin{aligned}
\frac{d}{dt}|\gamma_A(t+\tau) - y_\alpha(t)|^2 &= 2\left\langle b(\gamma_A(t+\tau), A) - b(y_\alpha(t), \alpha(t)), \gamma_A(t+\tau) - y_\alpha(t)\right\rangle \\
&= 2\left\langle b(\gamma_A(t+\tau), \alpha(t)) - b(y_\alpha(t), \alpha(t)), \gamma_A(t+\tau) - y_\alpha(t)\right\rangle \\
&\quad + 2(A - \alpha(t))\left\langle F\gamma_A(t+\tau), \gamma_A(t+\tau) - y_\alpha(t)\right\rangle \\
&\leq 2\|D_y b\| |\gamma_A(t+\tau) - y_\alpha(t)|^2 + C(A - \alpha(t))\,,
\end{aligned}$$

where C denotes some absolute constant. Therefore,

$$|\gamma_A(t+\tau) - y_\alpha(t)|^2 \leq e^{2\|D_y b\|(t-t_1)}|\gamma_A(t_1+\tau) - y_\alpha(t_1)|^2 + C\int_{t_1}^{t} e^{2\|D_y b\|(t'-t)}(A-\alpha(t'))\,dt'\,.$$

We choose a suitable time shift τ such that $\gamma_A(t_1+\tau) - y_\alpha(t_1) = O(\delta)$. This yields the estimate

$$\forall t \in (t_1, t_2) \quad |\gamma_A(t+\tau) - y_\alpha(t)|^2 \leq C\delta^2 e^{2\|D_y b\|(t-t_1)} + C\frac{\delta}{c_\eta}\,. \tag{3.20}$$

Because it is a converging trajectory, we know that γ_A spends at most a time T_η in the zone outside the ball $B(e_A, \eta)$. We choose δ so small that the following inequality holds,

$$C\delta^2 e^{2\|D_y b\|T_\eta} + C\frac{\delta}{c_\eta} < \eta^2\,.$$

From the tracking estimate (3.20) we deduce that during the time $t \in (t_1, t_1+T_\eta)$ we have the following estimate,

$$\sup_{t\in(t_1,t_1+T_\eta)} |y_\alpha(t) - \gamma_A(t+\tau)|^2 \leq C\delta^2 e^{2\|D_y b\|T_\eta} + C\frac{\delta}{c_\eta} < \eta^2\,.$$

Therefore we have $y_\alpha(t_1+T_\eta) \in B_A = B(e_A, 2\eta)$. And so we have $t_2 \leq t_1 + T_\eta$ (but this estimate is inessential).

Concerning the trajectory γ_A, exponential convergence towards the eigenvector e_A holds true:

$$\forall t > 0 \quad |\gamma_A(t+\tau) - e_A| \leq C|\gamma_A(\tau) - e_A|e^{-\mu_A t}\,, \tag{3.21}$$

where $\mu_A > 0$ is the spectral gap, and C denotes some absolute constant. In fact we replace μ_A by $\underline{\mu} = \min(\mu_a, \mu_A)$ in (3.21) for the sake of convenience. Plugging this estimate into the optimal reward, we get

$$\begin{aligned}
\epsilon\int_{t_1}^{t_2} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' &= \epsilon\int_{t_1}^{t_2} e^{-\epsilon t'} L(\gamma_A(t'+\tau))\,dt' + \epsilon\int_{t_1}^{t_2} e^{-\epsilon t'}\left[L(y_\alpha(t')) - L(\gamma_A(t'+\tau))\right]\,dt' \\
&\leq \epsilon\int_{t_1}^{t_2} e^{-\epsilon t'} L(e_A)\,dt' + \epsilon\int_{t_1}^{t_2} e^{-\epsilon t'}\left[L(\gamma_A(t'+\tau)) - L(e_A)\right]\,dt' \\
&\quad + \eta\|DL\|_\infty\left(e^{-\epsilon t_1} - e^{-\epsilon t_2}\right) \\
&\leq \lambda(A)\left(e^{-\epsilon t_1} - e^{-\epsilon t_2}\right) + C\|DL\|_\infty\frac{\epsilon}{\epsilon+\underline{\mu}}\left(e^{-(\epsilon+\underline{\mu})t_1} - e^{-(\epsilon+\underline{\mu})t_2}\right) \\
&\quad + \eta\|DL\|_\infty\left(e^{-\epsilon t_1} - e^{-\epsilon t_2}\right)\,.
\end{aligned}$$

We conclude that there exists some absolute constant C such that

$$\epsilon\int_{t_1}^{t_2} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' \leq \overline{\lambda}\left(e^{-\epsilon t_1} - e^{-\epsilon t_2}\right) + \epsilon C\left(e^{-(\epsilon+\underline{\mu})t_1} - e^{-(\epsilon+\underline{\mu})t_2}\right) + \eta C\left(e^{-\epsilon t_1} - e^{-\epsilon t_2}\right)\,, \tag{3.22}$$

where we have introduced the notation: $\overline{\lambda} = \max(\lambda(a), \lambda(A))$.

We now turn to the second (easier) case:

$$\forall t \in (t_2, t_3) \quad y_\alpha(t) \in \Delta_\delta \cap (B_a \cup B_A)\,.$$

We assume that $y_\alpha(t)$ lies inside the ball B_A, without loss of generality. We directly have:

$$\begin{aligned}\epsilon\int_{t_2}^{t_3} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' &= \epsilon\int_{t_2}^{t_3} e^{-\epsilon t'} L(e_A)\,dt' + \epsilon\int_{t_2}^{t_3} e^{-\epsilon t'}\left[L(y_\alpha(t')) - L(e_A)\right]dt' \\ &\leq \lambda(A)\left(e^{-\epsilon t_2} - e^{-\epsilon t_3}\right) + \eta\|DL\|_\infty\left(e^{-\epsilon t_2} - e^{-\epsilon t_3}\right).\end{aligned}$$

We conclude that there exists some absolute constant C such that we also have

$$\epsilon\int_{t_2}^{t_3} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' \leq \overline{\lambda}\left(e^{-\epsilon t_2} - e^{-\epsilon t_3}\right) + \epsilon C\left(e^{-(\epsilon+\underline{\mu})t_2} - e^{-(\epsilon+\underline{\mu})t_3}\right) + \eta C\left(e^{-\epsilon t_2} - e^{-\epsilon t_3}\right). \tag{3.23}$$

All in all we add successively estimations (3.22) and (3.23). Using telescopic cancellations, we end up with

$$\epsilon\int_{T_0}^{T_1} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' \leq \overline{\lambda}\left(e^{-\epsilon T_0} - e^{-\epsilon T_1}\right) + \epsilon C\left(e^{-(\epsilon+\underline{\mu})T_0} - e^{-(\epsilon+\underline{\mu})T_1}\right) + \eta C\left(e^{-\epsilon T_0} - e^{-\epsilon T_1}\right). \tag{3.24}$$

This proves that, for ϵ and η small enough, we necessarily have $T_1(y) < +\infty$. Otherwise the trajectory would not be optimal thanks to the estimate (3.19), and also the trivial bound:

$$\epsilon\int_0^{T_0} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' \leq \epsilon T_0(\delta)\|L\|_\infty.$$

This prove that close-to-optimal trajectories necessarily enter the set $\mathcal{Z}_{-\delta}$ before some maximal time $T_1(\epsilon,\delta)$.

In fact we can estimate better the maximal time $T_1(\epsilon,\delta)$, and prove that $\epsilon T_1(\epsilon,\delta) = O(\eta)$ as $\epsilon \to 0$. Let y_α be a close-to-optimal trajectory, starting from $y \in \mathcal{S}$. Following (3.24) we have

$$\begin{aligned}\epsilon\left(u_\epsilon(y) - O(1)\right) &\leq \epsilon\int_0^{T_1} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' + \epsilon e^{-\epsilon T_1} u_\epsilon(y_\alpha(T_1)) \\ &\leq \epsilon\int_0^{T_0} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' + \epsilon\int_{T_0}^{T_1} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' + e^{-\epsilon T_1(y)}\epsilon u_\epsilon(y_\alpha(T_1)) \\ &\leq \epsilon T_0(\delta)\|L\|_\infty + \overline{\lambda}\left(1 - e^{-\epsilon T_1(y)}\right) + O(\epsilon) + O(\eta) + e^{-\epsilon T_1(y)}\epsilon u_\epsilon(y_\alpha(T_1)).\end{aligned}$$

We deduce

$$e^{-\epsilon T_1(y)}\left(\epsilon u_\epsilon(y_\alpha(T_1)) - \overline{\lambda}\right) \geq \epsilon u_\epsilon(y) - \overline{\lambda} + O(\epsilon T_0(\delta)) + O(\epsilon) + O(\eta).$$

Therefore we have

$$\epsilon T_1(y) \leq \log\left(\frac{\epsilon u_\epsilon(y_\alpha(T_1)) - \overline{\lambda}}{\epsilon u_\epsilon(y) - \overline{\lambda} + O(\epsilon T_0(\delta)) + O(\epsilon) + O(\eta)}\right). \tag{3.25}$$

We strongly use the fact that $y_\alpha(T_1) \in \mathcal{Z}_{-\delta}$. There exists a time $T_2(\delta)$ (not depending on ϵ) such that we can send y onto $y_\alpha(T_1)$ within time $T_2(\delta)$. For this, simply send y onto a point $y' \in \mathcal{Z}_{-\delta}$ (for example with constant control $\alpha(t) \equiv \alpha^*$), then connect y' to $y_\alpha(T_1)$ thanks to Lemma 3.6. Notice that the time $T_2(\delta)$ may degenerate as η (and thus δ) goes to 0. Using the dynamic programming principle we get,

$$\epsilon u_\epsilon(y) \geq \epsilon\int_0^{T_2} e^{-\epsilon t'} L(y_\alpha(t'))\,dt' + \epsilon e^{-\epsilon T_2(\delta)} u_\epsilon(y_\alpha(T_1)).$$

Plugging this estimate in (3.25), we conclude

$$\begin{aligned}\epsilon T_1(y) &\le \log\left(\frac{\epsilon u_\epsilon(y_\alpha(T_1)) - \overline{\lambda}}{e^{-\epsilon T_2(\delta)}\epsilon u_\epsilon(y_\alpha(T_1)) - \overline{\lambda} + O(\epsilon T_2(\delta)) + O(\epsilon T_0(\delta)) + O(\epsilon) + O(\eta)}\right)\\ &\le -\log\left(1 + \frac{\left(e^{-\epsilon T_2(\delta)} - 1\right)\epsilon u_\epsilon(y_\alpha(T_1)) + O(\epsilon T_2(\delta)) + O(\epsilon T_0(\delta)) + O(\epsilon) + O(\eta)}{\epsilon u_\epsilon(y_\alpha(T_1)) - \overline{\lambda}}\right).\end{aligned}$$

Finally, we obtain the estimate

$$\epsilon T_1(y) = O(\eta) \quad \text{as } \epsilon \to 0\,, \quad \text{uniformly for } y \in \mathcal{S}\,.$$

This concludes the proof of Lemma 3.11. □

7- Conclusion. We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. First, the function $\epsilon u_\epsilon(y)$ is uniformly bounded on the simplex $\mathcal{S}$: $\|\epsilon u_\epsilon\|_\infty \le \|L\|_\infty$ (3.8). Second, we show equicontinuity of the family of functions $(\epsilon u_\epsilon)_\epsilon$. Let $y, y' \in \mathcal{S}$. We assume without loss of generality that y and y' lie outside the ergodic set $\mathcal{Z}_0$. From Steps 5 and 6 we deduce that the trajectories $y_\alpha(t)$ and $y'_\alpha(t)$ enter the approximated ergodic set $\mathcal{Z}_{-\delta}$ within times $T_1(y)$ and $T_1(y')$ respectively, such that $\epsilon T_1 = O(\eta)$. From Step 2 we can connect $z = y_\alpha(T_1(y))$ and $z' = y'_\alpha(T_1(y'))$ within time $T_2(\delta)$ which is independent of ϵ. From the dynamic programming principle, we have

$$\begin{aligned}\epsilon u_\epsilon(y') - \epsilon u_\epsilon(y) &\le \epsilon \int_0^{T_1(y')} e^{-\epsilon t} L(y'_\alpha(t))\,dt + \epsilon e^{-\epsilon T_1(y')} u_\epsilon(z')\\ &\qquad - \epsilon \int_0^{T_1(y)} e^{-\epsilon t} L(y_\alpha(t))\,dt - \epsilon e^{-\epsilon T_1(y)} u_\epsilon(z) + O(\epsilon)\\ &\le \epsilon u_\epsilon(z') - \epsilon u_\epsilon(z) + 2\epsilon T_1(y')\|L\|_\infty + 2\epsilon T_1(y)\|L\|_\infty + O(\epsilon)\\ &\le O(\epsilon T_2(\delta)) + O(\epsilon T_1) + O(\epsilon)\,.\end{aligned}$$

The last estimate was obtained at Step 3 (controllability argument). We end up with

$$\epsilon u_\epsilon(y') - \epsilon u_\epsilon(y) \le O(\epsilon T_2(\delta)) + O(\eta) + O(\epsilon)\,.$$

Exchanging the roles of y and y' we obtain the uniform bound,

$$\forall (y, y') \in \mathcal{S} \times \mathcal{S} \quad |\epsilon u_\epsilon(y') - \epsilon u_\epsilon(y)| \le O(\eta) \quad \text{as } \epsilon \to 0\,.$$

Since the parameter η can be chosen arbitrarily small, we get that the family $(\epsilon u_\epsilon)_\epsilon$ converges uniformly towards some constant (up to extraction). In fact the limit is unique [5, 3]. we re-do the proof of this statement for the sake of completeness.

Assume there exist two constant values $l_1 < l_2$ and to subsequences $(u_{\epsilon_1}), (u_{\epsilon_2})$ such that $\lim_{\epsilon_1 \to 0} \epsilon_1 u_{\epsilon_1} = l_1$ and $\lim_{\epsilon_2 \to 0} \epsilon_2 u_{\epsilon_2} = l_2$ uniformly in $\mathcal{S}$. We recall that $-u_\epsilon$ is the viscosity solution of the stationary Hamilton-Jacobi equation (3.4). Since the convergence is uniform, we can find ν, ϵ_1 and ϵ_2 small enough such that $-\epsilon_1 u_{\epsilon_1} \ge -l_1 - \nu$, and $-\epsilon_2 u_{\epsilon_2} \le -l_2 + \nu < -l_1 - \nu$. Therefore, $-u_{\epsilon_1}$ is a viscosity subsolution of

$$-l_1 - \nu + H(y, D_y u_{\epsilon_1}) \le 0\,, \tag{3.26}$$

and $-u_{\epsilon_2}$ is a viscosity supersolution of the same equation,

$$-l_1 - \nu + H(y, D_y u_{\epsilon_2}) \ge 0\,. \tag{3.27}$$

We deduce from standard comparison theorems [5, 12] that $\forall y \in \mathcal{S}\ -u_{\epsilon_1}(y) \leq -u_{\epsilon_2}(y)$. However we can add to u_{ϵ_2} a large positive constant such that $u_{\epsilon_2} > u_{\epsilon_1}$, and (3.26)–(3.27) are still verified. This is a contradiction.

We conclude that there is a unique possible limit for the sequence $(\epsilon u_\epsilon)_\epsilon$. This completes the proof of Theorem 3.2. □

3.4 Proof of Corollary 3.3

The proof of Corollary 3.3 following Theorem 3.2 is contained in [3, Theorem 5]. We repeat the argument for the sake of completeness. We fix $\delta > 0$, and we choose $T = \frac{\delta}{\epsilon}$. We split the rewards as follows:

$$\epsilon u_\epsilon(y) = \sup_\alpha \left\{ \epsilon \int_0^T L(y_\alpha(t))\, dt + \epsilon \int_0^T \left(e^{-\epsilon t} - 1\right) L(y_\alpha(t))\, dt + \epsilon e^{-\epsilon T} u_\epsilon(y_\alpha(T)) \right\}. \tag{3.28}$$

We have for the second contribution,

$$\left| \epsilon \int_0^T \left(e^{-\epsilon t} - 1\right) L(y_\alpha(t))\, dt \right| \leq \|L\|_\infty \left(e^{-\delta} - 1 + \delta\right).$$

Therefore, dividing (3.28) by δ we get as $\epsilon \to 0$ (or equivalently $T \to +\infty$),

$$\liminf_{T\to+\infty} \frac{1}{T} \sup_\alpha \left\{ \int_0^T L(y_\alpha(t))\, dt \right\} + \frac{\left(e^{-\delta} - 1\right)}{\delta} \lambda_{HJ} = O(\delta),$$

$$\limsup_{T\to+\infty} \frac{1}{T} \sup_\alpha \left\{ \int_0^T L(y_\alpha(t))\, dt \right\} + \frac{\left(e^{-\delta} - 1\right)}{\delta} \lambda_{HJ} = O(\delta),$$

where we have used the uniform convergence $\epsilon u_\epsilon \to \lambda_{HJ}$ in $\mathcal{S}$. Since $\delta > 0$ can be chosen arbitrarily small, Corollary 3.3 is proven.

4 Numerical simulations and perspectives

All the numerical simulations shown in Section 3 have been performed for the running example (1.4) with parameters listed in Table 1. We have solved numerically the Hamilton-Jacobi-Bellman equation (3.1) using a classical upwind scheme for the discretization of the Hamiltonian (3.2). We observe that the quantity $\frac{u(T,y)}{T}$ converges to a constant value λ_{HJ} which is close to the Perron eigenvalue $\lambda_P(\alpha^*)$. The discrepancy in the limit falls within the range of error due to the numerical scheme. Furthermore, we observe that the function $u(T, y) - \lambda_{HJ} T$ presumably converges to an eigenvector $\overline{u}$ (Figure 8).

We notice that the numerical scheme selects either a or A at each step because of the very definition of the hamiltonian (3.2). It is quite instructive to plot the line where the control switches from a to A, namely where $\langle Fy, D_y \overline{u} \rangle = 0$. We call it the *separation line* (Figure 8b). Apparently the optimal eigenvector e_{α^*} belongs to this line. We observe numerically that the optimal trajectories are glued to this separation line (Figure 9a). We observe fast oscillations between the extremal values a and A at the scale of the time step (see Figure 9b). The values

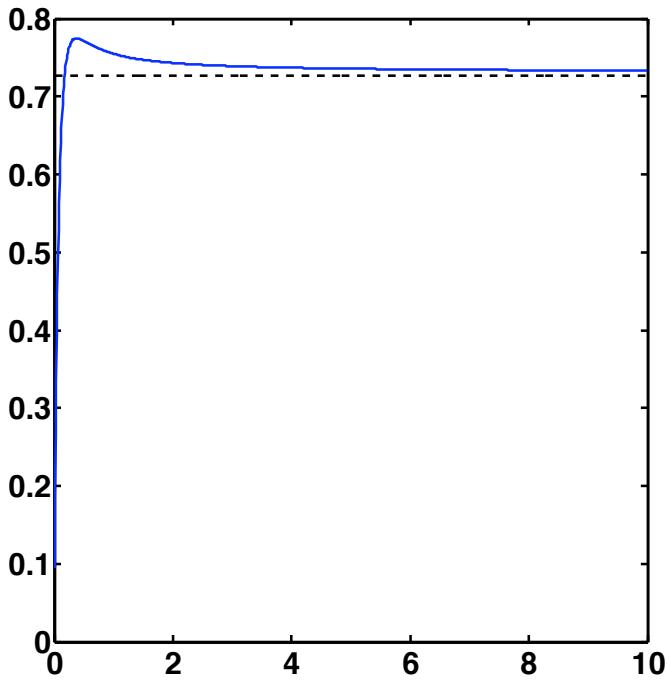


Figure 7: Convergence of $\frac{u(T,y)}{T}$ towards a constant value λ_{HJ}. We have plotted a specific value at an arbitrary point $y_0 \in \mathcal{S}$. We observe that the limiting value λ_{HJ} is close to the maximal Perron eigenvalue $\lambda_P \approx 0.7273$ (dashed line, see also Figure 1). The discrepancy between the limiting value and λ_P (dashed line) falls below the numerical error due to the scheme, which is of order one.

Growth in the first compartment	$\tau_1 = 0.5$
Growth in the second compartment	$\tau_2 = 5$
fragmentation in the second compartment	$\beta_2 = 1$
fragmentation in the third compartment	$\beta_3 = 2$
minimal rate of sonication	$a = 1$
maximal rate of sonication	$A = 6$
optimal rate of sonication (constant control)	$\alpha^* = 3.35$
space step in the simplex	$\Delta y = 10^{-2}$
time step	$\Delta t = 10^{-3}$
final time of computation	$T = 10$

Table 1: Set of parameters for illustrations and numerical simulations.

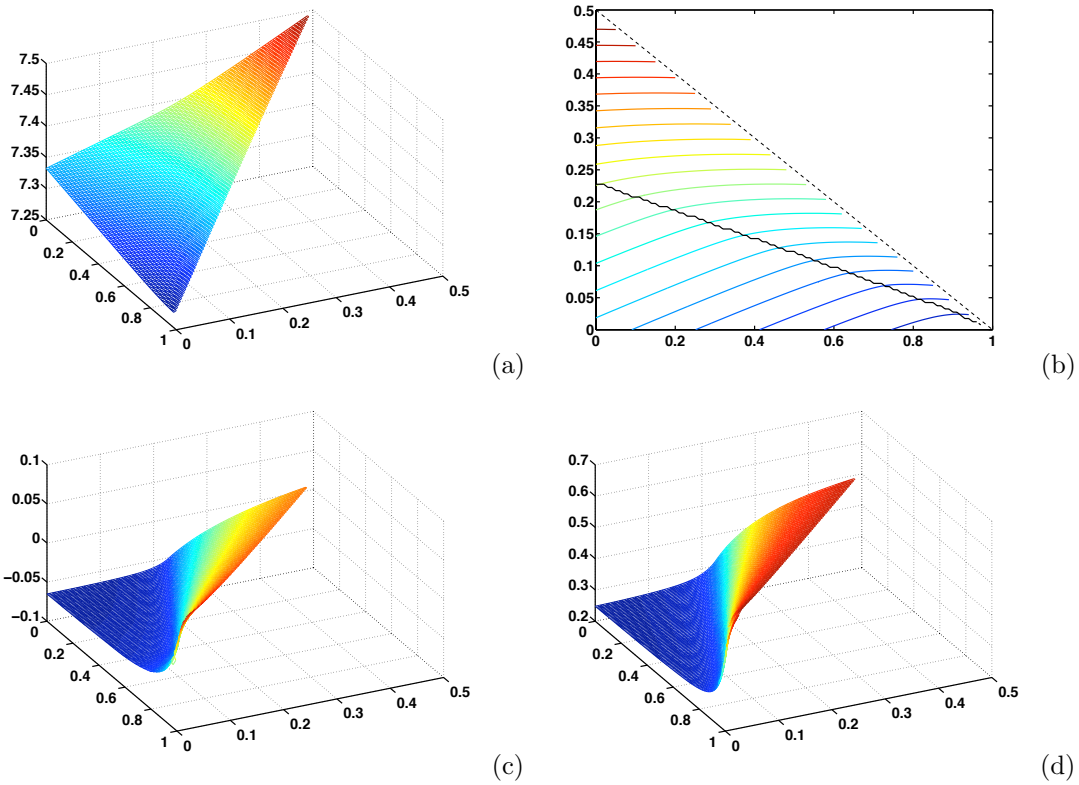
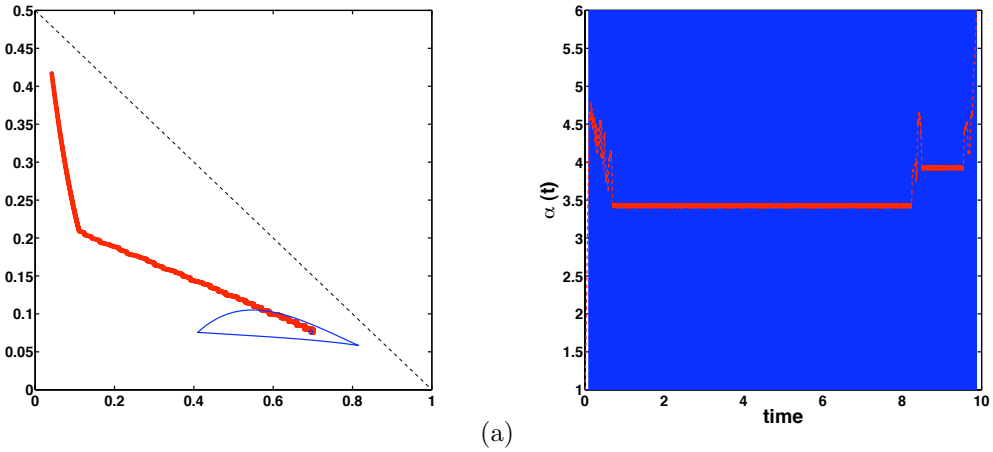


Figure 8: (a) The function $u(T, y)$ presumably converges to an eigenvector $\overline{u}$ as $T \to +\infty$, modulo a constant of order $\lambda_{HJ}T$. (b) Level sets of the function $\overline{u}$ and the *separation line* where the optimal control switches from a to A. (c) and (d) Derivatives of the function $\overline{u}$ with respect to the first and the second variable.



(a) (b)

Figure 9: (a) An optimal trajectory starting from the upper corner of the simplex. It presumably converges towards the optimal eigenvector e_{α^*}. The optimal trajectory appears to be glued to the separation line. (b) The corresponding optimal control $\alpha(t)$ is plotted in blue. It oscillates rapidly between the two extremal values $a = 1, A = 6$. It converges weakly to $\alpha^* = 3.35$ as the time step Δt goes to zero (here, $\Delta t = 10^{-3}$), as can be observed when averaging the function $\alpha(t)$ (in red: local average of $\alpha(t)$ with a span of $100\Delta t$).

a and A are chosen in such a way that local averaging over several time steps yields a constant control α^*. We conjecture that the control $\alpha(t)$ obtained through a bang-bang procedure (thus taking extremal values a and A) converges weakly to the constant control α^* in infinite horizon.

We now list perspectives for future attention, by discussing the assumptions made in the present work. The first question concerns the higher-dimensional case. Here the two dimensional structure of the simplex plays a crucial role. For instance, it is not even clear how to define properly the ergodic set in higher dimension of space. A natural extension would be to study the case where the control $\alpha(t)$ has $N - 1$ degree of freedom, where N is the dimension of the simplex (the original problem (1.1) being of dimension $N + 1$). Next we discuss ways to remove the hypotheses **(H1-2-3-4-5)**, separately.

(H1)-(H3) In the case where G or F is irreducible, the limiting eigenvectors do not lie on the boundary of the simplex. Therefore the line of Perron eigenvectors $\{e_\beta : 0 \leq \beta \leq \infty\}$ does not divide the simplex into two parts. However the line $\Phi_0 = \{y \in \mathcal{S} : \varphi(y) = 0\}$ does so. It is the zero level set of a cubic function. We have plotted in Figure 10 several examples, for the cases where G or F, or both, are irreducible.

(H2) When the Perron eigenvalue $\lambda_P(\alpha)$ is nondecreasing with respect to α and bounded (cf. Proposition 2.1 for the running example), we have $\max_{\alpha\in[a,A]} \lambda_P(\alpha) = \lambda_P(A)$. Our strategy of proof does not contain this particular example (see Step 6). Nevertheless, under an additional assumption (see below) we exhibit a particular solution to (3.1) with a different

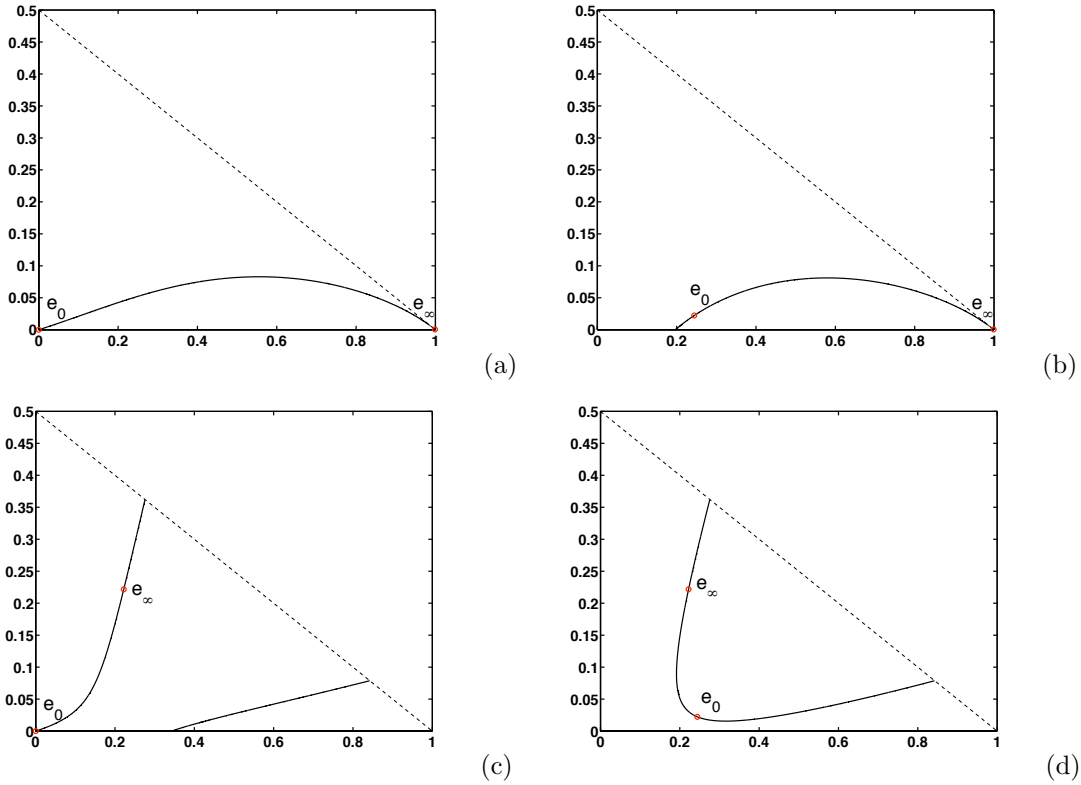


Figure 10: Discussion of Hypotheses **(H1)-(H3)**. We have plotted the zero level set of the cubic function $\Phi_0 = \{y \in \mathcal{S} \ : \ \varphi(y) = 0\}$ together with the extremal eigenvectors e_0 and e_∞ in four cases: (a) G and F are both reducible (*i.e.* the running example (1.4)), (b) G is irreducible and F is reducible, (c) G is reducible and F is irreducible, (d) G and F are both irreducible. We have modified F such that we still have $m^T F = 0$.

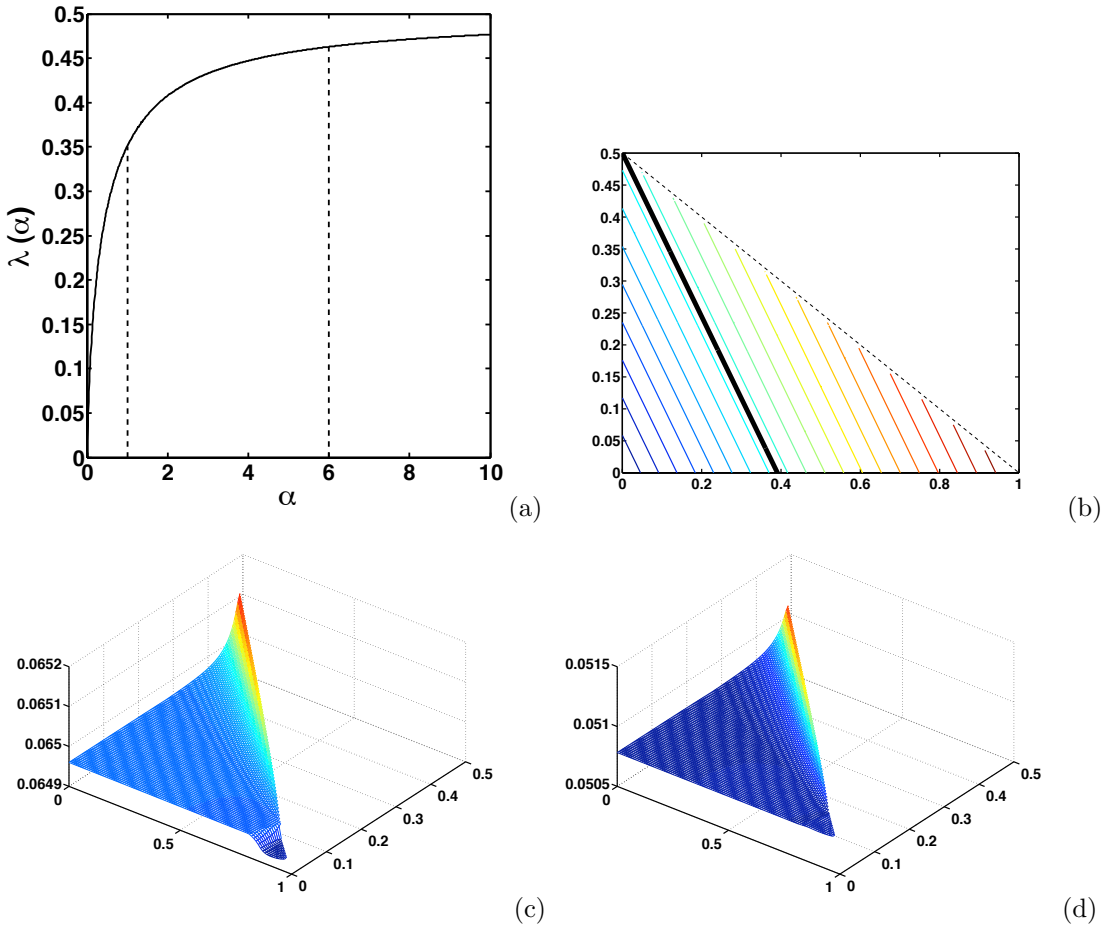


Figure 11: Discussion of Hypothesis **(H2)**. Here, $\tau_1 = \tau_2 = 0.5$, $a = 1$ and $A = 6$. The function $\lambda(\alpha)$ is increasing (a). The optimal control is $\alpha(t) = A$. Consequently $\lambda^* = \lambda_P(A)$. The eigenvector is also known: $\overline{u}(y) = \log\langle \phi_A, y\rangle$. This claim is confirmed by numerical simulations: the level sets of $u(T, y)$ are straight lines (b). The slope of the lines is in accordance with ϕ_A (the thick line is directed by $m \wedge \phi_A \in T\mathcal{S}$). The gradient of the function $\exp(u(T, y) - u(T, y_0))$ is almost constant: observe the small amplitude in (c) and (d).

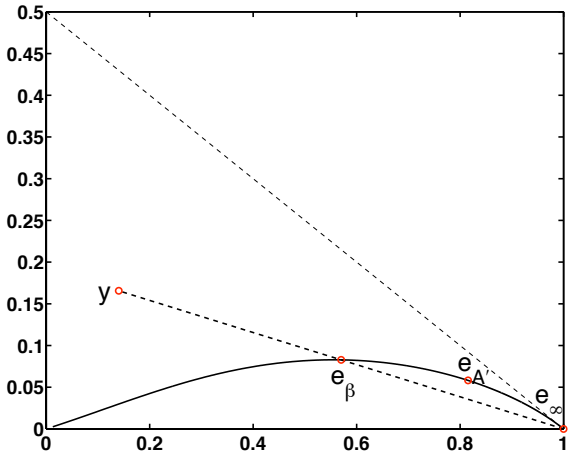


Figure 12: Construction of the subset $\mathcal{S}'$: the segment $[y, e_\infty]$ crosses the line Φ_0 on e_β with $\beta \leq A'$

initial condition, namely

$$u(t,y) = \lambda_P(A)t + \log \langle \phi_A, y \rangle \ . \tag{4.1}$$

Interestingly this coincides with the expansion of $r_\alpha(T,x)$ in the case of constant control $\alpha = A$ (2.1). Moreover the function $\overline{u}(y) = \log \langle \phi_A, y \rangle$ is a good candidate for being the Hamilton-Jacobi eigenvector. It is a viscosity solution of the ergodic stationary equation,

$$-\lambda_P(A) + H(y, D_y \overline{u}(y)) = 0 \, .$$

In Figure 11 we show numerical simulations which confirm that this particular solution describes well the asymptotic behaviour of solutions to (3.1).
To assert that $u(t,y)$ given by (4.1) is a particular solution of (3.1) we check that the optimality condition is verified for $\alpha = A$,

$$\begin{aligned} \max_{\alpha \in [a,A]} \langle b(y,\alpha), D_y u \rangle &= \max_{\alpha \in [a,A]} \left\{ \frac{\langle (G + \alpha F) y, \phi_A \rangle}{\langle y, \phi_A \rangle} - \langle m, Gy \rangle \right\} \\ &= \max_{\alpha \in [a,A]} \left\{ \lambda_P(A) + (\alpha - A) \frac{\langle Fy, \phi_A \rangle}{\langle y, \phi_A \rangle} \right\} \\ &= \lambda_P(A) \, . \end{aligned}$$

The optimality condition is satisfied if $\langle Fy, \phi_A \rangle \geq 0$ for all $y \in \mathcal{S}$. This condition is guaranteed under the two following conditions: (i) $\alpha \mapsto \lambda_P(\alpha)$ is nondecreasing, and (ii) for all $y \in \mathcal{S}$ the segment $[y, e_\infty]$ crosses the line of eigenvectors Φ_0, where e_∞ is a corner of the simplex (Figure 12). The second condition is satisfied for the running example since in this case $e_0 = (0\ 0\ 1/3)^T$ and $e_\infty = (1\ 0\ 0)^T$ are corners of the simplex $\mathcal{S}$ and the tangent to Φ_0 at the point e_∞, given by the vector $Ge_\infty - \langle m, Ge_\infty \rangle \, e_\infty = \tau_1(-2\ 1\ 0)^T$, coincides with the edge of the simplex which does not contain e_0. We introduce the intersection point $e_\beta \in \Phi_0$. We decompose $y = c_\beta e_\beta + c_\infty e_\infty$, with $c_\beta \geq 0$ (and $c_\infty \in \mathbb{R}$). We have on

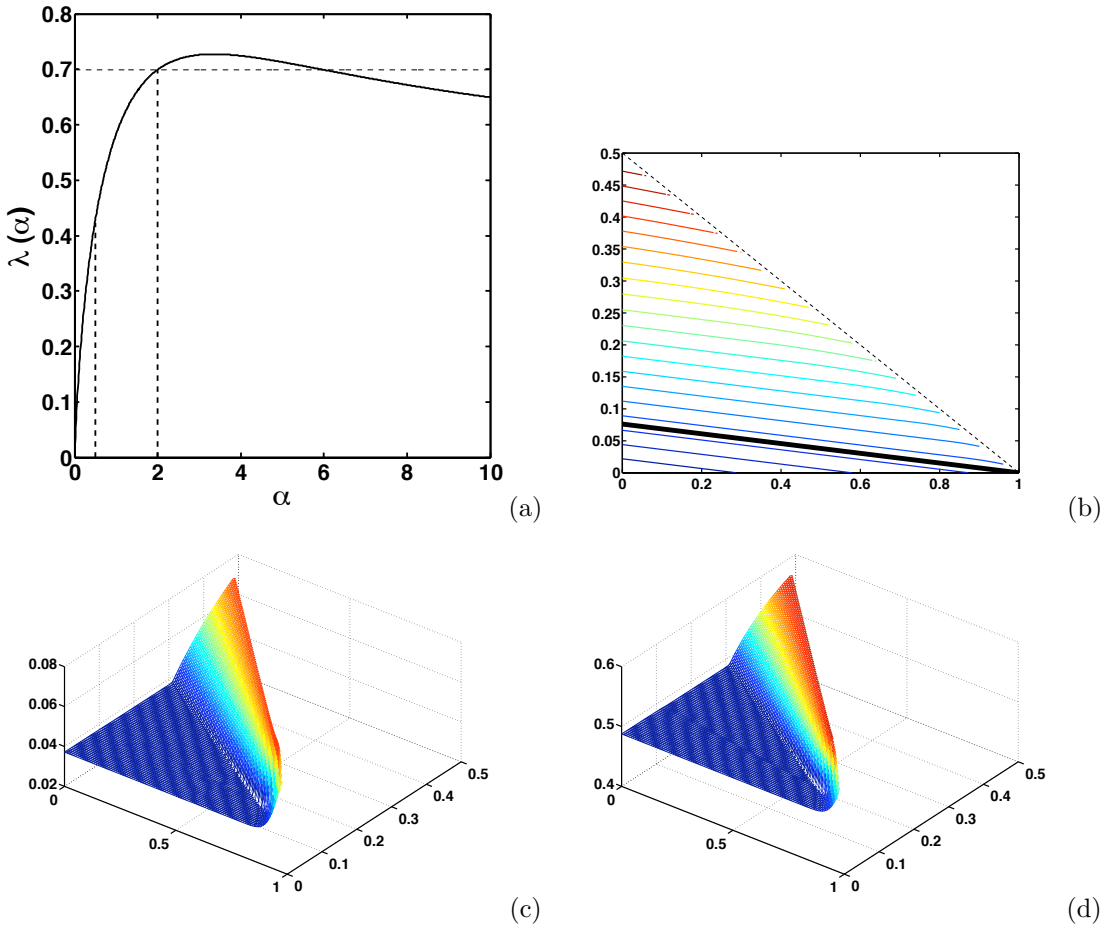


Figure 13: Discussion of Hypothesis **(H2)**. Here, $\tau_1 = 0.5$, $\tau_2 = 5$, $a = 0.5$ and $A = 2$. The function $\lambda(\alpha)$ is increasing up to $\alpha^* = 3.35$, then decreasing (a). The eigenvector is known on a subpart $\mathcal{S}' \subset \mathcal{S}$, where $\mathcal{S}'$ is defined in the text: $\forall y \in \mathcal{S}'\ \overline{u}(y) = \log \langle \phi_A, y \rangle$. This claim is confirmed by numerical simulations: the level sets of $u(T, y)$ are straight lines on $\mathcal{S}'$ (b). The slope of the lines is in accordance with ϕ_A (the thick line is directed by $m \wedge \phi_A \in T\mathcal{S}$). The gradient of the function $\exp(u(T, y) - u(T, y_0))$ is plotted in (c) and (d).

the one hand,

$$\lambda_P(A)\,\langle \phi_A, e_\beta\rangle = \phi_A(G+AF)e_\beta = \lambda_P(\beta)\,\langle \phi_A, e_\beta\rangle + (A-\beta)\,\langle \phi_A, Fe_\beta\rangle\ .$$

On the other hand we get since $Fe_\infty = 0$,

$$\begin{aligned}\langle Fy, \phi_A\rangle &= c_\beta\,\langle Fe_\beta, \phi_A\rangle + c_\infty\,\langle Fe_\infty, \phi_A\rangle \\ &= c_\beta \frac{\lambda_P(A)-\lambda_P(\beta)}{A-\beta}\,\langle \phi_A, e_\beta\rangle \geq 0\,.\end{aligned}$$

The same result holds true in the case where $\lambda_P(\alpha)$ is increasing up to A, and satisfies $\lambda_P(\beta) \geq \lambda_P(A)$ for all $\beta > A$.
Last but not least, in the case where $\lambda_P(\alpha)$ is increasing up to A, and there exists $A' > A$ such that $\lambda_P(A') = \lambda_P(A)$ the situation is quite different. We choose the smallest possible $A' > A$. The function defined by (4.1) is a particular solution only on the subset of the simplex $\mathcal{S}'$ defined by the following rule: $y \in \mathcal{S}'$ if the segment $[y, e_\infty]$ crosses the line Φ_0 on e_β with $\beta \leq A'$ (Figures 12 and 13).

(H4) When the quantity (3.5) does not have a constant sign, we cannot rule out the situation where the trajectories cross the line of eigenvectors Φ_0 anywhere. This causes problems on the proper definition of the ergodic set $\mathcal{Z}_0$ (see also discussion of how to remove **(H5)** below). This also causes problems on the monotonicity formulas (3.13)-(3.14). However we believe this is just a technical assumption. When the quantity $\left\langle \frac{de_\beta}{d\beta}, \frac{\Theta F e_\beta}{|\Theta F e_\beta|} \right\rangle$ changes sign then we may switch the parametrization from $A+\delta$ to $a-\delta$, or *vice-versa*, to preserve the monotonicity of $\beta(t)$.

(H5) This Hypothesis is essential to build properly the ergodic set $\mathcal{Z}_0$ in our way. In fact it guarantees that the trajectories which define the boundary $\partial\mathcal{Z}_0$ lie on the two opposite sides of Φ_0. This rules out a possible spiraling phenomenon. It could be possible to define the ergodic set $\mathcal{Z}_0$ in another way but this would lead to increasing complexity of the preceding steps.
We propose the following construction: starting from e_a, the trajectory γ_A^a crosses Φ_0 on e_{β_1} with $\beta_1 \geq A$ due to Assumption **(H4)**. If $\beta_1 > A$ we need to switch the control after crossing Φ_0 to preserve the monotonicity formula (Lemma 3.9). The trajectory starting from e_{β_1} with constant control a, $\gamma_a^{\beta_1}$ reaches Φ_0 on e_{β_2} with $\beta_2 \leq a$ due to Assumption **(H4)**. If $\beta_2 = a$ we are done: the two portions of trajectories γ_A^a and $\gamma_a^{\beta_1}$ enclose a stable set $\mathcal{Z}_0$ which is a good candidate to being the ergodic set.
If $\beta_2 < a$ however we switch again: the trajectory starting from e_{β_2} with constant control A, $\gamma_A^{\beta_2}$ reaches Φ_0 on e_{β_3} with $\beta_3 > \beta_1$ (because trajectories cannot cross). We define iteratively two monotonic sequences β_{2i} and β_{2i+1} (resp. decreasing and increasing) such that $\beta_{2i} < a$ and $\beta_{2(i+1)} > A$. The two limits $\underline{\beta} < \beta_{2i} < a < A < \beta_{2(i+1)} < \overline{\beta}$ define the boundary of a stable and controllable set $\mathcal{Z}_0$ through the following periodic cycle: starting from $e_{\underline{\beta}}$ the trajectory with constant control A reaches $e_{\overline{\beta}}$ in finite time, and the trajectory starting from $e_{\overline{\beta}}$ with constant control a reaches $e_{\underline{\beta}}$ in finite time.
We cannot rule out the existence of several such periodic cycles (with control a on Φ^+ and control A on Φ^-). In this case the set $\mathcal{Z}_0$ constructed above is certainly not attractive, because $\mathcal{Z}_0$ does not contain any other cycle than its boundary.

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Appendix

A Diagonalisation of $G + \alpha F$

In the case of the running example, we give a condition for which the matrix $G + \alpha F$ is diagonalisable (in $\mathbb{R}$).

Proposition A.1. *Consider the matrices G and F defined in* (1.4)*, with the condition $\tau_2 > 2\tau_1$. Then the matrix $G + \alpha F$ is diagonalisable in $\mathbb{R}$ for any $\alpha > 0$. Furthermore, the three real eigenvalues satisfy*

$$\lambda_1^* > 0 > \lambda_2 > \lambda_3.$$

Proof. We compute the value of the characteristic polynomial of $G + \alpha F$ at $x = 0$

$$P(0) = -\alpha\tau_1\tau_2\beta_3 - \alpha^2\tau_1\beta_2\beta_3 < 0$$

as for two well chosen negative values

$$P(-\alpha\beta_3) = \alpha\tau_2\beta_3(\alpha\beta_3 - 2\tau_1)$$

and

$$P(-\tau_2) = \alpha\beta_2(\tau_2^2 + \tau_1\tau_2) + \alpha\beta_3(\tau_2^2 - 2\tau_1\tau_2) - \alpha^2(\tau_1 + \tau_2)\beta_2\beta_3.$$

We notice that

$$P(-\alpha\beta_3) > 0 \quad \Leftrightarrow \quad \alpha > \frac{2\tau_1}{\beta_3}$$

and

$$P(-\tau_2) > 0 \quad \Leftrightarrow \quad \alpha < \frac{\tau_2}{\beta_3} + \frac{\tau_2}{\tau_1\tau_2}\frac{\tau_2 - 2\tau 1}{\beta_2}.$$

Because $\tau_2 > 2\tau_1$, we have

$$\frac{\tau_2}{\beta_3} + \frac{\tau_2}{\tau_1\tau_2}\frac{\tau_2 - 2\tau 1}{\beta_2} > \frac{\tau_2}{\beta_3} > \frac{2\tau_1}{\beta_3}$$

so if $P(-\alpha\beta_3) \leq 0$, then $P(-\tau_2) > 0$. Finally P is a third order polynomial which satisfies the following properties: (i) it tends to $-\infty$ at $-\infty$; (ii) it takes positive values for negative x; (iii) it is negative at $x = 0$; (iv) it tends to $+\infty$ at $+\infty$. Thus P has 3 real roots and it proves the proposition. □

B Criterion for Assumption (H4)

Proposition B.1. *Denoting (e_1, e_2, e_3) a basis of eigenvectors for $G + \alpha F$, so that $e_1 = e_\alpha$, and (ϕ_1, ϕ_2, ϕ_3) a basis of dual eigenvectors, we have the formula*

$$\left\langle \frac{de_\alpha}{d\alpha}, \Theta F e_\alpha \right\rangle = \frac{\lambda_2 - \lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}(\phi_2 F e_1)(\phi_3 F e_1) \left\langle e_2 - e_1, \Theta(e_3 - e_1) \right\rangle .$$

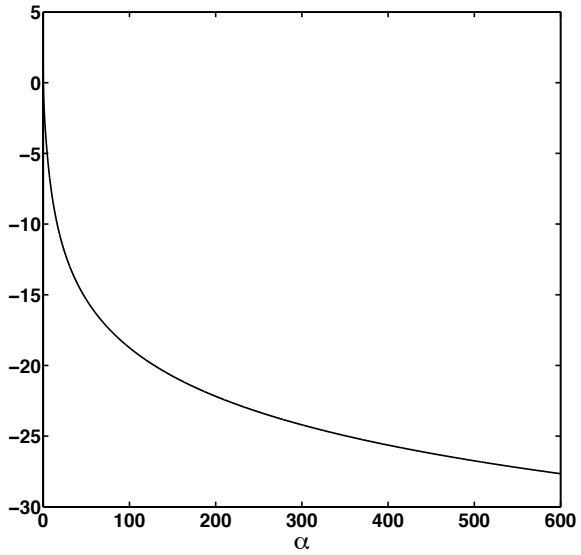

Figure 14: Numerical check of the Assumption **(H4)** for the running example, by using the formula in Proposition B.1. Plot of the quantity $-\left\langle \frac{de_\alpha}{d\alpha}, \Theta F e_\alpha \right\rangle$ as a function of α in log-scale.

Proof. Decompose $\frac{de_\alpha}{d\alpha}$ and Fe_α along the basis (e_1, e_2, e_3) :

$$\frac{de_\alpha}{d\alpha} = a_1 e_1 + a_2 e_2 + a_3 e_3, \qquad Fe_\alpha = b_1 e_1 + b_2 e_2 + b_3 e_3.$$

For all α we have $\langle m, e_\alpha \rangle = 1$ because $e_\alpha \in \mathcal{S}$, so $\left\langle m, \frac{de_\alpha}{d\alpha} \right\rangle = 0$. Because $m^T F = 0$, we also have $\langle m, Fe_\alpha \rangle = 0$. It leads to the relations

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0.$$

As a consequence we have

$$\begin{aligned}\left\langle \frac{de_\alpha}{d\alpha}, \Theta F e_\alpha \right\rangle &= \langle a_2(e_2 - e_1) + a_3(e_3 - e_1), b_2\Theta(e_2 - e_1) + b_3\Theta(e_3 - e_1)\rangle \\ &= a_2 b_3 \langle (e_2 - e_1), \Theta(e_3 - e_1)\rangle + a_3 b_2 \langle (e_3 - e_1), \Theta(e_2 - e_1)\rangle .\end{aligned}$$

Now we look for a relation between a_i and b_i and we start from

$$(G + \alpha F)e_\alpha = \lambda(\alpha) e_\alpha$$

which gives after differentiation with respect to α

$$(G + \alpha F)\frac{de_\alpha}{d\alpha} + Fe_\alpha = \lambda_\alpha \frac{de_\alpha}{d\alpha} + \lambda'(\alpha) e_\alpha.$$

Testing against ϕ_i with $i \in \{2, 3\}$, we obtain

$$\lambda_i \phi_i \frac{de_\alpha}{d\alpha} + \phi_i F e_1 = \lambda_1 \phi_i \frac{de_\alpha}{d\alpha}.$$

Finally we find

$$b_i = \phi_i F e_1 = (\lambda_1 - \lambda_i)\phi_i \frac{de_\alpha}{d\alpha} = (\lambda_1 - \lambda_i)a_i$$

and the result of the proposition follows. □

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